SPECTRAL STABILITY OF RELATIVE EQUILIBRIA

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Abstract. The spectral stability of synchronous circular orbits in a rotating conservative force field is treated using a recently developed Hamiltonian method. A complete set of necessary and sufficient conditions for spectral stability is derived in spherical geometry. The resulting theory provides a general unified framework that encompasses a wide class of relative equilibria, including the circular restricted three-body problem and synchronous satellite motion about an aspherical planet. In the latter case we find an interesting class of stable nonequatorial circular orbits. A new and simplified treatment of the stability of the Lagrange points is given for the restricted three-body problem.

Keywords : Stability, satellite orbits, three-body problem

1. Introduction

Rotating force fields are frequently encountered in celestial mechanics, for example geosynchronous satellite orbits (Blitzer *et al.*, 1962, 1985), planetary rings, galactic motion (Binney and Tremaine, 1987), the problem of two centers and the restricted three-body problem (Szebehely, 1970). The stability of equilibrium orbits in such systems is one of the central problems of celestial mechanics and a great deal of effort has been devoted to developing methods for calculating stability limits (Wintner, 1941, Siegel and Moser, 1971, Broucke, 1980).

It is useful to distinguish among three different kinds of local stability of autonomous flows (Holm *et al.*, 1985) :

(i) *Lyapunov Stability.* An equilibrium $\mathbf{z_0} \in \mathbf{R}^n$ of a flow $\dot{\mathbf{z}} = \mathbf{Z}(\mathbf{z})$ satisfying $\mathbf{Z}(\mathbf{z_0}) = 0$ is Lyapunov stable if for every neighborhood V of $\mathbf{z_0}$ there exists a subneighborhood $U \subset V$ such that $\mathbf{z}(\mathbf{0}) \in U \Rightarrow \exists \mathbf{z}(\mathbf{t}) \in V$ for all forward time.

(ii) *Linear Stability.* An equilibrium $\mathbf{z_0}$ is linearly stable if the orbits of the tangent map are bounded for all forward time.

(iii) *Spectral Stability.* An equilibrium $\mathbf{z_0}$ is spectrally stable if the spectrum of the tangent flow has no positive real part. For Hamiltonian systems this definition reduces to *neutral stability*, for which the spectrum is pure imaginary.

In general, *Lyapunov stability* $\Rightarrow$ *linear stability* $\Rightarrow$ *spectral stability*, but not conversely. It can be shown that an equilibrium is linearly stable iff it is spectrally stable and all Jordan blocks corresponding to eigenvalues on the imaginary axis are one-dimensional (Hirsch and Smale, 1974). Since the boundaries of linear and spectral stability are identical for Hamiltonian flows, the notion of spectral stability allows us to calculate stability limits without continually excluding multiple

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eigenvalue cases. In this paper "stable" will always mean "spectrally stable."

The important special case of axisymmetric potentials has received much attention in the literature, particularly motion about an oblate planet (Blitzer, 1962, Danby, 1968, Zare, 1983). Since the potential is cyclic in the longitude, the problem is fully autonomous and reduces to two degrees of freedom. Furthermore, *Dirichlet's Principle* now applies, which asserts that if the kinetic energy is positive definite, an equilibrium is *Lyapunov* stable if the potential energy U has a local minimum there (Abraham and Marsden, 1978). Thus it is possible to make much stronger statements about the stability of the motion for axisymmetric potentials. The stability of circular orbits in general axisymmetric gravitational (Howard, 1990a) and magnetic (Howard, 1990b) fields will be reported separately.

For nonaxisymmetric potentials it is advantageous to transform to a noninertial co-rotating coordinate system in which the equations of motion are autonomous. However, the kinetic energy is then no longer locally quadratic in the velocity components, so that the linearized kinetic and potential energies cannot be simultaneously diagonalized. For this reason most authors obtain a characteristic equation by linearizing the equations of motion, a procedure which often obscures the generic features of the stability analysis. We offer here a unified treatment of the spectral stability of circular orbits in arbitrary uniformly rotating potentials using a recently developed Hamiltonian method (Howard and MacKay, 1987a,b). This systematic approach yields explicit expressions in arbitrary dimension for the stability boundaries for tangent and Krein bifurcations in terms of the derivatives of the potential function. It turns out that harmonic potentials enjoy a privileged position in this theory, yielding especially simple stability conditions.

The paper is organized as follows. We begin by reviewing the basic features of Hamiltonian stability theory, which yields explicit stability boundaries for tangent bifurcations and Krein collisions. In Section 3 the results are applied to equilibrium orbits in an arbitrary three dimensional potential, for which the phase space is six dimensional. If the potential is symmetric with respect to an equatorial plane, then equatorial equilibrium orbits are possible. In such cases the latitudinal libration decouples from the equatorial modes, greatly simplifying the analysis. The resulting stability criteria are then applied to satellite motion about an aspherical planet, using a standard model potential which includes a J_2 term and an asymmetric longitude-dependent term proportional to the equatorial eccentricity ϵ. On the basis of available data the planets of our solar system are all shown to support stable synchronous satellite orbits.

In addition to equatorial equilibrium orbits we also find *nonequatorial* circular orbits for both oblate- and prolate-type potentials. For $J_2 > 0$ nonequatorial equilibrium orbits exist for $\epsilon \neq 0$, but are always unstable; for $J_2 < 0$, however, stable families of nonequatorial equilibria exist, even for $\epsilon = 0$. These results may have some bearing on observed ring structures around prolate galaxies (Binney and Tremaine, 1987). In Section 4 we revisit the venerable restricted three-body problem and obtain some familiar results directly from our general formulas. The present treatment not only puts this classic problem in perspective, but also yields a single simple condition for the stability of the colinear Lagrange points without the necessity of analyzing a single quintic equation.

2. Stability of Hamiltonian Flows

In this section we briefly review the elements of Hamiltonian stability theory. For a more complete treatment see Howard and MacKay (1987a,b). Hamilton's equations in n degrees of freedom may be written

$$\dot{\mathbf{z}} = J \cdot DH_z, \tag{2.1}$$

where $\mathbf{z} = (\mathbf{q}, \mathbf{p}) \in R^{2n}$, $H(\mathbf{z})$ is the Hamiltonian, DH_z is the derivative,

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{2.2}$$

and I_n is the $n \times n$ identity. The equilibria $\mathbf{z}_{0i}$ are given by setting $DH_z = 0$. The tangent flow (variational equations) near such an equilibrium point $\mathbf{z_0}$ is then

$$\dot{\xi} = L_0 \cdot \xi, \tag{2.3}$$

where $\xi = \mathbf{z} - \mathbf{z_0}$ and

$$L_0 = J \cdot D^2 H_0 \tag{2.4}$$

is an infinitesimally symplectic matrix (Abraham and Marsden, 1978).

The eigenvalues σ_i of L_0 are given by the characteristic equation

$$P(\sigma) = \det(L - \sigma I) = 0, \tag{2.5}$$

which is necessarily even;

$$P(\sigma) = \sigma^{2n} + A_1\sigma^{2n-2} + A_2\sigma^{2n-4} + \cdots + A_n. \tag{2.6}$$

The coefficients A_k are readily expressed as functions of the elements of L. Defining $\tau = -\sigma^2$, we obtain the reduced characteristic equation

$$Q(\tau) = \tau^n - A_1\tau^{n-1} + A_2\tau^{n-2} - \cdots + (-)^n A_n = 0. \tag{2.7}$$

The motion is said to be *spectrally stable* if all the zeroes of Q are non-negative real. The libration frequencies are then given by $\omega_i = \sqrt{\tau_i}, \quad i = 1, 2, ..., n$.

Spectral stability may be lost in just two ways :

(1) *Tangent Bifurcation :* a pair of pure imaginary eigenvalues $\pm\sigma$ coalesce at zero and split along the real axis (a stability index τ becomes negative).

(2) *Krein Collision :* a pair of imaginary eigenvalues σ_1 and σ_2 coalesce at a nonzero point and split off into the complex plane. Their complex conjugates

do likewise, forming a complex quadruplet. (Equivalently, two stability indices τ_1 and τ_2 merge at a nonzero point and become complex).

Thus, the transition boundary (in the space of polynomial coefficients) for tangent bifurcations is given by setting $\tau = 0$ in (2.7), while that for Krein collisions is given by the vanishing of the discriminant $\Delta(Q)$. In order for a pair of eigenvalues to actually leave the imaginary axis after a Krein collision it is also necessary that the double eigenvalue have mixed Krein signature (Moser, 1958). The stability region is thus bounded by the intersection of the transition boundaries defined by the plane $A_n = 0$ and the hypersurface $\Delta = 0$. Figures 1 and 2, taken from Howard and MacKay (1987b), depict the stability regions for four and six dimensional equilibria. The edges and corners of these simply connected regions correspond to various multiple eigenvalue configurations. It can be shown that two-and three degree of freedom equilibria are spectrally stable iff all $A_k \geq 0$ and $\Delta(Q) \geq 0$. A complete set of necessary and sufficient conditions for spectral stability in arbitrary dimension may be obtained by applying Sturm's theorem to the reduced characteristic equation (2.7), requiring that all its roots be non-negative real. The stability of a general periodic orbit may be treated by a similar analysis of the monodromy matrix (Howard and MacKay, 1987a).

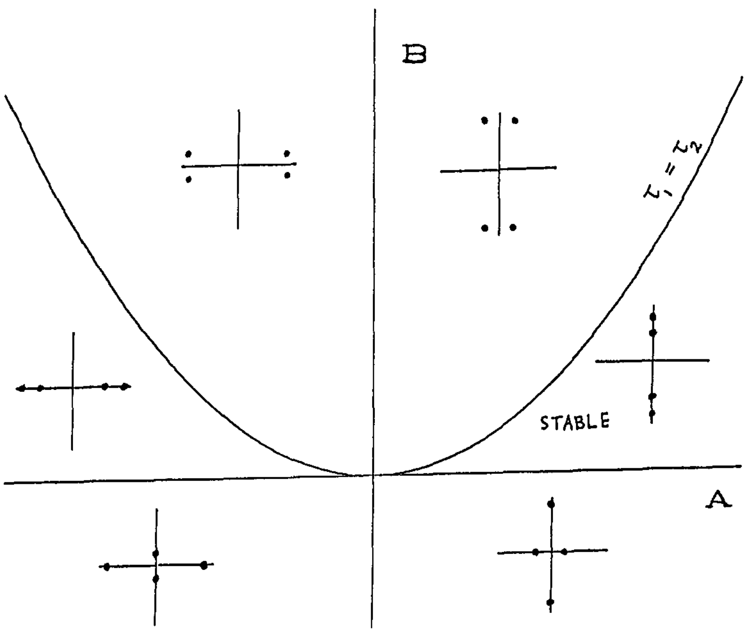

Fig. 1. Stability region for four dimensional equilibrium. A and B are the coefficients of the reduced characteristic polynomial.

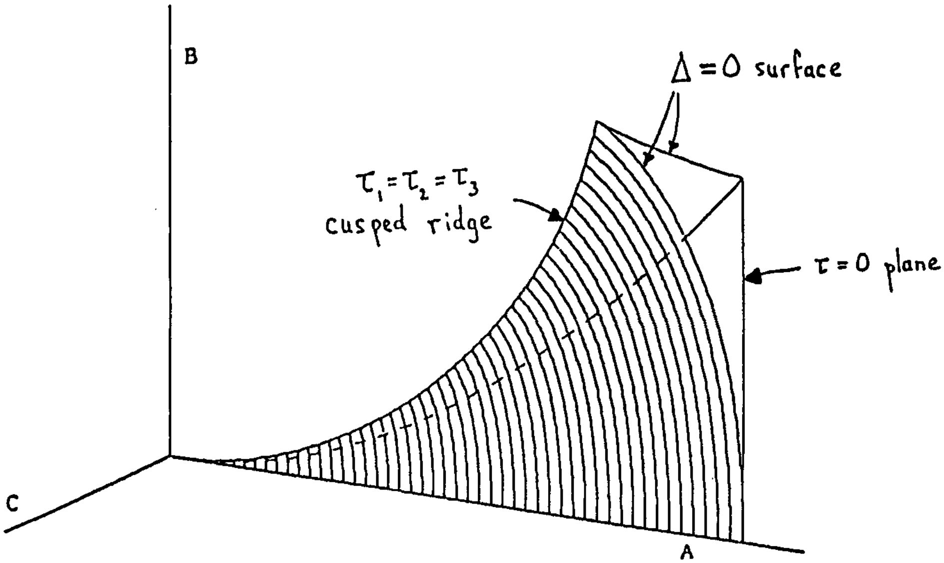

Fig. 2. Stability region for six dimensional equilibrium.

3. Equilibrium Orbits in a General Aspherical Potential

In this section we investigate the stability of circular orbits in an arbitrary three-dimensional rotating potential, for which the phase space is six-dimensional. Although we shall be primarily concerned with satellite orbits, the theory is quite general and not limited to gravitational systems. Thus, while nonharmonic potentials are allowed, restricting to harmonic potentials pays an immediate dividend in simplicity. Using a standard model for the Earth's gravitational potential, which includes longitudinal as well as latitudinal variations, Blitzer *et al.* (1962) demonstrated the existence of pairs of stable and unstable equatorial equilibrium orbits and showed that, for purely oblate potentials, the transverse (latitudinal) libration decoupled from the equatorial modes. However, their results were limited to a particular model in which the deviations from sphericity were very small, and they did not attempt to identify stability boundaries. In a later paper, Blitzer (1985) studied the stability of circular orbits in a general axisymmetric potential. Here we consider arbitrarily aspherical potentials and obtain general stability conditions in spherical polar coordinates.

The symmetric case is of central importance in celestial mechanics, as it includes equatorial orbits about an oblate planet, the predominant configuration for the natural satellites in our solar system. This problem has been studied by many authors (Danby, 1968, Zare, 1982), most often using the first two terms of the spherical harmonic expansion. For potentials even in the latitude we find that the cubic reduced characteristic polynomial factors for equatorial equilibria; the decoupling property is thus seen to be quite general. Using this factorization and the harmonic property, we obtain a greatly simplified set of conditions for spectral stability.

Next we apply the results to the standard planetary model with two variable coefficients (α, β), and derive simple explicit expressions for the transition boundaries for tangent bifurcations and Krein collisions in the $\alpha - \beta$ plane for each equilibrium orbit. For "obloidal" planets $(J_2 > 0)$ we find four basically different equatorial equilibria, only one of which is stable for a range of α and β. This "principal orbit" loses stability by a Krein collision of the equatorial modes upon crossing a stability boundary given by the vanishing of the discriminant of $Q(\tau)$. For purely oblate $(\beta = 0)$ potentials two pairs of eigenvalues merge, so that stability is lost via a double tangent bifurcation. For "proloidal" potentials $(J_2 < 0)$ we again find a single stable equatorial orbit. Working out the transition boundaries for this case, we find that stability can be lost through a tangent bifurcation of the transverse mode, as well as by Krein collision of the equatorial modes. Morever, stable nonequatorial orbits are born out of the transverse destabilization! The stability boundaries for these orbits are found analytically for the equatorial orbits and by a numerical eigenvalue calculation for the nonequatorial orbits.

3. 1. EQUATIONS OF MOTION

Consider a small mass moving in a smooth arbitrary potential U rotating at uniform angular speed ω. The Hamiltonian per unit mass is, in spherical polar coordinates (ϕ = longitude, θ = colatitude)

$$H = \frac{1}{2}(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{p_\phi^2}{r^2 \sin^2\theta}) + U(r, \theta, \phi - \omega t), \tag{3.1}$$

where the canonical momenta are

$$p_r = \dot{r}, \qquad p_\theta = r^2\dot{\theta}, \qquad p_\phi = r^2 \sin^2\theta \; \dot{\phi}.$$

To remove the time dependence, let us transform to a co-rotating coordinate system r', θ', ϕ' using the generating function

$$F = rp_r' + \theta p_\theta' + (\phi - \omega t)p_\phi'$$

so that $\phi' = \phi - \omega t$ and $p_\phi' = r^2 \sin^2\theta(\dot{\phi}' + \omega)$. The new Hamiltonian (Jacobi integral) is then autonomous,

$$H = \frac{1}{2}(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{p_\phi^2}{r^2 \sin^2\theta}) + U(r, \theta, \phi) - \omega p_\phi \tag{3.2}$$

where we have dropped the primes for convenience.

The equations of motion in the rotating system are

$$\begin{aligned} \dot{r} &= p_r \\ \dot{\theta} &= \frac{p_\theta}{r^2} \\ \dot{\phi} &= \frac{p_\phi}{r^2 \sin^2 \theta} - \omega \end{aligned}$$

$$\begin{aligned} \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{r^3} + \frac{p_\phi^2}{r^3 \sin^2 \theta} - \frac{\partial U}{\partial r} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{r^2 \sin^3 \theta} - \frac{\partial U}{\partial \theta} \\ \dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = -\frac{\partial U}{\partial \phi}, \end{aligned} \tag{3.3}$$

so that the relative equilibria (r_0, θ_0, ϕ_0) are given by

$$p_r = p_\theta = 0, \qquad p_\phi = r_0^2 \omega \sin^2 \theta_0$$

and the simultaneous solutions of

$$\begin{aligned} U_r &= r\omega^2 \sin^2 \theta \\ U_\theta &= \frac{1}{2} r^2 \omega^2 \sin 2\theta \\ U_\phi &= 0, \end{aligned} \tag{3.4}$$

where the subscripts denote partial derivatives. In general there will be more than one equilibrium orbit, not necessarily equatorial.

In writing H in the form (3.1) we have implicitly assumed the existence of a spin axis fixed in the inertial frame. In the case of a rotating planet this is always possible if the spin axis coincides with a principal inertial axis (I_3, say) which is not an intermediate axis. Thus, an axisymmetric body has a stable rotational mode with the angular momentum vector **L** coincident with the axis of symmetry. In general, however, ω precesses around **L** at the Chandler frequency $\Omega = (1 - I_1/I_3)\omega_3$ with an amplitude depending on the initial conditions. If $I_1 \neq I_2$ the motion is more complicated (Goldstein, 1980). Thus, strictly speaking our analysis is valid only when I_3 is a stable spin axis, i.e. $I_3 \notin [I_1, I_2]$. If the precession is large and rapid, then U in the inertial frame will contain additional time-dependences which cannot be removed by a simple transformation to rotating coordinates. For the Earth the period of the Chandler precession is about 420 days and the amplitude only a few tenths of an arc-second, so that the effect on satellite orbits is quite negligible.

3. 2. STABILITY

Note that in the vicinity of a relative equilibrium the kinetic energy T is not a positive-definite quadratic form. Consequently, we cannot employ the customary Lagrangian technique of simultaneously diagonalizing T and U (Goldstein, 1980). Following instead the Hamiltonian program outlined in Section 2 and evaluating the variational matrix $L = J \cdot D^2 H_0$, we find

$$-L = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{r^2} & 0 \\ 2\omega/r & 2\omega\cot\theta & 0 & 0 & 0 & \frac{-1}{r^2}\sin^2\theta \\ U_{rr} + 3\omega^2\sin^2\theta & U_{r\theta} + r\omega^2\sin 2\theta & U_{r\phi} & 0 & 0 & -2\omega/r \\ U_{r\theta} + r\omega^2\sin 2\theta & U_{\theta\theta} + r^2\omega^2(2+\cos 2\theta) & U_{\theta\phi} & 0 & 0 & -2\omega\cot\theta \\ U_{r\phi} & U_{\theta\phi} & U_{\phi\phi} & 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

The reduced characteristic equation is

$$Q(\tau) = \tau^3 - A\tau^2 + B\tau - C \tag{3.6}$$

where

$$\begin{aligned} A &= -\frac{1}{2}Tr(L^2) \\ -4B &= Tr(L^4) - \frac{1}{2}(TrL^2)^2 \\ C &= \det L. \end{aligned} \tag{3.7}$$

The motion is spectrally stable iff

$$A, B, C \geq 0 \tag{3.8}$$

and

$$\Delta = 4(A^2 - 3B)(B^2 - 3AC) - (AB - 9C)^2 \geq 0. \tag{3.9}$$

If any one of these conditions is violated, the motion will become linearly unstable. It should be noted, however, that an orbit reaching the boundary for Krein collisions will not actually destablilize unless the eigenvalues have mixed Krein signature (Howard and MacKay, 1987a).

The algebra required to evaluate these conditions is fairly daunting, especially (3.9) for Krein collisions. Fortunately, for equatorial orbits the latitudinal libration separates out, so that it is only necessary to examine the discriminant of a simple quadratic. The stability of nonequatorial orbits will be determined numerically from the eigenvalues of L.

3. 3. EQUATORIAL ORBITS

If U is even in the latitude $\pi/2 - \theta$, inspection of (3.4) shows that there is the possibility of equatorial equilibria. Taking $\theta = \pi/2$ and $U_\theta = U_{r\theta} = 0$ in (3.5), we find, after some manipulation,

$$\begin{aligned} A &= U_{rr} + \frac{1}{r^2}\left(U_{\theta\theta} + U_{\phi\phi}\right) + 4\omega^2 \\ B &= (U_{rr} + 3\omega^2)(\frac{1}{r^2}U_{\theta\theta} + \omega^2) + \frac{1}{r^2}U_{\phi\phi}(U_{rr} + \frac{1}{r^2}U_{\theta\theta}) - \frac{1}{r^2}U_{r\phi}^2 \\ C &= \left(\frac{1}{r^2}U_{\theta\theta} + \omega^2\right)\left[\frac{1}{r^2}U_{\phi\phi}\left(U_{rr} - \omega^2\right) - \frac{1}{r^2}U_{r\phi}^2\right]. \end{aligned} \tag{3.10}$$

For harmonic potentials, A reduces to a constant,

$$A = 2\omega^2. \tag{3.11}$$

Minor simplifications also occur in the coefficients B and C in the case of harmonic potentials. These expressions may then be used to calculate stability boundaries for tangent bifurcations and Krein collisions.

Inspection of the variational matrix L shows that the θ-motion is entirely decoupled from the r- and ϕ- librations, so that (3.6) factors;

$$Q(\tau) = (\tau - \tau_\theta)(\tau^2 - A'\tau + B') = 0 \tag{3.12}$$

where

$$\tau_\theta = \frac{1}{r^2}U_{\theta\theta} + \omega^2 \tag{3.13}$$

and

$$\begin{aligned} A' &= U_{rr} + \frac{1}{r^2}U_{\phi\phi} + 3\omega^2 \\ B' &= \frac{1}{r^2}U_{\phi\phi}\left(U_{rr} - \omega^2\right) - \frac{1}{r^2}U_{r\phi}^2. \end{aligned} \tag{3.14}$$

The libration frequencies $\omega_i = \sqrt{\tau_i}$ easily follow from (3.12). For harmonic potentials, at equilibrium,

$$A' = \omega^2 - \frac{1}{r^2}U_{\theta\theta}. \tag{3.15}$$

The original coefficients are related to (A', B') as follows :

$$\begin{aligned} A &= A' + \tau_\theta \\ B &= B' + A'\tau_\theta \\ C &= B'\tau_\theta, \end{aligned} \tag{3.16}$$

in agreement with (3.10) and (3.14). The discriminant of Q is

$$\Delta = (\tau_r - \tau_\theta)^2(\tau_\phi - \tau_\theta)^2\Delta', \tag{3.17}$$

where τ_r and τ_ϕ are the remaining zeroes of Q (not necessarily purely radial or azimuthal), and

$$\Delta' = A'^2 - 4B' = (\tau_r - \tau_\phi)^2 \tag{3.18}$$

or

$$\Delta' = (U_{rr} + \frac{1}{r^2}U_{\phi\phi} + 3\omega^2)^2 + \frac{4}{r^2}U_{r\phi}^2 + \frac{4}{r^2}U_{\phi\phi}(\omega^2 - U_{rr}). \tag{3.19}$$

Now observe that since τ_θ is real, either τ_r and τ_ϕ are both real, or complex conjugates. In either case,

$$\text{sgn } \Delta = \text{sgn } \Delta' \tag{3.20}$$

so that our stability conditions are $A', B', \tau_\theta, \Delta' \geq 0$, which are equivalent to, but much simpler than, the original set $B, C, \Delta \geq 0$. Since $A' + \tau_\theta = 2\omega^2$ for harmonic potentials, the conditions $A', \tau_\theta \geq 0$ may be replaced by

$$|U_{\theta\theta}| \leq r^2\omega^2. \tag{3.21a}$$

Alternatively,

$$0 \leq \tau_\theta \leq 2\omega^2. \tag{3.21b}$$

According to the general theory, spectral stability may be lost in just two ways; (i) a tangent bifurcation, in which the coefficient C becomes negative, and (ii) a Krein collision, in which the discriminant Δ passes through zero. From (3.16) we see that a tangent bifurcation occurs when either B' or τ_θ becomes negative. The role of A' is to exclude the unstable portion of the $\Delta > 0$ manifold, as will be seen below.

3. 4. MODEL POTENTIAL

A popular model gravitational potential which includes both longitudinal and latitudinal variations is the truncated spherical harmonic expansion (Blitzer *et al.*, 1962),

$$U(r, \theta, \phi) = -\frac{\mu}{r} + \frac{\sigma\mu}{r^3}\left[a(3\cos^2\theta - 1) + b\sin^2\theta\cos 2\phi\right] \tag{3.22}$$

where $\mu = GM, \sigma = \pm 1, a = \frac{1}{2}J_2R^2, b = \frac{1}{2}\epsilon R^2, R$ is the average planetary radius, and ϵ is the mean ellipticity of the equator. For $\sigma = +1$ we shall speak of "obloidal" potentials while the case $\sigma = -1$ will be dubbed "proloidal." The terms oblate and prolate will be reserved for purely axisymmetric potentials. We begin with the obloidal case, $\sigma = +1$.

A. Obloidal Planet

The equilibrium conditions (3.4) are

$$\begin{aligned}
U_r &= \frac{\mu}{r^2} - \frac{3\mu}{r^4}\left[a(3\cos^2\theta - 1) + b\sin^2\theta\cos 2\phi\right] = r\omega^2\sin^2\theta \\
U_\theta &= -\frac{\mu}{r^3}(3a - b\cos 2\phi)\sin 2\theta = \frac{1}{2}r^2\omega^2\sin 2\theta \\
U_\phi &= -\frac{2\mu b}{r^3}\sin^2\theta\sin 2\phi = 0.
\end{aligned} \tag{3.23}$$

From (3.23c) it follows that *all* equilibrium orbits (proloidal as well as obloidal, nonequatorial as well as equatorial) have azimuth

$$\phi_0 = 0,\ \pi/2,\ \pi,\ 3\pi/2. \tag{3.24}$$

With $\cos 2\phi_0 = \sigma_1 = \pm 1$, the first two conditions (3.23) become

$$\omega^2 \sin^2\theta\, r^5 - \mu r^2 + 3\mu[2a - (3a - \sigma_1 b)\sin^2\theta] = 0 \tag{3.25}$$

$$[\omega^2 r^5 + 2\mu(3a - \sigma_1 b)]\sin 2\theta = 0. \tag{3.26}$$

Note that these equations are independent of any assumptions about the symmetry of L, so that the possibility of nonequatorial orbits must be considered.

Equatorial Orbits

Since b always occurs in the combination $\sigma_1 b$, we may without loss of generality take it to be positive; negative values of b then simply correspond to a 90^0 rotation in ϕ_0. If $\sigma_1 = -1$, (3.26) shows that only equatorial orbits are allowed. If $\sigma_1 = +1$, then we still get equatorial orbits, with the intriguing possibility of nonequatorial orbits for $b > 3a$. In the latter case the simplified theory of Section 3.3 does not apply and the question of stability is best settled by calculating the eigenvalues of L numerically; this is done in Section 3.4 below. In the present case we put $\theta_0 = \pi/2$ and $\sigma_1 = +1$ in (3.25) to obtain the radial equilibrium equation

$$\omega^2 r_0^5 - \mu r_0^2 - 3\mu(a - b) = 0, \tag{3.27}$$

which is the appropriate generalization of Kepler's third law. It is interesting that when $a = b$ this reduces to Kepler's third law for a spherical planet, with the equilibrium radius independent of both a and b.

At this point it is convenient to introduce dimensionless scaled variables $x = (r/R)\hat{\mu}^{-1/3}$, $\alpha = (a/R^2)\hat{\mu}^{-2/3}$, and $\beta = (b/R^2)\hat{\mu}^{-2/3}$, where $\hat{\mu} = \mu/\omega^2 R^3$ is the average ratio of the gravitational acceleration to the centrifugal acceleration at the equator. Equation (3.27) then simplifies to

$$x^5 - x^2 - 3(\alpha - \beta) = 0. \tag{3.27a}$$

For $|\alpha - \beta| << 1$ the solution near $x = 1$ is approximately

$$x \approx 1 + (\alpha - \beta) - 3(\alpha - \beta)^2 + \cdots.$$

All the stability boundaries will turn out to depend only on the two dimensionless parameters $\alpha = \frac{1}{2}J_2\hat{\mu}^{-2/3}$ and $\beta = \frac{1}{2}\epsilon\hat{\mu}^{-2/3}$.

For $\beta < \alpha$ Descarte's rule of signs implies that there is a single positive real root r_0. For $\beta > \alpha$ there are either two or no positive roots, depending on the sign of the discriminant. It is easily seen that (3.27a) has a double root when

$$\beta - \alpha = \frac{1}{5}\left(\frac{2}{5}\right)^{2/3} = \beta_0 = 0.1086 \tag{3.28}$$

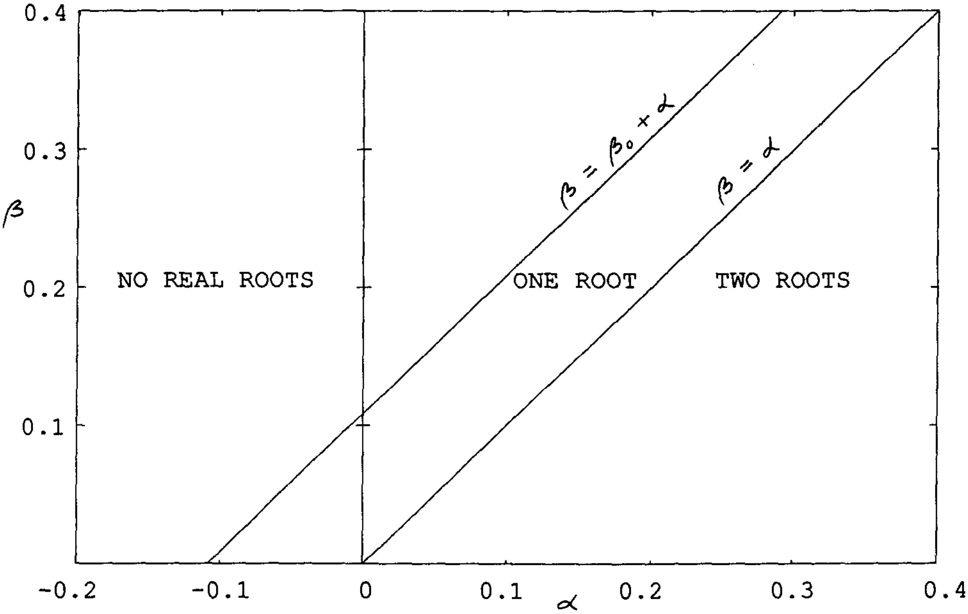

Fig. 3. Regions in the $\alpha - \beta$ plane where the radial equation (3.27) has zero, one and two roots for $\epsilon > 0$.

so that the number of equilibrium orbits for $\sigma_1 = +1$ are as follows :

$$\begin{aligned} \beta < \alpha &: 1 \text{ positive root} \\ \alpha < \beta < \alpha + \beta_0 &: 2 \text{ positive roots} \\ \beta > \alpha + \beta_0 &: 0 \text{ positive roots,} \end{aligned} \tag{3.29}$$

as shown in Figure 3.

When $\beta < 0$ ($\sigma_1 = -1$), Eq.(3.27) has a single positive root $\forall \alpha, \beta$, which will be seen to be always unstable. Let us now work out the transition boundaries (3.13), (3.14) and (3.19) in the $\alpha - \beta$ plane.

(i) $\tau_\theta \geq 0$: Using (3.22) in (3.13), we find

$$\tau_\theta = \omega^2[1 + \frac{2}{x^5}(3\alpha - \beta)] \tag{3.30}$$

where $x(\alpha, \beta)$ is one of the positive roots of (3.27a). The single positive root for $\beta < \alpha$ is clearly stable to transverse librations; in fact the two positive roots r_{01} and r_{02} of (3.27), which exist for $\alpha < \beta < \alpha + \beta_0$, are both stable to transverse librations, provided that $\beta < 3\alpha$. It remains to investigate the triangular region between the β-axis and the line $\beta = 3\alpha$ (Figure 4).

Eliminating x between (3.27a) and (3.30) yields the following locus of points where $\tau_\theta = 0$:

$$(5\beta - 9\alpha)^5 = 4(\beta - 3\alpha)^2. \tag{3.31}$$

It can be shown that $\tau_\theta < 0$ for the smaller root r_{01} in the region bounded by this curve and the β-axis. The outer orbit is stable to transverse librations wherever it is defined in the $\alpha - \beta$ plane. Finally, we observe that the single root for $\sigma_1 = -1$ is stable to transverse librations $\forall \alpha, \beta$.

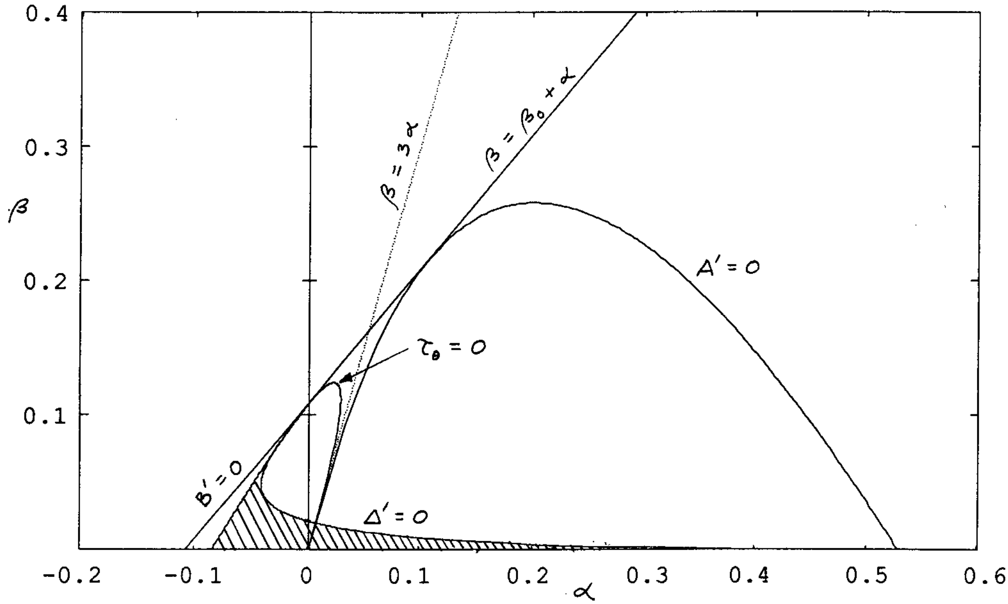

Fig. 4. Stability diagram for principal equilibrium orbit in model geophysical potential. The stable region is crosshatched. For $\alpha > 0$ (oblate planet) stability is lost via Krein collisions except for $\beta = 0$, when a double tangent bifurcation occurs at $\alpha = \alpha_0$. For $\alpha < 0$ (mythological prolate planet) stability can be lost either by an equatorial Krein collision or a transverse tangent bifurcation; when $\beta = 0$ stability is again lost via a double tangent bifurcation, at $\alpha = -1.04$ (off scale).

(*ii*) $A' \geq 0$: Using (3.22) in (3.14a) gives

$$A' = \omega^2[1 - \frac{2}{x^5}(3\alpha - \beta)]. \tag{3.32}$$

In this case we see immediately that both orbits have $A' > 0$ in the wedge where $\beta \geq 3\alpha$. To investigate the region outside the wedge we first calculate the locus of points where $A' = 0$. The result is

$$(3\alpha + \beta)^5 = 4(3\alpha - \beta)^2, \qquad \beta < 3\alpha. \tag{3.33}$$

(*iii*) $B' \geq 0$: Using (3.22) in (3.14b) gives

$$B' = \frac{12\omega^4\beta}{x^{10}}[x^2 + 5(\alpha - \beta)]. \tag{3.34}$$

This coefficient is crucial because, together with τ_θ, its vanishing yields the boundary for tangent bifurcations. Thus, its sign determines the character of the four fixed points (3.24), which are seen to alternate in type with σ_1 (which we have incorporated into the sign of β). For $\sigma_1 = +1$, setting $B' = 0$ and using (3.27a) to eliminate x we find that the stability boundary is just the line $\beta = \alpha + \beta_0$, at which the two positive roots merge and become complex. For $\sigma_1 = -1$, B' is always negative, so that this orbit is unstable $\forall \alpha, \beta$.

(iv) $\Delta' \geq 0$: Using (3.22) in (3.19) gives

$$\frac{x^{10}\Delta'}{\omega^4} = [x^2 - (3\alpha + \beta)]^2 - 48\beta[x^2 + 5(\alpha - \beta)]. \tag{3.35}$$

Setting $\Delta' = 0$ and solving the resulting quadratic in x^2, we find

$$x^2 = 3\alpha + 25\beta \pm 8[6\beta(\alpha + \beta)]^{1/2}. \tag{3.36}$$

Employing (3.27a) to eliminate x then gives the transition boundary for Krein collisions

$$4\{3\alpha + 11\beta \pm 4[6\beta(\alpha + \beta)]^{1/2}\}^2 = \{3\alpha + 25\beta \pm 8[6\beta(\alpha + \beta)]^{1/2}\}^5. \tag{3.37}$$

To implement these results we must identify each of the transition boundaries with the appropriate root of (3.22) and determine on which side of the curve the corresponding coefficient is positive. This was accomplished partly by analysis and partly by numerically scanning the $\alpha - \beta$ plane for each root. The results for the principal orbit r_{02} are shown in Figure 5. The transition boundary for tangent bifurcations is the line $\beta = \alpha + \beta_0$, at which r_{02} merges with r_{01}. B' is positive for r_{02} everywhere below this line. However, it is *not* a stability boundary, because Δ' is negative above its transition boundary, which is seen to lie near the α-axis. The stable region is the triangular area below the $\Delta' = 0$ curve, which is therefore a true stability boundary. The $\Delta' = 0$ curve is tangent to the α-axis at

$$\alpha_0 = (\frac{4}{27})^{1/3} = 0.5291,$$

where the $A' = 0$ curve intersects the α-axis. The role of the condition $A' = 0$ is thus to exclude the unstable portion of the $\Delta' > 0$ manifold, in this case the region where $\alpha > \alpha_0$. (The $\pm$ signs in (3.37) correspond to the stable and unstable portions, respectively.) The $B' = 0$ line is tangent to the $A' = 0$ curve (3.33) at the point $(\alpha_1, 2\alpha_1)$, where $\alpha_1 = \beta_0$. For $\alpha > \alpha_1$ the $A' = 0$ curve is a transition boundary for r_{02}; for $\alpha < \alpha_1$ it applies to r_{01}. Since $B' < 0$ everywhere for r_{01}, the inner orbit is always unstable and we will not discuss it further here. These results have all been verified by numerical computation of the eigenvalues of the matrix L.

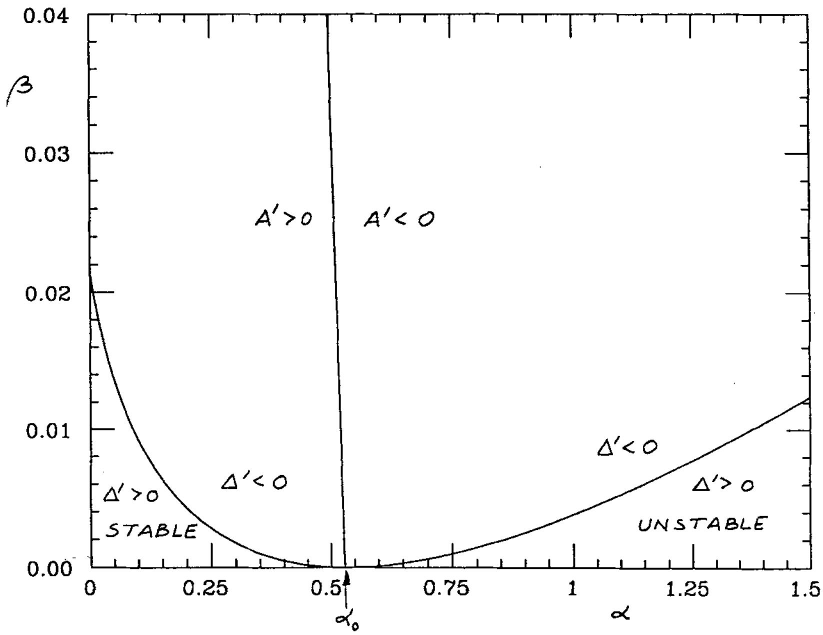

Fig. 5. Enlargement of the stability diagram of Figure 4 in the region near the boundary for Krein collisions for oblate potentials, showing how the $A' = 0$ curve excludes the unstable portion of the $\Delta' > 0$ region.

For the Earth, $J_2 = 1.08 \times 10^{-3}$ and $\epsilon = 3.21 \times 10^{-5}$, so that $\hat{\mu} = 289, \alpha = 1.24 \times 10^{-5}$ and $\beta = 3.67 \times 10^{-7}$. Thus, geosynchronous satellites are orders of magnitude away from the stability boundary in Figure 5! Even Saturn, which has $J_2 = 0.0165$ (Hubbard, 1989) and presumably $\epsilon \to 0$, so that $\hat{\mu} = 5.91$, we find $\alpha = 0.0025$, still a long way from the stability boundary. Does this mean that stable synchronous orbits always exist? Clearly, on physical grounds $\hat{\mu}$ is limited from below by a factor of a few. Further, it seems unlikely to encounter conditions where α could exceed 0.25. Yet a value of $\beta > 0.01$ is not at all inconceivable, especially for asteroids, which are typically quite irregular. Indeed, we could well argue that orbits about asteroids would be typically unstable. It is also worth noting that since α and β scale as $\omega^{4/3}$, larger rotational speeds correspond to less stable synchronous orbits.

Nonequatorial Orbits

When $\sigma_1 = +1$ and $\beta > 3\alpha$, the expression in brackets in Eq. (3.26) can vanish for $\theta \neq \pi/2$, which yields an equation for the equilibrium radius,

$$x^5 = 2(\beta - 3\alpha). \tag{3.38}$$

Note that although typically $x << 1$, the corresponding physical radius $r = \hat{\mu}^{1/3} R x$ is typically greater than R. Substituting this result into (3.25) then gives an expression for θ_0 :

$$\sin^2\theta_0 = \frac{x^2 - 6\alpha}{5(\beta - 3\alpha)}. \tag{3.39}$$

These orbits exist for $0 \le \sin^2\theta_0 \le 1$. The latter condition turns out to be exactly (3.31), the locus of points where $\tau_\theta = 0$, while the former condition is equivalent to

$$\beta = 3\alpha + \frac{1}{2}(6\alpha)^{5/2}. \tag{3.40}$$

The nonequatorial equilibria exist everywhere above this curve and outside the $\tau_\theta = 0$ lobe. However, numerical calculation of the eigenvalues of L suggests that this family of nonequatorial orbits is always unstable.

B. Proloidal Potentials

While proloidal planets do not exist in our solar system, proloidal moons and asteroids are not at all uncommon and are capable of supporting satellites. Ring structures have also been observed around prolate galaxies (Binney and Tremaine, 1987). Moreover, one cannot fully understand the obloidal case without some knowledge of the proloidal stability boundaries.

Equatorial Orbits

Again we may assume without loss of generality that $\beta > 0$. Replacing α by $-\alpha$ in (3.27), we see that there are either two or no positive roots. We shall ignore the unstable inner orbit and work out the transition boundaries for the outer orbit. The transition boundary for transverse tangent bifurcations ($\tau_\theta = 0$) is obtained by simply reversing the sign of α in (3.31) and turns out to apply to the outer orbit (in contrast to the obloidal case, where this boundary applied to the already unstable inner orbit). The $A' = 0$ curve does not exist in this quadrant. The transition boundary for equatorial tangent bifurcations is just the extension of the $B' = 0$ line from the first quadrant, shown as the dashed line in Figures 4 and 6. Finally, the transition boundary for Krein collisions is given by (3.37) where again both branches are required to draw the entire curve. The resulting stability region for equatorial orbits is thus the fin-shaped area between the $\Delta' = 0$ and the $\tau_\theta = 0$ curves, as depicted in Figure 6; the $B' = 0$ curve is not accessible from the stable region.

Nonequatorial Orbits

Setting $\alpha \to -\alpha$ in (3.38) and (3.39), we find that nonequatorial equilibria exist everywhere in the second quadrant above the τ_θ curve. While these stability boundaries may be calculated in principal by setting $\Delta = 0$ in (3.9) the results are too unwieldy to be useful. A numerical eigenvalue calculation shows these orbits to be stable in two regions : a narrow band above the $\tau_\theta = 0$ curve and a second layer above and along the negative α-axis to the left of the point $\alpha = -\alpha_2$, where

$$\alpha_2 = \frac{1}{9}(\frac{4}{9})^{1/3} = 0.0848 \tag{3.41}$$

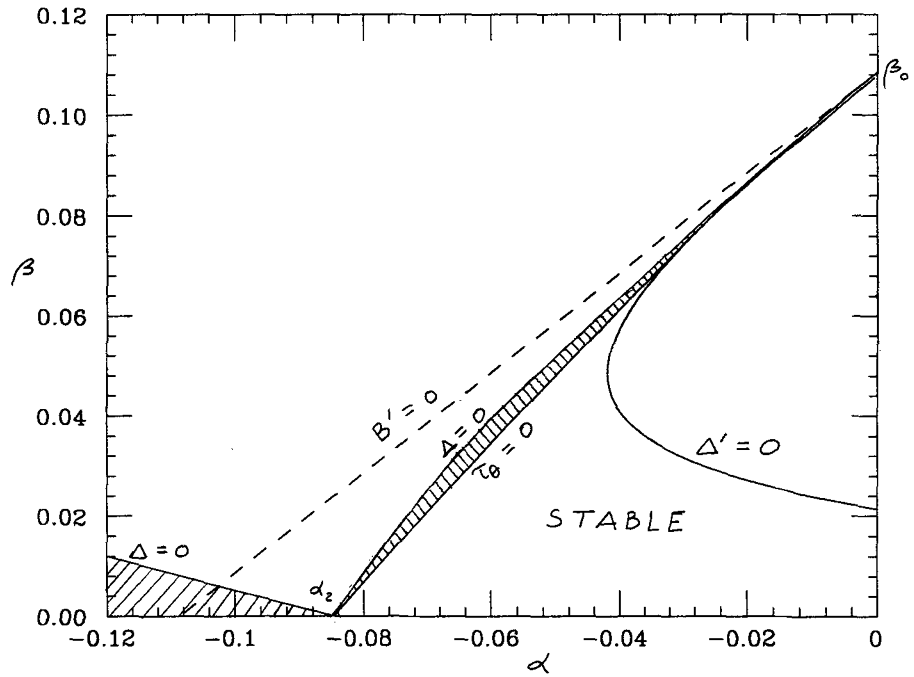

Fig. 6. Enlargement of the stability diagram of Figure 4 for prolate potentials. Stability can be lost either by crossing the boundary for Krein collisions ($\Delta' = 0$) or via a transverse tangent bifurcation, in which stable nonequatorial orbits are born. These orbits are stable in the shaded areas, losing stability via a Krein collision ($\Delta = 0$).

as indicated by the shaded areas in Figure 6. The thickness of the first layer shrinks to zero at the points $(0, \beta_0)$ and $(-\alpha_2, 0)$, while the second reaches a maximum near $\alpha = -0.2$, again shrinking to zero as $\alpha \to -\alpha_3 \approx -1.04$. Stability is lost via an equatorial Krein collision upon crossing this boundary upwards or by a double tangent bifurcation as α decreases through $-\alpha_3$ with β held equal to zero. Thus, if we imagine the shape of the planet to be changed adiabatically by some giant hand, a librating equatorial orbit would change smoothly into a nonequatorial one above or below the equatorial plane, depending on the phase of its libration.

C. Axisymmetric Potentials

It is instructive to compare these results with the stability boundaries for a purely oblate or prolate potential. In this case one can construct a 4D reduced Hamiltonian from the original 6D one by incorporating the longitudinal kinetic energy into an effective potential. In the reduced system, Krein collisions are impossible, and $A = 2\omega^2$ for harmonic potentials, so that a necessary and sufficient condition for spectral stability is that $B \geq 0$, or

$$-[U_{rr} + 3\omega^2 \sin^2\theta]\ [U_{rr} + \omega^2(1 - 3\cos^2\theta)] \geq [\frac{1}{r}U_{r\theta} + \omega^2 \sin 2\theta]^2. \qquad (3.42)$$

For the model potential (3.22), with $\theta = \pi/2$ and $\beta = 0$, it is easily shown that stability is lost by a tangent bifurcation when $\alpha > \alpha_0$ or $\alpha < \alpha_2$. Thus, in the full 6D treatment, the α-axis is a singular direction, corresponding to reduction from 6D to 4D. In this event it is readily seen that A' is the appropriate coefficient to demark tangent bifurcations. In the purely prolate case we also find stable nonequatorial equilibria which are born when α decreases through $-\alpha_2$ and subsequently lose stability by a tangent bifurcation at $-\alpha_3$. To see this, set $\beta = 0$ in (3.42) to obtain $-1 \leq \cos\theta_0 \leq -1/15$, or $\theta_0^* \leq \theta \leq \pi/2$, where $\sin\theta_0^* = \sqrt{8/15}$. The corresponding range of α for stable motion is then $-\alpha_3 \leq \alpha \leq -\alpha_2$, where

$$\alpha_3 = (\frac{9}{8})^{1/3} = 1.040. \qquad (3.43)$$

4. Restricted Three-Body Problem

It is also of interest to apply the Hamiltonian method to the restricted three-body problem. As usual, we adopt the idealized model of a negligibly small mass m moving in the gravitation field of two large bodies M_1 and M_2 revolving in circular orbits about their center of mass, as depicted in Figure 7. We wish to find all circular equilibrium orbits and determine their spectral stability. This problem is almost invariably solved in cartesian coordinates, a singularly inappropriate way to treat a rotating system! Here we employ the more natural spherical polar coordinate system, thereby obtaining the five equilibria and their stability boundaries directly and economically, without any hand-waving.

First observe that the present problem is a just a particular case of the general problem treated in Section 3, so that many of the results of that section apply here. Thus, the Hamiltonian (Jacobi integral) in a coordinate system rotating with M_1 and M_2 is (3.2), with angular velocity given by Kepler's third law,

$$\omega^2 = \frac{GM}{d^3}, \qquad (4.1)$$

where $d = d_1 + d_2$ is the distance between M_1 and M_2. Taking m as the unit of mass, let us scale time by setting $GM = 1$, where $M = M_1 + M_2$. The potential energy then takes the form,

$$U(r,\theta,\phi) = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}, \qquad (4.2)$$

where $\mu_1 = M_1/M$, $\mu_2 = M_2/M$, and, of course, $\mu_1 + \mu_2 = 1$. From Figure 7 we have

$$\begin{aligned} r_1^2 &= r^2 + d_1^2 - 2rd_1 \sin\theta\cos\phi \\ r_2^2 &= r^2 + d_2^2 + 2rd_2 \sin\theta\cos\phi. \end{aligned} \qquad (4.3)$$

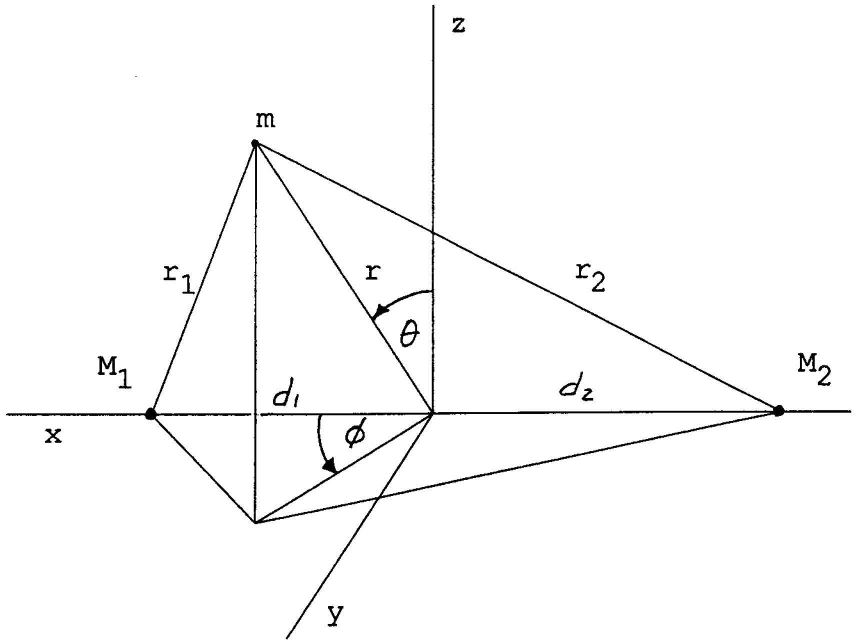

Fig. 7. Geometry of restricted three-body problem.

Since $U(r,\theta,\phi)$ is symmetric with respect to the orbital plane of M_1 and M_2, we know that the transverse libration decouples from the equatorial modes, so that the simplified stability criteria may be employed. Further, while equatorial orbits are allowed (and realized), we also know that nonequatorial orbits cannot be excluded *a priori.* To determine the possible equilibria, use (4.3) in (4.2) and evaluate eqs.(3.4) :

$$
\begin{gathered}
r\left(\frac{\mu_1}{r_1^3}+\frac{\mu_2}{r_2^3}\right)-\mu_1\mu_2\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)\sin\theta\cos\phi = r\sin^2\theta \\
\left[r\sin\theta+\mu_1\mu_2\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)\cos\phi\right]\cos\theta = 0 \\
\mu_1\mu_2\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)r\sin\theta\sin\phi = 0.
\end{gathered}
\tag{4.4}
$$

where we have used the fact that $\mu_1 d_1 = \mu_2 d_2$, scaled r in units of d and used Kepler's third law to eliminate ω. From the last condition we see that there are two basic cases :

(i) $r_1 = r_2$: triangular solutions.

(ii) $\phi_0 = 0, \pi$: straight line solutions.

Consider first the triangular solutions (i). Putting $r_1 = r_2$ in (4.4b) we find $\cos\theta = 0 \to \theta_0 = \pi/2$, showing that the triangular solutions are equatorial. Then, setting $\theta_0 = \pi/2$ and $r_1 = r_2$ in (4.4a) gives $r_1 = r_2 = \omega^{2/3} = d$ by (4.1). To complete the solution, we have, with $\mu = \mu_1$, say,

$$r_0 = [1 + \mu(\mu - 1)]^{1/2} d, \tag{4.5}$$

$$\sin \phi_0 = \frac{\sqrt{3}d}{2r_0}. \tag{4.6}$$

We now turn to the calculation of the stability boundaries, for which we need the following derivatives :

$$\begin{aligned} U_{rr} &= 1 - 3(r_0^2 + \mu_1\mu_2 \cos^2 \phi_0) \\ U_{\theta\theta} &= 0 \\ U_{\phi\phi} &= -3\mu_1\mu_2\, r_0^2 \sin^2 \phi_0 = -\frac{9}{4}\mu_1\mu_2 \\ U_{r\phi} &= 3\mu_1\mu_2\, r_0 \sin \phi_0 \cos \phi_0, \end{aligned} \tag{4.7}$$

from which

$$\begin{aligned} \tau_\theta &= A' = 1 \\ B' &= \frac{27}{4}\mu(1-\mu) \\ \Delta' &= (A')^2 - 4B' = 1 - 27\mu(1-\mu). \end{aligned} \tag{4.8}$$

Thus, in addition to the familiar Krein Collision, AKA "Trojan Bifurcation" at $\mu_c = 0.03852...$ we see that there is a tangent bifurcation at $\mu = 0$ and 1, when one of the masses M_1 or M_2 vanishes. However, as the orbiting body is assumed to have vanishingly small mass, this limit is not well defined. (There is no compelling reason to restrict μ to the range $(0, \frac{1}{2}]$.)

Now consider the colinear equilibria (*ii*). Setting $\sigma = \cos \phi_0 = \pm 1$ in (4.4b) gives

$$\left[\sigma\mu_1\mu_2 \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) + r \sin \theta\right] \cos \theta = 0. \tag{4.9}$$

It can be shown that the expression in brackets is always positive, so that again $\theta_0 = \pi/2$. The radial coordinate is then given by (4.4a), which may be written

$$(r_1^3 r_2^3 - \mu_1 r_2^3 - \mu_2 r_1^3) r_0 = \sigma\mu_1\mu_2(r_1^3 - r_2^3), \tag{4.10}$$

where $r_1^2 = (r_0 - \sigma d_1)^2$ and $r_2^2 = (r_0 + \sigma d_2)^2$. It is shown in many places (e.g. Moulton, 1914, p 291) that (4.10) has three distinct positive real roots r_{0i}, $\forall \mu \in (0, 1)$. However, the exact location or even the existence of these equilibria need not concern us! For we shall show that if *any* colinear equilibria exist, then they must all be unstable.

To do so it is sufficient to show that any one of the four stability conditions is violated. It is easily seen that $U_{r\phi} = 0$ and

$$U_{\theta\theta} = U_{\phi\phi} = \mu(1-\mu) r_0 \sigma \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right), \tag{4.11}$$

so that

$$B' = -\frac{1}{r_0^2}U_{\phi\phi}\left(3 + \frac{2}{r_0^2}U_{\phi\phi}\right). \tag{4.12}$$

Hence, we need only show that $U_{\phi\phi} > 0$ to prove that $B' < 0$. But this follows from (4.11), which implies that

$$\operatorname{sgn} U_{\phi\phi} = \sigma \operatorname{sgn}(r_2 - r_1), \tag{4.13}$$

which by Figure 7 is positive for all possible colinear equilibria. QED.

5. Discussion

We have derived explicit necessary and sufficient conditions for the spectral stability of circular equilibrium orbits in a general three-dimensional rotating potential. The method is quite general and applies to any Hamiltonian equilibrium for which the variational equations are known analytically. Stability boundaries for tangent bifurcations and Krein collisions are given in terms of the derivatives of the potential function. Significant simplifications are found for harmonic potentials, such as those occurring in gravitational problems. The result is a complete description of all possible linear instabilities in a potential of arbitrary complexity. In previous treatments, each such problem was analyzed via the equations of motion and gave only limited information under particular assumptions.

Applying the results to circular orbits around an aspherical planet we find stable equatorial equilibria for both oblate and prolate potentials, thus generalizing the results of Blitzer *et al.* (1962, 1985). In addition, we find stable nonequatorial orbits, which turn out to be born out of the equatorial orbits. For purely oblate or prolate planets, stability is lost only by tangent bifurcations, Krein collisions not being possible. In the full 6D case, on the other hand, with longitudinal as well as latitudinal variations in the potential, Krein collisions of the equatorial modes are the only destabilization route for obloidal potentials ($J_2 > 0$). For proloidal potentials, however, stability can be lost through tangent bifurcation of the transverse mode as well. Stable nonequatorial orbits are also found as a result of the latter process, which subsequently lose their stability via a Krein collision. In each case, a tangent bifurcation of a purely oblate/prolate potential may be viewed as a degenerate Krein collision or a tangent bifurcation of double eigenvalues. In general a Krein collision does not signal a loss of stability unless the corresponding eigenvalues have mixed Krein signature. Rather than carry out this calculation, the loss of stability is verified by calculating eigenvalues on both sides of the boundary.

Applying the results to synchronous satellite orbits about the planets in our solar system, we find that stable orbits exist, with parameters several orders of magnitude distant from the stability boundaries. In order to lose stability, a planet or asteroid would have to be very irregular, with J_2 on the order of several tenths or equatorial eccentricity exceeding 0.01. We also apply the same formalism to the circular restricted three body problem, obtaining an improved treatment of the stability of the Lagrange points. Finally, we remark that noncircular orbits may treated by an analogous analysis of the monodromy matrix (Howard and MacKay, 1987a).

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