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ON THE EVALUATION OF STATIONARY SEA SURFACE TOPOGRAPHY USING GEODETIC TECHNIQUES

Abstract

One of the principal problems in separating the non–tidal Newtonian gravitational effects from other forces acting on the ocean surface with a resolution approaching the 10 cm level arises as a consequence of *all* measurements of a geodetic nature being taken either ***at*** or ***to the ocean surface.*** The latter could be displaced by as much as ±2 m from the equipotential surface of the Earth's gravity field corresponding to the mean level of the oceans at the epoch of observation – i.e., the geoid. A secondary problem of no less importance is the likelihood of all datums for geodetic levelling in different parts of the world not coinciding with the geoid as defined above.

It is likely that conditions will be favourable for the resolution of this problem in the next decade as part of the activities of NASA's Earth and Ocean Physics Applications Program (EOPAP). It is planned to launch a series of spacecraft fitted with altimeters for ranging to the ocean surface as part of this program.

Possible techniques for overcoming the problems mentioned above are outlined within the framework of a solution of the geodetic boundary value problem to ±5 cm in the height anomaly. The latter is referred to a "higher" reference surface obtained by incorporating the gravity field model used in the orbital analysis with that afforded by the conventional equipotential ellipsoidal model (Mather 1974 b). The input data for the solution outlined are ocean surface heights as estimated from satellite altimetry and gravity anomalies on land and continental shelf areas. The solution calls for a quadratures evaluation in the first instance.

The probability of success will be enhanced if care were paid to the elimination of sources of systematic error of long wavelength in both types of data as detailed in (Mather 1973 a ; Mather 1974 b) prior to its collection and assembly for quadratures evaluations.

1. Introduction

1.1. Preamble

On the basis of both temperature and salinity measurements as well as current estimations in the oceans, oceanographers have maintained for some time that the free ocean surface, apart from periodic tidal and wind effects, does not lie along an equipotential surface of the Earth's gravity field. The latter concept is based on the principles of Newtonian gravitation and, in ocean areas, is referred to as the geoid. These deviations of the ocean surface from the geoid are referred to as ***sea surface topography*** using the analogy that exists on land. The ocean surface can be expected to deviate from the geoid due to tidal and wind effects, which are essentially of short term in the context of the active life of an altimeter equipped spacecraft. The United States Government's National Aeronautics & Space Administration (NASA) plans to launch a series of such spacecraft as part of its Earth and Ocean Physics Applications Program (EOPAP) (NASA 1972).

In addition, concepts used in physical oceanography point to the existence of extensive sea surface topography of a time dependent nature, though on a different scale to that due to the wind and tides, as a consequence of currents like the Gulf Stream (IBID, Vol. 1, figure 4–3), with contour variations of up to $1\frac{1}{2}$ m . The study of temperature and salinity effects led Stommel (1965) to indicate the possible existence of quasi–stationary sea surface topography. A study of his results shows a significant stationary component correlated with latitude. This is confirmed on surface harmonic analysis of Stommel's map when the even degree harmonics (see Table 1) are substantially larger than the other terms. It should be pointed out that the data analyzed was banded in latitude between $+65°$ and $-65°$. The existence of such a phenomenon is, of course, not equivalent to an increase in the Earth's flattening as deduced from the orbital perturbations of near–Earth satellites by about 2 parts in 10^7 , with attendant effects on $\overline{C}_{20}$, *as no mass accretion is implied.*

These observations based on physical oceanographic principles, have some bearing on the widely reported discrepancies between equipotential surfaces of the Earth's gravity field as deduced from the results of geodetic levelling, and mean sea level as read at tide gauges. These discrepancies appear to occur along coastlines correlated with significant changes in latitude. See (Mather 1973 c, Appendices 1 & 2) for an up to date summary. The Appendices referred to also show that, apart from the Brazilian levelling, the stationary sea surface topography as deduced from steric levelling based on physical oceanographic principles (e.g., Sturges 1973) does not agree with that indicated by geodetic levelling. The discrepancies obtained are as large as $1\frac{1}{4}$ m . The possibility of errors being introduced in extrapolating the results of steric levelling to coastal areas cannot be ruled out, though this has been estimated as being a few cm (IBID).

The Earth and Ocean Physics Applications Program that NASA plans to implement in the next decade includes the launching of three spacecraft fitted with altimeters for ranging to the ocean surface (NASA 1972). The first of these is GEOS–C due for launch in late–1974. It is equipped with a dual–mode altimeter capable of providing global coverage with a resolution of 1–2 m in the low energy mode, and 50 cm is the high energy mode. The latter data type is likely to be collected in limited areas of special interest due to the substantial energy

Table 1

Surface Harmonic Analysis of Sea Surface Topography after (Stommel 1965) between 65° N and 65° S (units – metres)

Degree	Order (m)						
n	$\bar{C}_{n0}$	$\bar{C}_{n1}$	$\bar{S}_{n1}$	$\bar{C}_{n2}$	$\bar{S}_{n2}$	$\bar{C}_{n3}$	$\bar{S}_{n3}$
0	1.2(8)						
1	0.0(7)	– 0.1(3)	0.0(6)				
2	– 0.4(0)	– 0.0(6)	0.0(4)	0.0(2)	– 0.0(1)		
3	0.0(5)	– 0.0(3)	– 0.0(2)	0.0(1)	– 0.0(2)	– 0.0(7)	0.0(3)
4	– 0.1(4)						

requirements and spacecraft tracking limitations. In geodetic terms, the altimeter provides a valuable ***independent*** means of de–coupling the non–tidal gravitational forces from other forces acting on the ocean surface. If this information were of adequate resolution and obtained under properly controlled circumstances, it has the potential to effect the de–coupling referred to above. The non–periodic Newtonian effects will be described as "gravitational" in this paper. In the conventional parlance of physical geodesy, evaluation of these effects can be described as determining the ocean geoid.

The effect of the non–gravitational effects described above on the geometry of the ocean surface can only be obtained if the problem were formulated with a precision approaching ± 5 cm, as these effects are estimated as being of the order of $\pm 1–2$ m. In the present development, the effect of tide producing forces is considered to have been excluded from the gravitational effects. This can be achieved in principle by adopting a model for the tides and removing the resultant gravitational effects from all observations.

It has been traditional to base the principles underlying "geoid determinations" on solving the geodetic boundary value problem. These solutions as obtained at the present time, are based on techniques which, in the general case, define the vector $\vec{d}$ of separation of the general point $P\ (\phi_g, \lambda_g, W_0 + \Delta W + \delta W)$ at the Earth's surface from the equivalent point $Q\ (\phi_a, \lambda_a, U_0 + \Delta W)$ with a precision approaching ± 1 m assuming there was no error in the data. All symbols are described in section 1.2. The difference between the "true" latitude ϕ_g and longitude λ_g of P and the assumed values ϕ_a and λ_a are not of geodetic interest as obtained by this method at the present time because position determination on a global basis with the highest possible precision, both at present and in the foreseeable future, is most likely to be achieved by extra–terrestrial techniques not influenced by concepts based on the direction of the vertical, on the basis of the technology available at present. δW is the discrepancy in geopotential between the datum for geodetic levelling and the adopted definition of the geoid, while ΔW is the observed difference in geopotential of P with respect to the datum of the levelling. δW is an unknown quantity whose effect on the solution of the geodetic boundary value problem can be formulated (Mather 1973 b). An earlier attempt at formulating a solution of this problem which had a resolution at the 5 cm level (Mather 1973 a) ignored the existence of the effects δW and circumvented the problem by proposing an iterative solution. This solution was based on the assumption that gravity anomalies would be available for use in such a solution on a global basis.

This latter contention is unlikely to be realized during the time span of EOPAP. On the other hand, there is a strong possibility that altimetry data of adequate precision which can be reduced to elevations of the ocean above the geodetic reference surface, can provide information which could be used in the quadratures evaluation of the geodetic boundary value problem, and thereby define sea surface topography. A revised solution of this problem based on :

a) satellite altimetry data in ocean regions ; and

b) surface gravity information in continental (land) and shelf areas has been formulated (Mather 1974 a ; Mather 1974 b).

This solution is based on three assumptions :

1) Short period tidal effects have been eliminated from the observations at the 5 cm (50 μGal) level.

2) An adequate overlap was available between the altimetry and surface gravity data in shelf areas, thus permitting the prediction of gravity anomalies for the computation of height anomaly data for land areas.

3) Any significant long–wave stationary sea surface topography was known and capable of modeling.

If the first is not satisfied, the data would be subject to periodic effects whose influence on the evaluation of sea surface topography would depend on the extent to which the global sampling of data were correlated with the time dependent tidal effects. A lack of correlation over significant wavelengths would cause the ensuing errors to behave as random errors in the quadratures evaluation as shown in (Mather 1974 a, table 1).

The third assumption is of dubious validity and it would be far more realistic to ***admit to this limitation that all data is related to the ocean surface and not the geoid.*** The consequence of taking this step is that the geodetic boundary value problem is no longer evaluated by quadratures methods alone due to the existence of unknown quantities δW in the formulation. The introduction of such a concept requires a revision of the solution given in (Mather 1974 b). The following summary outlines an updated version of the solution based on the principles outlined in (Mather 1973 b). This would provide a consistent basis for the evaluation of quasi-stationary sea surface topography without making any assumptions regarding its magnitude. It would also allow for the fact that all datums for geodetic levelling do not necessarily lie on the geoid.

1.2. A Guide to Notation

i) Some frequently used symbols

A_n = Surface harmonic of degree n in the spherical harmonic representation of T''.

c_R = $f(1/3 - \sin^2\phi) + h/R + o\{f^2\}$; subscript $_p$ refers to value at point of computation

c_r = $\left((R_p - R)/r_0\right)^2 + c_{Rp} + c_R + o\{f^2\}$; $c_{\bar{r}}$ is value on Brillouin sphere

$c_\Delta = \left(1 + 2(dR/R)\right)/\left(1 + c_{\bar{r}}\right)^{\frac{1}{2}} - 1$

$c_\phi = f + m - 3f\sin^2\phi$

$dR = \bar{R} - R$

dS = Element of surface area at the surface of measurement

$d\sigma$ = Element of solid angle

f = Flattening of meridian ellipse

$f(\psi)$ = Stokes' function

G_n = Surface harmonic of degree n in the representation of Δg_c

g = Observed gravity at the surface of measurement (Earth's surface)

h = Normal height

$\vec{i}$ = Set of unit vectors along the x_i axes

$M\{X\}$ = Global mean value of X

$M(\psi)$ = $-(1 + 3\cos\psi)$

m = Ratio of centrifugal force to normal gravity at equator of equipotential ellipsoid

$\vec{N}$ = Unit vector normal to the surface of measurement

N_c = Indirect effet, defined by equation 21

N_s = Stokesian contribution to ζ'.

R = Distance from centre of mass of solid Earth and oceans ; subscript p refers to point of computation

$\overline{R}$ = Radius of Brillouin sphere

R_{anm} = $\overline{P}_{nm}\,(\sin\phi) \begin{vmatrix} \cos m\lambda \\ \sin m\lambda \end{vmatrix}$

r = Distance of element of surface area dS from the point of computation P

$\overline{r}$ = $2\overline{R}\sin\frac{1}{2}\,\psi$

r_0 = $2R\sin\frac{1}{2}\,\psi + o\,\{fr_0\}$

S_{anm} = $\begin{vmatrix} S_{1nm} & S_{2nm} \end{vmatrix}$

T_c = $T'' - (W_0 - U_0) - \delta W$; see equation 19

T' = Disturbing potential for "higher" reference system ; see equation 1

T'' = $T' - V$; overbar refers to values on Brillouin sphere

U = Spheropotential due to gravitating rotating equipotential ellipsoid

U_0 = Value of U on the equipotential ellipsoid

U_r = Potential on the higher system of reference

V = Potential of the atmosphere

W_0 = Potential of the geoid (unknown)

x_i = Local rectangular Cartesian coordinate system $x_1\ x_2\ x_3$ with the x_3 axis along local spherop normal, the $x_1\ x_2$ plane defining the local horizon and completing the local Laplacian triad (x_1 north, x_2 east)

β = Ground slope ; β_1, β_2 are components in the meridian and prime vertical directions respectively

γ = Normal gravity on the "higher" reference surface

Δg = The gravity anomaly at the surface of the Earth

$\Delta g'$ = $\Delta g - \delta\gamma$

$\Delta g''$ = $\Delta g' + 2V/R + \partial V/\partial h + \frac{2T''}{R}\,c_\phi - \frac{1}{2}g\,\zeta_d^2 - 2(W_0 - U_0)/R - 2\,\delta W/R$; overbar refers to value on Brillouin sphere

Δg_c = $\overline{\Delta g}'' + 2(W_0 - U_0)/R + 2\,\delta W/R$; see equation 20

ΔW = Geopotential difference between the Earth's surface and the levelling datum

δW = Geopotential difference between sea surface / levelling datum and the geoid

$\delta\gamma$ = Change in normal gravity from equipotential ellipsoid to "higher" reference model ; $\delta\gamma = \gamma \sum_{n=2}^{n'} (n-1)\,R_{anm}\,S_{anm} + o\{f\,\delta\gamma\}$

$\delta \Delta g''$ = Change in $\Delta g''$ between Earth's surface and Brillouin sphere ;

$$\delta \Delta g'' = -\gamma\left(\sum_{a=1}^{2} \partial \xi_a / \partial x_a - (\xi_1 \tan \phi)/R - 2\zeta'/R^2 + \right.$$

$$\left. + o\,\{f(\partial \Delta g''/\partial h)\}\right) dR + o\,\{dR^2\,(\partial^2 \Delta g/\partial h^2\}$$

δ = $f \sin 2\phi \cos$ (azimuth of P from $d\sigma$) $+ o\,\{f^2\}$

θ = $\left((R_p - R)/R\right) \cot \frac{1}{2}\psi - \delta + o\,\{f^2\}$ for $\psi > 10°$

λ = Longitude, positive east

ξ_a = Components of the deflections of the vertical at the surface of the Earth in the meridian and prime vertical directions respectively with reference to the "higher" reference surface, positive if the outward vertical is north/east of the spherop normal

ζ' = Height anomaly on "higher" reference surface

ζ_d = Deflection of the vertical, positive if the vertical is north/east of spherop normal.

Φ = $2R\,(R \cos \delta - R_p \cos(\psi + \delta))/r^2 - 1$

ϕ = Geocentric latitude, positive north

ψ = Angle between geocentric radii to the element of surface area dS and the point of computation P

$$\vec{\nabla} = \frac{\partial}{\partial x_i} \vec{i}$$

$$\frac{\partial \gamma}{\partial h} = -2\,\frac{\gamma}{R}\,(1 + c_\phi)$$

ii) Conventions

$x_a y_a = x_1 y_1 + x_2 y_2$

$x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots$, i taking all possible values

x_a = The two values x_1 and x_2

x_i = The three quantities x_1, x_2 and x_3

2. Formulation of the Problem

The updating of the solution described earlier (Mather 1973 a ; Mather 1974 b) is supplementary to the earlier work and should be read in conjunction with these references. The disturbing potential T'_p at the general point P at the Earth's surface with respect to the "higher" reference model proposed earlier (equation 15 in the second reference) is given according to the principles defined in (Mather 1973 b) in the generalized Bruns' equation by

$$T'_p = W_p - U_{rp} = (W_0 + \Delta W + \delta W) - (U_0 + \Delta W) + \gamma_p \zeta' + o\,\{10^{-3}\,\mathrm{kGal\,m}\}$$
$$= (W_0 - U_0) + \delta W + \gamma \zeta' + o\,\{10^{-3}\,\mathrm{kGal\,m}\} \qquad (1)$$

where the subscript $_p$ refers to values at the general point P or at the point on the associated spherop with the same surface coordinates as P in the case of quantities on the reference system. The gradient of the disturbing potential $(\partial T'/\partial h)_p$ remains unchanged as (Mather 1974 b, Equation 20)

$$\left(\frac{\partial T'}{\partial h}\right)_p = -\Delta g' + \zeta \left(\frac{\partial \gamma}{\partial h}\right)_p + \frac{1}{2} g_p \zeta^2_{dp} + o\{10^{-4}\,\Delta g\} \qquad (2)$$

where

$$\Delta g' = \Delta g - \delta \gamma \qquad (3)$$

All notation is defined in section 1.2. These relations are adequate as the use of the "higher" reference model which incorporates the shape of the equipotential surfaces of the Earth's gravity field as implied in the low degree harmonic representations (up to, say, degree 20), can be expected to reduce the magnitude of the height anomaly by at least an order of magnitude.

The boundary value conditions can be expressed in terms of the disturbing potential of the solid Earth and oceans (T") which has the property of satisfying Laplace's equation exterior to and on the surface of measurement S, in the form

$$2\pi T'' = \iint [T'' \vec{\nabla} \cdot \vec{N} \frac{1}{r} - \frac{1}{r} \vec{\nabla} \cdot \vec{N}\, T''] \, ds \qquad (4)$$

This equation can be written as

$$T'' = I_1 + I_2 \qquad (5)$$

where

$$I_1 = \frac{1}{2\pi} \iint \frac{R^2}{r} \left(-\frac{\partial T''}{\partial h} - T'' \frac{x_3}{r^2} \right) d\sigma \qquad (6)$$

and

$$I_2 = \frac{1}{2\pi}\iint \frac{R^2}{r}\left(\frac{x_a \tan\beta_a}{r^2}\,T'' + \frac{\partial T''}{\partial x_a}\,\tan\beta_a\right) d\sigma \tag{7}$$

As

$$T'' = T' - V \tag{8}$$

V being the potential of the atmosphere, it follows from equations 1, 2 and the definition of $\partial\gamma/\partial h$ (Mather 1973 a, p. 32) that

$$\frac{\partial T''}{\partial h} = \frac{\partial T'}{\partial h} - \frac{\partial V}{\partial h} = -\Delta g' - \left(\frac{2V}{R} + \frac{\partial V}{\partial h}\right) - \frac{2T''}{R}(1 + c_\phi) + \tfrac{1}{2} g \zeta_d^2$$

$$+ \frac{2}{R}(W_0 - U_0) + 2\frac{\delta W}{R} + o\{10^{-8}\, g\} = -\Delta g'' - \frac{2T''}{R} \tag{9}$$

where

$$\Delta g'' = \Delta g' + \left(\frac{2V}{R} + \frac{\partial V}{\partial h}\right) + \frac{2T''}{R}\, c_\phi - \tfrac{1}{2} g \zeta_d^2 - \frac{2}{R}(W_0 - U_0) - 2\frac{\delta W}{R} + o\{10^{-8} g\} \tag{10}$$

As T'' satisfies Laplace's equation exterior to and on the surface of measurement, it can be represented completely by the spherical harmonic expansion

$$T'' = \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}}, \quad n \neq 1 \tag{11}$$

where the centre of the system of reference coincides with the centre of mass of the solid Earth and oceans, as $n \neq 1$. Consequently, it follows from equation 9 that $\Delta g''$ can be completely represented by the spherical harmonic expansion

$$\Delta g'' = \sum_{n=0}^{\infty} (n-1)\frac{A_n}{R^{n+2}} + o\{f^2 \Delta g''\}, \quad n \neq 1 \tag{12}$$

in the space exterior to the surface of measurement and on it. Thus I_1, defined by equation 6, can be expressed as

$$I_1 = I_{11} + I_{12} + I_{13} \tag{13}$$

where

$$I_{11} = \frac{1}{2\pi}\iint \frac{\overline{R}^2}{\overline{r}}\left(\overline{\Delta g}'' + \frac{3\overline{T}''}{2\overline{R}}\right) d\sigma \tag{14}$$

the overbar referring to values on the Brillouin sphere centered on the centre of mass of the solid Earth and oceans and including all the Earth's topography,

$$I_{12} = \frac{1}{2\pi}\iint \frac{R^2}{r}\left(\Delta g''\left(c_\Delta + \frac{3}{2}\frac{dR}{R}\right) + \frac{3T''}{2R}\left(c_\Delta + 3\frac{dR}{R}\right) - \delta\,\Delta g'' + o\{f^2\,\Delta g\}\right) d\sigma \quad (15)$$

provided the amplitude of long wavelength terms in $\Delta g''$ do not, in confluence, exceed 15 mGal (the contribution of long–wave effects through the dominant term – see section 1.2 – is more likely to be one order of magnitude smaller), and

$$I_{13} = \frac{1}{4\pi}\iint -\frac{R}{r}\,\Phi\,T''\,d\sigma \quad (16)$$

The technique used in (Mather 1973 b) can be adopted in attempting to separate the effect of the stationary sea surface topography from that of the data used in the solution of the geodetic boundary value problem, noting that its magnitude is of order $\frac{1}{2}$ mGal but possibly influencing quadratures evaluations through long wave lengths.

The problem as described in (Mather 1974 b) is largely a revision of the solution of equation 14, as effects introduced through equations 15 and 16 are possibly an order of magnitude smaller, except in the case of some ***high frequency effects*** in continental areas which should not seriously disrupt a successfull iteration to convergence. The separation of the unknown sea surface topography effects from the rest of the solution on the lines described in (Mather 1973 b) gives a final solution in the form

$$\zeta' = N_s + N_c \quad (17)$$

where the Stokesian term N_s is given by its alternate forms

$$N_s = \begin{cases} \dfrac{1}{\gamma}[M\{T_c\} + (M\{\delta W\} - \delta W)] + \dfrac{1}{4\pi\gamma}\displaystyle\iint M(\psi)\,T_c\,d\sigma \\ \quad + \dfrac{1}{4\pi\gamma}\displaystyle\iint M(\psi)\,\delta W\,d\sigma + o\{fN_s\} \\ \\ \dfrac{1}{\gamma}[(W_0 - U_0) - \bar{R}\,M\{\Delta g_c\} + (2M\{\delta W\} - \delta W)] + \\ \quad + \dfrac{\bar{R}}{4\pi\gamma}\displaystyle\iint f(\psi)\,\Delta g_c\,d\sigma - \dfrac{1}{2\pi\gamma}\displaystyle\iint f(\psi)\,\delta W\,d\sigma + o\{fN_s\} \end{cases} \quad (18)$$

T_c and Δg_c being given by

$$T_c = \gamma \zeta' - V + dR \frac{\partial T''}{\partial h} + o\{ fT'' \} \tag{19}$$

and

$$\Delta g_c = \Delta g' + \left(\frac{\partial V}{\partial h} + \frac{2V}{R} \right) + \delta \Delta g'' + \frac{2T''}{R} c_\phi - \tfrac{1}{2} g \zeta_d^2 + o\{10^{-8} g\} \tag{20}$$

The indirect effect N_c is given by

$$N_c = \frac{V}{\gamma} + \frac{1}{2\pi\gamma} \iint \frac{R^2}{r} \left(\frac{\partial T''}{\partial x_a} \tan \beta_a + T'' \left(\frac{x_a \tan \beta_a}{r^2} - \frac{1}{2R} \left(\Phi - 3 \left[c_\Delta + 3 \frac{dR}{R} \right] \right) \right) - \delta \Delta g'' + \Delta g'' \left(c_\Delta + \frac{3}{2} \frac{dR}{R} \right) + o\{ f^2 \Delta g \} \right) d\sigma \tag{21}$$

Notes :

1) All values of normal gravity (γ) are "surface" values wherever such distinction is necessary. Values of R are geocentric distances to the surface elements.

2) The alternative form at 18 are not interchangeable in the representation of elements of surface area because they are based on the orthogonal property of surface harmonics. The values of T_c in continental areas would have to be based on values of ζ' computed from equation 17 for use in the first alternative in equation 18. The contribution of N_s to ζ' is obtained from the modified form of the second alternative

$$N_s = \frac{1}{\gamma} [W_0 - U_0) - \bar{R} M\{ \Delta g_c \} + (2 M\{ \delta W \} - \delta W)] + \frac{\bar{R}}{4\pi\gamma} \int_0^{\psi_0} \int_0^{2\pi} f(\psi) \left[\Delta g_c - 2 \frac{\delta W}{R} \right] \sin \psi \, d\psi \, d\alpha + \frac{\bar{R}}{2\gamma} \sum_{n=2}^{\infty} Q_n \left[G_n - \frac{2}{R} \delta W_n \right] \tag{22}$$

Q_n being the Molodenskii truncation function (Molodenskii et al., 1962, p. 147) of degree n, and G_n, δW_n being the n–th degree surface harmonic in the representation of Δg_c and the effect of the stationary sea surface topography, the latter being unknown. As pointed out in (Mather 1974 b), only a finite number n' of G_n are known. Usually, $n' \doteqdot 20$. If the "higher" reference model used were indeed representative, these contributions would tend to zero. N_s in such a case will be estimated from a spherical cap calculation to angular distance ψ_0 together with the unknown effect of the stationary sea surface topography.

3) It is estimated that if $n' \doteqdot 20$, then ψ_0 should be at least $10°$ if systematic effects due to truncation errors are to be avoided. The effect of this error on the evaluation of stationary sea surface topography will be a function of the extent of the continental area. The contribution of island and small land areas to quadratures evaluations would tend to be negligible. A truncation error of ± 20 cm could be considered to be tolerable in the context of sea surface topography determinations when estimating the contribution of coastal areas on continents (e.g., Eurasia, the Americas and Africa). This error could be as great as ± 50 cm for smaller extents (e.g., islands in the Pacific) with intermediate figures for areas like Antarctica and Australia.

4) From the discussion at 3), it follows that gravity anomalies have to be defined at sea for the evaluation of equation 22. An important area of research is the development of optimization techniques for strengthening the gravity field representation in these areas from the available gravity data, satellite altimetry and other manifestations of the gravity field, taking into account the systematic biases which exist between the various data types.

5) For a generalized approach, it must be assumed that δW is an unknown quantity with a global distribution. The input *from continental areas* will be the difference in geopotential between the geodetic levelling datum and the geoid as defined earlier. This input would hold its magnitude over large areas. The input *from oceanic regions* would represent the difference in potential between the ocean surface to or at which the altimeter range or gravity measurement was made, as reduced for the adopted tidal model and, where relevant, other periodic effects on the ocean surface which have been eliminated in the averaging procedure used in the orbital analysis, and the geoid.

6) In the case of the altimeter data, the estimate T'' in equation 21 from satellite altimetry has to be obtained from equations 1 and 8 as

$$T'' = (W_0 - U_0) + \delta W + \gamma \zeta' - V \tag{23}$$

where the first two terms are unknown. $\partial T'' / \partial x_\alpha$ will be largely unaffected by this defect because the more dominant term $(W_0 - U_0)$ has no effect on the gradient while δW is constant for considerable continental extents. In oceanic areas, where δW is more variable, the effect is scaled by the (unknown) term $\tan \beta_a$ which is of order 10^{-4} or less and is thus negligible. The only effect of potential consequence is through the term Φ in the expression multiplying T'' where some distorting effects at the 30 cm level may well occur. These magnitudes will only be of consequence where the great mountain ranges occur (less than 3 °/o of the Earth's surface area) and the effect on quadratures in oceanic regions is probably less than ± 5 cm . As the other terms in T'' and $\Delta g''$ in equation 21 are all f times smaller than the Stokesian contribution, they can be computed by excluding the unknown terms in equations 1 and 10 without introducing errors in excess of 5–10 cm , even acknowledging the present uncertainty in the estimate of W_0 .

It is therefore reasonable to conclude that the computation of the indirect effect N_c through equation 21 without considering the unknown terms $(W_0 - U_0)$ and δW is unlikely to introduce errors in excess of 5 cm in the final result, and certainly less than 10 cm.

7) The term $\delta \Delta g''$, introduced to designate the change due to the upward continuation of $\Delta g''$ from the Earth's surface to the Brillouin sphere, appears in the overall expression for ζ' in the form

$$\frac{1}{2\pi\gamma}\iint \delta\Delta g''\left(\frac{R}{2}f(\psi) - \frac{R^2}{r}\right)d\sigma = \frac{R}{4\pi\gamma}\iint \delta\,\Delta g''\Big(1 - 5\cos\psi - \tag{24}$$

$$- 6\sin\tfrac{1}{2}\psi - 3\cos\psi \log[\sin\tfrac{1}{2}\psi(1+\sin\tfrac{1}{2}\psi)]\Big)\times(1+o\{f\})d\sigma$$

on using the relations

$$f(\psi) = \sum_{n=2}^{\infty}\left(\frac{R_p}{R}\right)^n \frac{2n+1}{n-1} P_{no}(\cos\psi) \quad \text{and} \tag{25}$$

$$\frac{1}{r} = \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{R_p}{R}\right)^n P_{no}(\cos\psi)$$

in a manner similar to the procedure described in (Mather 1973 a, 38–40). As the kernel is slowly varying, the nett effect on the quadratures evaluation of equation 17 is dominated by long wavelength characteristics in $\delta\Delta g''$ which, in turn, are a function of those in $\partial \xi_a / \partial x_a$. These third derivative characteristics of the gravity field are likely to have significant magnitude through terms of very short wavelength. Thus, it would appear that even a 10 °/o error could be tolerated in the evaluation of $\delta\Delta g''$ in very disturbed areas provided the wavelengths do not exceed 200 km in such regions and 2000 km in other areas.

8) *The evaluation of the stationary sea surface topography*

Given the total formulation of the problem described in equations 17, 18, 21 & 22, and using the "higher" reference model, the optimum manner for obtaining the (quasi–) stationary sea surface topography appears to be the following.

It would be reasonable to define the geoid as ***the equipotential surface of the Earth's gravity field obtained when*** $M\{\delta W\}$ ***as defined in equation 18 is zero.*** Also see note 10) below. Assume that T_c is defined for all land areas using equation 22 after appropriate optimization of data in the shelf and near coastal oceanic areas as described at 4). Equations 17, 18 and 21 can be expressed as observation equations for all oceanic areas which take the form

$$\zeta'_{oi} + v_i = \frac{1}{\gamma} M\{T_c\} - \frac{1}{\gamma} \delta W_i + \sum_j A_{ij} \; \delta W_j + K_i \tag{26}$$

where, in the main,

$$A_{ij} = \frac{1}{4\pi} \; M(\psi_{ij}) \, d\sigma_j \tag{27}$$

and

$$K_i = \frac{1}{4\pi\gamma} \iint M(\psi_i) T_c \, d\sigma + N_{ci} \tag{28}$$

T_c is given by equation 19 and N_c by equation 21, evaluated in terms of the discussion at 6) above. It should be noted that terms of the form

$$(W_0 - U_0) - \bar{R} \; M\{\Delta g_c\} - \delta W - \frac{1}{2\pi} \iint_{cap} \delta W \; d\sigma$$

are introduced whenever T_c is estimated for land areas for use in equation 28, ignoring the existence of stationary sea surface topography. The equations at 26 can be solved in the usual manner for the δW_j, $(W_0 - U_0)$, $M\{\Delta g_c\}$ and $M\{T_c\}$. The latter terms contain information of zero degree about the quality of fit between the orbits, the surface gravity data and the "higher" reference model adopted.

Correlation effects

The ζ'_{oi} are the heights of the ocean above the "higher" reference surface and therefore the v_i are correlated with position due to both orbital errors and quadratures errors in the K_i. A solution of this type is similar in principle to the deduction of geocentric orientation vectors from the comparison of gravimetric and astro–geodetic data (e.g., Mather 1971). The ***stability of the solution*** does not seem to be impaired due to the v_i being correlated ***provided the entire area is sampled.*** In the case of sea surface topography determinations, the observation equations are solved on the basis of comparisons at sea, which covers only 70 °/o of the globe. The effect of this bias on the solutions will have to be carefully considered when accepting results from this technique. It is not felt that this threatens the validity of the technique proposed.

9) The principal concern in the first instance is ascertaining whether stationary sea surface topography of significance exists with long wavelengths. Hence it may be more meaningful to define the global distribution of δW by a surface harmonic expansion

$$\delta W_j = \sum_{n=0}^{n'} \sum_{m=0}^{n} \overline{P}_{nm} (\sin \Phi_j) \left(\overline{\delta W}_{1nm} \cos m\lambda_j + \overline{\delta W}_{2nm} \sin m\lambda_j \right) \quad (29)$$

The block of equations of the type at 26 will thus be solved for the $(n'+1)^2$ coefficients $\overline{\delta W}_{anm}$.

10) *Datums for Geopotential*

The δW for 30 °/o of the surface area of the Earth will be represented by corrections to datums for geopotential in relation to the geoid. As explained at 8) above, the adoption of $M\{\delta W\}$ as zero in equation 18 ***provides a unique definition for the geoid.*** This definition would be unacceptably biased if the datum for geopotential for a particular continental area were inappropriately chosen. To illustrate this point, consider Australia. The internally consistent national level net departs significantly from mean sea level as defined by a well distributed net of tide gauges, the departures being correlated with position.

If a stable definition of the geoid is desired at the 5–10 cm level from the procedure outlined above – ***and it should be pointed out that no other unique definition has been proposed*** – the computation of geopotential for gravity stations in Australia to be used in the solution of equation 22, should be referred to the *mean* of the mean sea levels for the epoch as indicated by the tide gauges in the network. This procedure should be adopted when effecting the quadratures solutions described by equations 17–22.

3. Conclusions

The discussion in the previous shows that neither the existence of (quasi–) stationary sea surface topography nor the possibility that geodetic levelling datums do not lie on a unique equipotential surface of the Earth's gravity field need inhibit the definition of the former with a resolution at the 10 cm level.

The evaluation of this phenomenon using a solution of the geodetic boundary value problem as formulated in equations 17–22 has the following significant advantages;

a) It provides a unique definition of the geoid, provided each datum for geopotential were based on the *mean* of the regional mean sea level datums for the epoch of observation.

b) Effects of zero degree can be isolated and further analysed for dynamic, gravitational and scale effects.

Obviously, the procedure for evaluation described above would give meaningful results only if the data is of adequate quality. The criteria to be satisfied by gravity anomalies can be summarized as follows (IBID, p. 68) :

i) Global gravity standardization networks should aim to have a station spacing of 1000 km in continental areas. Determinations should be based on techniques where the values of gravity were established with a precision of 0.1 mGal and no significant correlation existed between errors in adjacent values.

ii) Geopotential values of land gravity stations in the network should be based on levelling data such that correlated errors were held to below 0.15 kGal m , but where individual connections could have uncorrelated errors at the 1–2 kGal m level.

iii) The gravity station spacing in the representative network on land was such that the error of representation was ± 3 mGal (10 km spacing in non–mountainous areas).

iv) Values of normal gravity were based on geocentric and nor regional geodetic coordinates. The effect of the curvature of the vertical in high mountains is at the noise level of the observations and is unlikely to have an impact of any significance on quadratures evaluations.

The requirements to be met by satellite altimetry data has been outlined in (Mather 1974 a ; Mather 1974 b). In summary, it would be difficult to obtain solutions from altimetry data where the errors had wavelengths in excess of $100°$. The principal sources of error are

1) the resolution of the orbit ; and

2) the precision of the altimetry.

In the case of high precision determinations, the former can be viewed as being largely a function of the station spacing in the tracking station network used to monitor the positional variations of the altimeter–equipped spacecraft and the nature of the instrumentation used. It would appear feasible for a spacecraft equipped with a 10 cm altimeter (like the one proposed for SEASAT), and tracked on *a continuing basis* by a network of at least 25 well distributed 10 cm tracking systems, to provide the coverage necessary to ensure altimetry data of adequate quality for the determination of sea surface topography at the 10 cm level from the quadratures evaluation of the geodetic boundary value problem as outlined in equations 17–22.

It is therefore of great importance to ensure that the following objectives are met for the successful evaluation of (quasi–) stationary sea surface topography from satellite altimetry and gravity anomalies by solving the geodetic boundary value problem within the time frame of EOPAP :

Firstly, the tracking schedules for the GEOS–C and SEASAT missions be adequately designed to ensure that the altimetry is obtained under optimum conditions for a favourable solution.

Secondly, global gravity standardization networks should be established to meet the conditions outlined at i) above. Global gravity data banks should attempt to meet the requirements outlined at ii) to iv) above.

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