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A NEW FORMULA FOR EVALUATING THE TRUNCATION ERROR COEFFICIENT

Abstract

In this paper, a new formula for evaluating the truncation coefficient Q_n *is derived from recurrence relations of Legendre polynomials. The present formula has been conveniently processed by an electronic computer, providing the value of* Q_n *up to a degree* $n = 49$ *which are exactly equal to those of Paul (1973).*

The geoidal height is obtained by performing an integration of gravity anomaly weighted by Stokes' function over the whole surface of the spherical earth. When the integration is not extended over the whole surface but restricted to a spherical cap, the effect of neglecting the remote zones on the computation of the geoidal height should be evaluated. The effect is called "truncation error" (de Witte, 1967). Molodenskii et al. (1962) developed the effect in a series of zonal spherical harmonics with the truncation coefficients Q_n (n : degree of polynomial), which have later been called "Molodenskii's truncation coefficients" by de Witte (1967).

Molodenskii at al. (1962) gave Q_n up to $n = 8$ in the forms of power series. The higher–degree values of Q_n, however, are required for their practical applications to evaluating the truncation errors for satellite gravity anomaly, Hagiwara (1972) obtained a general formula for evaluating Q_n, and published a table of Q_n up to $n = 18$. Later, Paul (1973) pointed out Hagiwara's values of degree higher than 16 to be somewhat invalid due to round–off errors cumulative in the computation process. Paul (1973) also proposed another formula, by which he computed Q_n's up to $n = 49$. The main achievement with Paul's formula is that the computation deals with only a fixed and finite number of terms for all values of n.

The present work aims at deriving another convenient expression of Q_n from recurrence relations of Legendre functions.

The Stokes function expresses $S(x)$, where $-1 \leqslant x \leqslant 1$, as a closed formula

$$S(x) = 1 - 5x - 3\sqrt{2(1-x)} + \sqrt{\frac{2}{1-x}} - 3x \log \frac{\sqrt{1-x}\,(\sqrt{2}+\sqrt{1-x})}{2} \qquad \cdots\cdots\cdots (1)$$

Bull. Geod. 50 (1976) pp. 131–135.

This satisfies a differential equation given by

$$\frac{d}{dx}\left\{(1-x^2)\frac{dS(x)}{dx}\right\} + 2S(x) = 2 + 9x + \frac{1}{\sqrt{2}\,(1-x)^{3/2}}, \qquad (2)$$

which can easily be proved by substituting (1) into the left–hand side of (2). (2) corresponds to the following differential equation of a Legendre function :

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\} + n(n+1)\,P_n(x) = 0. \qquad (3)$$

Molodenskii's truncation coefficient Q_n is expressed as

$$Q_n(t) = \int_{-1}^{t} S(x)\,P_n(x)\,dx \qquad (4)$$

where $x \leqslant t \leqslant 1$.

In the previous paper (Hagiwara, 1972), a series expansion of $P_n(x)$ was utilized for performing the above integration; the corresponding formula of Q_n involved double summations with number of terms increasing with n, which caused round–off error accumulation in the computation process. In the present paper, we integrate the differential equation (2) weighted by $P_n(x)$, instead of (4).

Integrating the left–hand side of (2) weighted by $P_n(x)$ over an integral $[-1, t]$, then we get

$$\int_{-1}^{t}\frac{d}{dx}\left\{(1-x^2)\frac{dS(x)}{dx}\right\}P_n(x)\,dx + 2Q_n(t)$$

$$= -(n-1)(n+2)\,Q_n(t) - nS(t)\left\{P_{n-1}(t) - tP_n(t)\right\} + (1-t^2)P_n(t)\frac{dS(t)}{dt} \qquad \ldots\ldots\ldots (5)$$

Here, for the evaluation of the integral in (5), we use recurrence relations of Legendre function

$$\left.\begin{aligned} x\frac{dP_n(x)}{dx} - \frac{dP_{n-1}(x)}{dx} &= n\,P_n(x) \\ (1-x^2)\frac{dP_n(x)}{dx} &= n\left\{P_{n-1}(x) - x\,P_n(x)\right\} \end{aligned}\right\} \qquad (6)$$

Meanwhile the integration of the right–hand side of (2) weighted by $P_n(x)$ is evaluated as follows. First, by denoting

$$I_n(t) = \int_{-1}^{t} P_n(x)\,dx = \frac{1}{2n+1}\left\{ P_{n+1}(t) - P_{n-1}(t) \right\} , \quad (7)$$

we have

$$J_n(t) = \int_{-1}^{t} xP_n(x)\,dx = \frac{1}{2n+1}\left\{ (n+1)\, I_{n+1}(t) + n\, I_{n-1}(t) \right\}. \quad (8)$$

For the case of $n = 0$, we have $J_0 = I_1$.

Furthermore, we consider the integration of the last term in the right–hand side of (2), such as

$$K_n(t) = \frac{1}{2\sqrt{2}}\int_{-1}^{t} \frac{P_n(x)}{(1-x)^{3/2}}\,dx. \quad (9)$$

The zero and first–degree integrations are easily performed as

$$\left.\begin{aligned} K_0(t) &= -\frac{1}{2}\left(1 - \sqrt{\frac{2}{1-t}}\right) \\ K_1(t) &= K_0(t) - \left(1 - \sqrt{\frac{1-t}{2}}\right) \end{aligned}\right\} \quad (10)$$

The difficulty in evaluating (9) is overcome if we obtain a recurrence relation between K_n's. From the known values of K_0 and K_1, the second and higher–degree K_n's are obtained through such a recurrence relation.

Applying a recurrence relation of Legendre function

$$(n+1)\, P_{n+1}(x) + n\, P_{n-1}(x) = (2n+1)\, x\, P_n(x) \quad (11)$$

to (9), we can derive

$$(n+1)\, K_{n+1}(t) + n\, K_{n-1}(t) = (2n+1)\left\{ K_n(t) - \frac{L_n(t)}{2} \right\} \quad (12)$$

where we define

$$L_n(t) = \frac{1}{\sqrt{2}}\int_{-1}^{t} \frac{P_n(x)}{\sqrt{1-x}}\,dx. \quad (13)$$

Another recurrence relation

$$P_n(x) = \frac{1}{2n+1}\left\{\frac{dP_{n+1}(x)}{dx} - \frac{dP_{n-1}(x)}{dx}\right\} \tag{14}$$

corresponding to (7) is substituted into (13), then we obtain

$$L_n(t) = \frac{I_n(t)}{\sqrt{2(1-t)}} - K_{n+1}(t) + K_{n-1}(t). \tag{15}$$

Eliminating L_n from both (12) and (15), a recurrence relation between K_n's is finally obtained in the form :

$$K_{n+1}(t) - 2K_n(t) + K_{n-1}(t) = -\frac{I_n(t)}{\sqrt{2(1-t)}}. \tag{16}$$

Thus, we arrive at the final expression of Q_n for $n \neq 1$, i.e.

$$Q_n(t) = -\frac{1}{(n-1)(n+2)}\left[nS(t)\left\{P_{n-1}(t) - tP_n(t)\right\} - (1-t^2)P_n(t)\frac{dS(t)}{dt} + 2K_n(t) + 2I_n(t) + 9J_n(t)\right], \tag{17}$$

where the derivative of the Stokes function is given as

$$\frac{dS(t)}{dt} = -8 + \frac{3\sqrt{2}}{\sqrt{1-t}} + \frac{1}{\sqrt{2}(1-t)^{3/2}} + \frac{3(\sqrt{2}-\sqrt{1-t})}{\sqrt{2}(1-t^2)} - 3\log\frac{\sqrt{1-t}(\sqrt{2}+\sqrt{1-t})}{2} \tag{18}$$

It is confirmed that the computation results up to $n = 49$ by (17) are exactly equal to the results obtained by Paul (1973). The advantage of the present method lies in the fact that (17) has a simple mathematical form including terms of the Stokes function and its derivatives.

The truncation for the deflection of the vertical is similarly made on the Vening Meinesz integral. This problem was first treated by Cook (1951). In this case, instead of Q_n, Cook's truncation coefficient

$$q_n(t) = \frac{1}{2}\int_{-1}^{t}\sqrt{1-x^2}\,\frac{dS(x)}{dx}\,P_n^1(x)\,dx \tag{19}$$

is used. The relation between Q_n and q_n can be readily derived as

$$q_n(t) = -\frac{n(n+1)}{2} Q_n(t) - \frac{1}{2} S(t) P_n^1(t) \sqrt{1-t^2} . \qquad (20)$$

Substitution of (17) into (20) gives us the general formula for evaluating Cook's truncation coefficient :

$$q_n(t) = \frac{n(n+1)}{2(n-1)(n+2)} \left[\frac{2}{n+1} S(t) \left\{ P_{n-1}(t) - tP_n(t) \right\} - (1-t^2) P_n(t) \frac{dS(t)}{dt} \right.$$

$$\left. + 2 K_n(t) + 2 I_n(t) + 9 J_n(t) \right] . \qquad (21)$$

Acknowledgment

The author wishes to thank Dr. Mrinal K. Paul of the Earth Physics Branch, Ottawa, Canada, for his valuable suggestions and comments on the present work.

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Received : 10.10.1975
Revised : 26.11.1975

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