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THE COMPUTATION OF LONG GEODESICS ON THE ELLIPSOID BY NON–SERIES EXPANDING PROCEDURE

Abstract

In this paper the author shows a procedure to settle the computation of very long geodesic lines on the ellipsoid without using the series expansion. The integration of elliptic integrals appearing in the procedure is numerically carried out by means of a mechanical quadrature–the method of Repeated Interval Halving.

The author also devises formulae for the numerical solution of the problem, in order to make the amount of significance error least and determine the kind of quadrant for the computation of inverse trigonometric function.

The anti–podal problem for the direct and inverse solution is rigorously solved by this method.

I. – Introduction

The computation of very long geodesic line on the biaxial ellipsoid is one of the main problems in geometric geodesy.

Generally speaking, it is the power series of trigonometric functions with intricate coefficients that has so far played a main role in computing geometrical quantities on the surface of the ellipsoid. It is true that we can readily compute the numerical value of the power series by using logarithmic tables or by desk calculators. However, the power series of trigonometric functions used in computing the geodesic line is constructed so ingeniously that we cannot easily verify whether it is mathematically right or not. Thus, we cannot correct wrong numerals in the coefficients of the power series, which may come from printing miss, for example. Moreover, the series itself is so complicated that it is extremely difficult ro recast it for the solution of such a singular problem as the anti–podal problem.

On the other hand, it is essential for the computation of very long geodesic line to numerically solve the integral equation including an elliptic integral. In the world of mathematics there are various methods of numerical quadrature being revived by the impact of the use of the electronic computer. In fact, before the advent of the high speed computer, these methods were regarded as the impractical for the rigorous computation of mechanical quadrature.

The author introduces a method for the solution of the integral equations, which appear in the Helmert's classical formulae for the computation of very long geodesic line by means of the conformal sphere.

The method of mechanical quadrature is named as Repeated Interval Halving. By using this method together with the iterative method for the solution of the integral equation, the author makes the procedure of computing very long geodesic line brief. Because of the brevity of this procedure, the procedure for solving the anti–podal problem is readily derived by slightly modifying the procedure for the computation of geodesic lines of the ordinary length.

II. — Recasting of the formulae for mechanical quadrature

In this paper we take the Helmert's classical method for the computation of a geodesic line on the ellipsoid. By comparing the ellipsoid with its conformal sphere, we can describe the relationship of differential quantities on their respective surfaces by the following formulae :

$$\frac{\mathbf{ds}}{\mathbf{d}\sigma} = \mathbf{M}\frac{\mathbf{d}\phi}{\mathbf{d}\beta} = \frac{\mathbf{N}'\cos\phi'\,\mathbf{dL}}{\cos\beta'\,\mathbf{d}\lambda}\ , \tag{2.1}$$

whose symbols denote as follows :

ϕ = geodetic latitude on the ellipsoid,

β = reduced latitude on its conformal sphere,

ds = differential geodetic distance on the surface of the ellipsoid,

dσ= differential angular distance on the conformal sphere corresponding to **ds**,

M = the radius of curvature along the meridian of the ellipsoid,

N' = the radius of curvature along the prime vertical of the ellipsoid at the latitude ϕ',

$\phi' = \phi + \mathbf{d}\phi$

$\beta' = \beta + \mathbf{d}\beta$

dL = differential longitude on the ellipsoid, and

dλ= differential longitude on the conformal sphere corresponding to **dL**.

From the relationship between geodetic latitude and its reduced latitude we can readily derive the following equation :

$$N' \cos \phi' = a \cos \beta'$$

where a denotes the equatorial radius of the ellipsoid.

By applying the equation mentioned above to the differential equation (2.1), we obtain the following equations :

$$\frac{dL}{d\lambda} = \frac{1}{a} \frac{ds}{d\sigma} \tag{2.2}$$

$$\frac{dL}{d\lambda} = \frac{M}{a} \frac{d\phi}{d\beta} \tag{2.3}$$

Then, by differentiating the equation, $\tan \beta = (1 - e^2)^{\frac{1}{2}} \tan \phi$, which is one of the fundamental relations between geodetic latitude and its reduced latitude, by β, we obtain the following equation :

$$\frac{d\phi}{d\beta} = \frac{1}{(1 - e^2)^{\frac{1}{2}}} \frac{\cos^2 \phi}{\cos^2 \beta} , \tag{2.4}$$

where e denotes the first eccentricity of the ellipsoid. By plugging (2.4) into (2.3), we also obtain the following equation :

$$\frac{dL}{d\lambda} = \frac{M/a}{(1 - e^2)^{\frac{1}{2}}} \frac{\cos^2 \phi}{\cos^2 \beta} \tag{2.5}$$

By rewriting the equation, $\tan \beta = (1 - e^2)^{\frac{1}{2}} \tan \phi$, with the notation $V = (1 - e^2 \cos^2 \beta)^{-\frac{1}{2}}$, we obtain the following equation :

$$a \cos \beta = \frac{c}{V} \cos \phi \ , \tag{2.6}$$

where c denotes the radius of curvature at the pole of the ellipsoid. By applying (2.6) to (2.5), we finally obtain the following equation :

$$\frac{dL}{d\lambda} = (1 - e^2 \cos^2 \beta)^{\frac{1}{2}} \tag{2.7}$$

Additionally, from (2.2) we can readily derive the following equation :

$$\frac{ds}{d\sigma} = a (1 - e^2 \cos^2 \beta)^{\frac{1}{2}} \tag{2.8}$$

Here, we make the total differential expression from (2.7) as follows :

$$dL = (1 - e^2 \cos^2 \beta)^{\frac{1}{2}} \, d\lambda \tag{2.9}$$

On the other hand, according to Napier's rule we can readily derive the following equation from Fig, 2–1 :

$$\sin \beta_2 = \sin (\sigma_1 + \sigma_t) \sin \beta_0 , \tag{2.10}$$

where β_0 denotes the reduced latitude of the highest point on the geodesic line.

By denoting the reduced latitude and σ_t in Fig. 2–1 about an arbitrary point on the geodesic line by β and σ, respectively, we can rewrite (2,10) as follows :

$$\sin \beta = \sin (\sigma_1 + \sigma) \sin \beta_0 \tag{2.11}$$

Here, we can readily rewrite (2.11) as follows :

$$\cos^2 \beta = 1 - \sin^2 (\sigma_1 + \sigma) \sin^2 \beta_0 \tag{2.12}$$

By plugging (2.12) into (2.8) and by denoting the second eccentricity of the ellipsoid by e', we can readily obtain the following :

$$\frac{ds}{dx} = b (1 + k^2 \sin^2 x)^{\frac{1}{2}}, \text{ where } k^2 = e'^2 \sin^2 \beta_0 \; , \; x = \sigma_1 + \sigma \; , \tag{2.13}$$

and b denotes the radius along the minor axis of the ellipsoid. On the other hand, from the small spherical triangle **PAB** in Fig. 2–2 we can readily derive the following equation :

$$d\lambda \cos \beta = d\sigma \sin \alpha_{12} \tag{2.14}$$

Because of the conformal condition of the sphere, α_{12}, the azimuth of the geodesic line on the sphere, is identical to its corresponding azimuth on the ellipsoid. Thus, from Fig. 2–1 we can readily derive the following equation :

$$\sin \alpha_{12} = \frac{\cos \beta_0}{\cos \beta_1} \tag{2.15}$$

By applying (2.15) to (2.14), we obtain the following equation :

$$d\lambda = \frac{\cos \beta_0}{\cos^2 \beta} \, d\sigma \tag{2.16}$$

By plugging (2.16) again into the total differential equation (2.9), we obtain the following :

$$\mathbf{d\,L} = \cos\beta_0 \,\frac{(1 - e^2 \cos^2 \beta)^{\frac{1}{2}}}{\cos^2 \beta_0}\, \mathbf{d}\,\sigma \tag{2.17}$$

Let us subtract (2.16) from (2.17), and we obtain the following :

$$\mathbf{d\,L} - \mathbf{d}\,\lambda = \cos\beta_0 \left[\frac{(1 - e^2 \cos^2 \beta)^{\frac{1}{2}}}{\cos^2 \beta} - \frac{1}{\cos^2 \beta}\right] \mathbf{d}\,\sigma \tag{2.18}$$

This expression is the final form if a closed formula for expanding into series. The author, however, transforms the right–hand side of (2.18) for the convenience of mechanical quadrature as follows :

We at first rewrite the terms inside the bracket of (2.18) as :

$$\frac{(1 - e^2 \cos^2 \beta)^{\frac{1}{2}}}{\cos^2 \beta} - \frac{1}{\cos^2 \beta} = \frac{(1 - e^2 \cos^2 \beta)^{\frac{1}{2}} - 1}{\cos^2 \beta} \tag{2.19}$$

Let us multiply both the numerator and the denominator in the right–hand side of (2.18) by $(1 - e^2 \cos^2 \beta)^{\frac{1}{2}} + 1$, respectively, and we obtain the following :

$$\mathbf{d}\,\lambda - \mathbf{d\,L} = \frac{e^2 \cos\beta_0 \,\mathbf{d}\,\sigma}{(1 - e^2 \cos^2 \beta)^{\frac{1}{2}} + 1} \tag{2.20}$$

By applying (2.12) to (2.20), we finally obtain the following :

$$\mathbf{d}\,\lambda - \mathbf{d\,L} = \frac{e^2 \cos^2 \beta_0 \,\mathbf{d\,x}}{1 + \frac{b}{a}\,(1 + k^2 \sin^2 x)^{\frac{1}{2}}} \tag{2.21}$$

By integrating (2.13) and (2.21) at an interval from σ_1 to $\sigma_1 + \sigma$, respectively, we obtain the followings :

$$s = b \int_{\sigma_1}^{\sigma_1+\sigma} (1 + k^2 \sin^2 x)^{\frac{1}{2}}\, \mathbf{d\,x} \tag{2.22}$$

and

$$\lambda - L = e^2 \cos\beta_0 \int_{\sigma_1}^{\sigma_1+\sigma} \frac{dx}{1 + \frac{b}{a}\,(1 + k^2 \sin^2 x)^{\frac{1}{2}}} \tag{2.23}$$

In order to normalize the interval of the integration, we put an equation as follows :

$$x = \sigma_1 + \sigma_t\, z \tag{2.24}$$

Then, by differentiating both sides of (2.24) with respect to **x** and **z** , respectively, we obtain the following :

$$dx = \sigma_t\, dz \tag{2.25}$$

By plugging (2.24) and (2.25) into (2.22) and (2.23), respectively, we finally obtain the following equations :

$$s = b\,\sigma_t \int_0^1 \sqrt{1 + k^2 \sin^2 (\sigma_1 + \sigma_t z)\, dz} \tag{2.26}$$

$$\lambda - L = e^2 \cos\beta_0\ \sigma_t \int_0^1 \frac{dz}{1 + \frac{b}{a}\sqrt{1 + k^2 \sin^2 (\sigma_1 + \sigma_t z)}} \tag{2.27}$$

$$\lambda = L + e^2 \cos\beta_0\ \sigma_t \int_0^1 \frac{dz}{1 + \frac{b}{a}\sqrt{1 + k^2 \sin^2 (\sigma_1 + \sigma_t z)}} \tag{2.28}$$

Here, the integration in (2.26) and (2.27) can respectively be carried out by a method of mechanical quadrature, which will be described in the next chapter.

III. – The subroutine of the mechanical quadrature

We have several ways of computing the definite integrals, which are contained in (2.26) and (2.27). The conventional way of computing the integrals is to expand the integrands into binomial series and to integrate them term by term. This method is simple, but it lacks generality.

In this paper, we at first construct the subroutine of numerical integration for the function of single variable – mechanical quadrature. There are many kinds of formulae about the mechanical quadrature such as Trapezoidal rule, Simpson's rule, Gauss' formula, Lobatto's formula, and so on. In view of the convergency, however, the method of Repeated Interval Halving is the most suitable for the numerical solution of the integrals, which are contained in (2.26) and (2.27). By using this method the author constructs the subroutine of the numerical integration.

The principle of the Repeated Interval Halving is briefly to be described as follows :

In order to evaluate such a definite integral with the normalized interval of integration as :

$$\int_0^1 f(x)\,dx\ ,$$

we make trapezoids such as shown in Fig, 3–1, by halving the interval of the abscissa of the coordinate successively. By applying the Trapezoidal rule to each one of the subintervals, we can compute the value of the trapezoid area in each step of the halving as follows :

$$T_1 = \frac{f(0) + f(1)}{2}$$

$$T_2 = \frac{T_1 + f(\frac{1}{2})}{2}$$

$$T_3 = \frac{1}{2}\left[T_2 + \frac{f(\frac{1}{4}) + f(\frac{3}{4})}{2}\right]$$

$$T_4 = \frac{1}{2}\left[T_3 + \frac{f(\frac{1}{8}) + f(\frac{3}{8}) + f(\frac{5}{8}) + f(\frac{7}{8})}{4}\right]$$

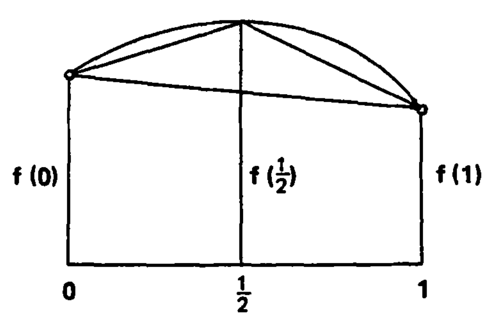

Fig. 3–1

Accordingly, in the **n–th** step of the halving we can express a general formula for computing the value of the trapezoid area as follows :

$$T_n = \frac{1}{2}\left[T_{n-1} + \frac{\sum_{i=1}^{i=n-1} f\left(\frac{2i-1}{2^{n-2}}\right)}{2^{n-2}} \right]$$

with the exception that

$$T_1 = \frac{f(0) + f(1)}{2} .$$

Consequently, we have a sequence of the approximation which tends to the value of the integral, provided that the integrand $f(x)$ is a continuous function. Then, by constructing a linear combination of T_{n-1} and T_n , we obtain the resulting approximation, which is correct even for a quadratic polynomial. For example, if the integrand has the following form : $f(x) = x^2$, we can easily compute the following values :

$$\int_0^1 x^2\,dx = \frac{1}{3}, \quad T_1 = \frac{1}{2}, \quad \text{and} \quad T_2 = \frac{3}{8} .$$

We make the linear combination of T_1 and T_2 with weights, which denote by p and q , as follows :

$$pT_1 + qT_2 = \frac{-1}{3}, \quad \text{on condition that} \quad p + q = 1 .$$

By solving the equations mentioned above with respect to p and q, we obtain the following results :

$$p = \frac{1}{3} \text{ and } q = \frac{4}{3} .$$

Therefore, we can express the integration by using T_1 and T_2 as follows :

$$S = \frac{4T_2 - T_1}{3} .$$

In similar manner, we can derive a general formula, which exactly holds, if the integrand $f(x)$ consists of polynomials of third degree, as follows :

$$S_n = \frac{4T_{n+1} - T_n}{3}$$

To extend the technique of using linear combinations to be applicable to a polynomial of higher degree, we try to determine the coefficients of a linear combination such as :

$$C_1 = pS_1 + qS_2 ,$$

the equation of which is exactly correct for polynomials of fourth degree. By repeating the way of the manipulation mentioned above, we can derive the following formula :

$$C_1 = \frac{4^2 S_2 - S_1}{4^2 - 1}$$

Furthermore, we can describe the general formula as follows :

$$C_n = \frac{4^2 S_{n+1} - S_n}{4^2 - 1}$$

By continuing these procedures further more, we obtain the following formulas :

$$D_n = \frac{4^3 C_{n+1} - C_n}{4^3 - 1} , \quad E_n = \frac{4^4 D_{n+1} - D_n}{4^4 - 1} , \text{ and}$$

$$F_n = \frac{4^5 E_{n+1} - E_n}{4^5 - 1} .$$

These formulas of approximation for the mechanical quadrature can be expressed by the following table :

T_1 ,

$$T_2 \, , S_1 = \frac{4\,T_2 - T_1}{4-1} \, ,$$

$$T_3 \, , S_2 = \frac{4\,T_3 - T_2}{4-1} \, , \quad C_1 = \frac{4^2\,S_2 - S_1}{4^2-1} \, ,$$

$$T_4 \, , S_3 = \frac{4\,T_4 - T_3}{4-1} \, , \quad C_2 = \frac{4^2\,S_3 - S_2}{4^2-1} \, , D_1 = \frac{4^3\,C_2 - C_1}{4^3-1} \, ,$$

$$T_5 \, , S_4 = \frac{4\,T_5 - T_4}{4-1} \, , \quad C_3 = \frac{4^2\,S_4 - S_3}{4^2-1} \, , D_2 = \frac{4^3\,C_3 - C_2}{4^3-1} \, , E_1 = \frac{4^4\,D_2 - D_1}{4^4-1}$$

The author constructs the subroutine program of the mechanical quadrature for the computation, which is based on the integration table described above.

IV. – Algorithm for the computation of very long geodesic lines

Inverse Problem

In the inverse problem the geographical coordinates of two points on the ellipsoid are given to find the length of the geodesic line and azimuths between the points. In other words, two geodetic latitudes and the difference of longitudes are given to solve an ellipsoidal triangle shown in Fig, 2–1.

As stated in chapter II, the ellipsoidal triangles can be transferred on the surface of the conformal sphere, on which the corresponding azimuths are preserved unchanged. To solve the triangle on the sphere, we should at first compute the difference of the longitudes on the sphere by using the integral equation (2.27), in which the difference of the longitudes denotes by λ .

Since σ_t in (2.27) is the implicit function of λ , the equation (2.27) is a typical example of non–linear equation which can be solved by using the method of iteration. Therefore, we can express the right–hand side of the equation (2.27) as follows :

$$\phi(\lambda) = \sigma_t \int_0^1 \frac{dz}{1 + \frac{b}{a}\sqrt{1 + k^2 \sin^2(\sigma_1 + \sigma_t z)}} \qquad (4.1)$$

The condition, on which the iterative process to solve the equation (4.1) is convergent, can be described as follows :

$$\frac{d\phi}{d\lambda} < 1 \ . \tag{4.2}$$

To begin with, by using the fundamental equation described as follows :

$$\tan\beta_1 = \frac{b}{a}\tan\varphi_1 \quad \text{and} \quad \tan\beta_2 = \frac{b}{a}\tan\varphi_2 \ , \tag{4.3}$$

we can convert the given geodetic latitudes into the reduced latitudes on the conformal sphere. Here, from the spherical triangle $PP_1\,P_2$ in Fig. 2–1 we can easily describe the following equation :

$$\cos\sigma_t = \sin\beta_1\ \sin\beta_2 + \cos\beta_1\cos\beta_2\cos\lambda \tag{4.4}$$

Furthermore, by applying Napier's rule to the same triangle as described above, we can derive the following equation :

$$\cos\beta_0 = \frac{\cos\beta_1\cos\beta_2\sin\lambda}{\sin\sigma_t}$$

By denoting k by the following equation :

$$k^2 = e'^2\sin^2\beta_0 \ ,$$

it should be noted that k is also the implicit function of λ .

We usually use the difference of longitudes on the ellipsoid as the first approximation of λ in the process of iteration described at the beginning of this chapter. Strictly, however, it should be noted that the value of the first approximation can arbitrarily be taken, provided that it satisfies the condition of the inequality (4.2).

Obviously, the method of the mechanical quadrature described in chapter III is available for the computation of the integral in the right–hand side of the equation (2.27).

Provided that such an inequality as

$$\left|\lambda_{i+1} - \lambda_i\right| < \epsilon \ , \tag{4.7}$$

holds in the process of the iteration, it is obvious that λ_i converges within the specified interval ϵ . In practice, in order to avoid the loss of significant digits, we at first compute the value of $\lambda - L$ instead of that of λ , and then we compute the value of λ by using the following equation :

$$\lambda = (\lambda - L) + L \,. \tag{4.8}$$

By plugging the converged value of λ into (4.4) and (4.5), respectively, we can directly compute the value of σ_t and β_0 . By using these values of β_0 and σ_t , the length of geodesic line which we pursue, can readily be derived by means of the mechanical quadrature from the integral equation (2.26). Here, it should be noted that the value of σ_1 can be computed by using (2.10) . that is, the following formula can directly be derived from (2.9) :

$$\sigma_1 = \arcsin\left(\frac{\sin\beta_2}{\sin\beta_0}\right) - \sigma_t \tag{4.9}$$

As for the azimuth, we can compute it by substituting the value of β_0 for (2.15). Similarly, the back azimuth can be computed by using the following equation :

$$\sin a_{21} = \frac{\cos\beta_0}{\cos\beta_2} \tag{4.10}$$

Direct Problem

In the direct problem the geographical coordinates of a point, the distance, and azimuth from it to another on the surface of the ellipsoid, are given to find the coordinates of the second point. In other words, a geodetic latitude, a geodetic longitude, the length of a geodesic line, and its azimuth, are given to solve an ellipsoidal triangle shown in Fig. 2–1. In similar way to the procedure described in the section of the inverse problem, we will solve a triangle on the conformal sphere, which corresponds to the ellipsoidal one.

To begin with, we rewrite the integral equation (2.26) in the following form :

$$\sigma_t = \frac{s}{b\displaystyle\int_0^1 \sqrt{1 + k^2 \sin^2(\sigma_1 + \sigma_t z)}\, dz}, \tag{4.11}$$

by dividing both sides of the integral equation (2.26) by the following :

$$b\int_0^1 \sqrt{1 + k^2 \sin^2(\sigma_1 + \sigma_t z)}\; dz$$

In the equation mentioned above, the length of the geodesic line s and the radius along the minor axis of the ellipsoid **b** are given in advance. Furthermore, k^2 is defined by the following equation :

$$k^2 = e'^2 \sin^2 \beta_0 \ ,$$

where e' denotes the second eccentricity of the ellipsoid, and β_0 can be computed by the following equation :

$$\cos \beta_0 = \sin \alpha_{12} \cos \beta_1 \qquad (4.12)$$

Here, α_{12} denotes the given azimuth, and β_1 denotes the reduced latitude which can be computed by using the following equation previously mentioned :

$$\tan \beta_1 = \frac{b}{a} \tan \varphi_1 \ ,$$

where φ_1 denotes the given latitude on the ellipsoid.

The equation (4.12) can readily be derived by applying Napier's rule to the triangle $P_0 \, P_1 \, P_2$ in Fig. 2–1.

As for σ_1 in the equation (4.11), we can compute the value of σ_1 by using the following equation, which can be derived in similar way as mentioned above :

$$\tan \sigma_1 = \frac{\tan \beta_1}{\cos \alpha_{12}} \qquad (4.13)$$

Consequently, we can compute the value of σ_t by applying the method of iteration to the equation (4.11). The first approximation of σ_t in the process of the iteration is conveniently taken as s/b in this paper. The condition for the convergency of the procedure of the iteration can be expressed as follows :

$$\left| \frac{d\phi}{d\sigma_t} \right| < 1 \ ,$$

where ϕ denotes as follows :

$$\phi(\sigma_t) = \frac{s}{b \int_0^1 \sqrt{1 + k^2 \sin^2 (\sigma_1 + \sigma_t z)} \, dz} \qquad (4.14)$$

In similar manner to the computational procedure of the inverse problem, the iterative computation of the integral equation (4.11) should be carried out, until the following inequality holds :

$$\left| \sigma_{i+1} - \sigma_i \right| < \epsilon \ ,$$

where ϵ denotes the previously specified interval.

The final value of σ_i through the process of the iteration is equal to that of σ_t shown in Fig. 2–1. By plugging the value of σ_t into the right–hand side of the equation (2.10), we can compute the reduced latitude of the second point. Then, we can obtain the geodetic latitude of the second point by using the following equation previously mentioned :

$$\tan \varphi_2 = \frac{a}{b} \tan \beta_2$$

As for the difference of the longitudes between the first point and the second one, we at first compute λ , the difference of the longitudes on the conformal sphere by using the following equation :

$$\sin \lambda = \frac{\sin \sigma_t \sin \alpha_{12}}{\cos \beta_2} \qquad (4.15)$$

The equation (4.15) can readily be derived by applying Napier's rule to the triangle $PP_1 P_2$ in Fig, 2–1.

In order to transform λ into L , the difference of longitudes on the ellipsoid, we whould compute the value of $\lambda - L$ by using the equation (2.27) with the method of the mechanical quadrature previously mentioned. By subtracting the value of $\lambda - L$ from that of λ , we obtain the value of L as follows :

$$L = \lambda - (\lambda - L) \qquad (4.16)$$

It should be noted here that the right–hand side of the equation (2.27) does not contain the unknown $\lambda - L$, so the method of iteration previously mentioned need not be applied.

Finally, the longitude of the second point can be derived by using the following, brief equation :

$$L_2 = L_1 + L \ , \qquad (4.17)$$

where L_1 denotes the longitude of the first point.

In the similar way as that of the inverse problem we can compute the value of back azimuth by using the equation (4.10).

V. – Procedure to avoid the loss of significant digits

Generals

In this section we consider the arrangement of algorithms for the purpose of avoiding the loss of significant digits in computing very long geodesic lines. The loss of significant digits in the process of computation is usually called significance errors. It is clear that the significance error usually arises in subtracting operation between two nearly equal numbers due to the finite length of stored numbers in computers. For example, we take such numbers as :

$$x = 0.13659476$$

$$y = 0.13654125.$$

Here, we set these two numbers of eight decimal places in the storage of a computer. By making a subtracting operation about the numbers mentioned above in the computer, we obviously obtain the following number :

$$x - y = 0.00005351$$

Here, it should be noted that the number of significant digits decreases from eight decimal places to four through the process of the subtracting operation mentioned above.

As far as the computation of very long geodesic lines is concerned, the author devices the following remedies for the loss of significant digits due to the subtraction :

The computation of b/a

As the indecies of the figure of a reference ellipsoid, an equatorial radius and a flattening are usually shown. As for the flattening, it is usually expressed in the following form :

$$f = 1/299.152813\text{ , i.e.} \tag{5.1}$$

By making this division, we obtain the following value :

$$f = 0.003\,342\,773\,182\ \ldots \tag{5.2}$$

with the significant digits of ten decimal places.

On the other hand, we have another formula of the flattening as follows :

$$f = \frac{a - b}{a}\ , \tag{5.3}$$

where $\mathbf{a}$ denotes the equatorial radius of the ellipsoid. Here, $\mathbf{b}$, the radius along the minor axis of the ellipsoid is usually defined by the following equation :

$$\mathbf{b} = \mathbf{a}\,(1 - \mathbf{f}) \tag{5.4}$$

From the equation mentioned above we can readily derive the following equation :

$$\mathbf{b/a} = 1 - \mathbf{f} \tag{5.5}$$

By substituting the numerical value in (5.2) for $\mathbf{f}$ in the equation (5.5), we obtain the following value :

$$\mathbf{b/a} = \mathbf{0.996\,657\,226\,817} \tag{5.6}$$

Actually, however, the number of the last two digits **17** in the right–hand side of the equation (5.6) is to be lost through the operation of the electronic computer, for the length of floating–point number in the storage of the computer is fixed ten decimal places. In other words, the number of the first two digits **99** in the equation (5.6) can substantially be equivalent to numbers **00** . Generally, the row of zeroes at the head of floating–point number is eliminated in the storage of computers, and the digit of the number is slid to the left–direction. Therefore, the number of the last digits there does not vanish. On the other hand, the row of nines in the storage of computers is not eliminated. So that the row of nines at the head of the number is one of factors for the loss of significant digits.

A remedy for the loss of the significant digits is as follows :

Since the flattening previously mentioned is usually given in the following form :

$$\mathbf{f} = 1/\mathbf{f'} \ .$$

where $\mathbf{f'}$ is a number, nearly **300** , the quantity $\mathbf{b/a}$ can obviously be expressed in the following form :

$$\mathbf{b/a} = (\mathbf{f'} - 1)/\mathbf{f'} \tag{5.7}$$

Here, $\mathbf{f'} - 1$, the numerator in the right–hand side of the equation (5.7) is not affected by the significance error, for the value of $\mathbf{f'}$ of the reference ellipsoid is generally far larger than one. Therefore, it is obvious that the formula (5.7) is free from the significance error.

The computation of the eccentricities

In case that we compute the value of the eccentricity of ellipse by using the following formula :

$$e^2 = f(2-f) \; , \tag{5.8}$$

it is possible that the significance error arise through the process of the computation. The reason is that the term $2-f$ generally the value $1.996\ldots$, so the length of significant digits of the value of e^2 decreases comparing with that of the flattening f.

However, the following formula, which is identical to that mentioned above :

$$e^2 = 2f - f^2 \tag{5.9}$$

is, in contrast with the former, not affected by this kind of error.

The reason is that the value of the first term in the right–hand side of the equation (5.9) is obviously far larger than that of the second one.

The author derives the following formula in order to compute the first eccentricity from the flattening :

$$e^2 = (2f' - 1)/f'^2 \tag{5.10}$$

Since the value of f' is nearly 300, the numerator of the right–hand side in (5.10) $2f' - 1$ is obviously not affected by the significance error.

As for the computation of the second eccentricity, the denominator of the following formula is obviously affected by the significance error for the computation :

$$e'^2 = e^2/(e^2 - 1) \tag{5.11}$$

The author derives the following formula by dividing both the numerator and the denominator of the equation mentioned above by the square of the first eccentricity e^2 :

$$e'^2 = 1/\left(\frac{1}{e^2} - 1\right) \tag{5.12}$$

Here, the following inequality obviously holds :

$$1/e^2 >> 1 \; ,$$

so that the denominator of the right–hand side of the equation (5.12) is obviously not affected by the significance error.

Significance errors resulting from the evaluation of trigonometric functions.

There is another kind of significance errors resulting from the computation of very long geodesic lines.

Generally speaking, for computing the inverse value of a trigonometric function it is necessary to avoid a cosine function, whose argument has a small value, and a sine function or a tangent function, the value of which argument is nearly **90°** . The reason is as follows :

It is clear that such a trigonometric function as is described above, has a value of **0.999 . . .** The first several digits of this number **999** consisting of all nines, are insignificant for the computation, as is previously described.

Consequently, the following procedures should be taken :

1. In case that σ_t has a small value, the formula (4.3)

$$\cos \sigma_t = \sin \beta_1 \sin \beta_2 + \cos \beta_1 \cos \beta_2 \cos \lambda$$

is not suitable to determine the value of σ_t , because the value of the $\cos \sigma_t$ is **0.99 . . .** In this case, the following formula should be used :

$$\sin \sigma_t = \left[(\sin \lambda \cos \beta_2)^2 + (\sin \beta_2 \cos \beta_1 - \sin \beta_1 \cos \beta_2 \cos \lambda)^2 \right]^{\frac{1}{2}} \quad (5.13)$$

However, if the value of λ is small, or if the following inequality holds, $\cos \lambda$ in (5.13) may be affected by the significance error. Furthermore, in that case the quantities in the second parenthesis of the right–hand side of the formula (5.13), that is, $\sin \beta_2 \cos \beta_1 - \sin \beta_1 \cos \beta_2 \cos \lambda$, have also the possibility of being suffered from the significance error. For this reason, the author devises the following formula :

$$\sin \frac{\sigma_t}{2} = \left(\sin^2 \frac{\beta_1 - \beta_2}{2} + \cos \beta_1 \cos \beta_2 \sin^2 \frac{\lambda}{2} \right)^{\frac{1}{2}} \quad (5.14)$$

This formula is also available in such a case as $\cos \sigma_t \geq 0.9$.

2. In similar manner, by using the following formula :

$$1 - \cos x = 2 \sin^2 \frac{x}{2} = \text{vers}\, x ,$$

the author devises the following formulas, in each of which the value of the argument of the left–hand side is small :

(i) By applying the formula mentioned above to (4.5), we can readily obtain the following formula :

$$\sin\frac{\beta_0}{2} = \frac{1}{\sqrt{2}}\left[\cos\beta_1\cos\beta_2\,(\cos\lambda\sin\Delta\sigma_t - 2\sin\lambda\sin^2\frac{\Delta\sigma_t}{2})\,/\sin\sigma_t + \right.$$

$$\left. + 2\sin^2\frac{\Delta\beta}{2} + \sin^2\beta_1\cos\Delta\beta + \cos\beta_1\sin\beta_1\sin\Delta\beta\right]^{\frac{1}{2}}, \tag{5.15}$$

where

$$\Delta\sigma_t = \sigma_t - \lambda\,, \quad \Delta\beta = \beta_2 - \beta_1\,.$$

(ii) We can readily derive the following trigonometric formula :

$$\cos(\sigma_1 + \sigma_2) = \cos(\sigma_2 - \sigma_1) - 2\sin\sigma_1\sin\sigma_2 \tag{5.16}$$

By applying Napier's rule to the triangle in Fig, 2–1, $P_0\ PP_1$ and $P_0\ PP_2$, respectively, we obtain the following equations :

$$\sin\sigma_1 = \sin\beta_1/\cos\alpha\,, \tag{5.17}$$

$$\sin\sigma_2 = \sin\beta_2/\cos\alpha \tag{5.18}$$

By substituting both (5.17) and (5.18) for (5.16), we finally obtain the following equation :

$$\cos 2\sigma_m = \cos\sigma_t - 2\sin\beta_1\sin\beta_2/\cos^2\alpha\,, \tag{5.19}$$

where

$$\sigma_m = (\sigma_1 + \sigma_2)/2 \quad \text{and} \quad \sigma_t = \sigma_2 - \sigma_1\,.$$

If the value of σ_m is small, or if the following inequality holds :

$$\cos^2 2\sigma_m > 0.9\,,$$

the author devises the following formula, by applying similar procedure as described above to the formula (5.19) :

$$\sin\sigma_m = (\sin^2\frac{\sigma_t}{2} + \sin\beta_1\sin\beta_2/\cos^2\alpha)^{\frac{1}{2}} \tag{5.20}$$

(iii) By applying the procedure mentioned in section **(i)** to (2.15) and to (4.10), respectively, we obtain the following equations :

$$\sin \frac{a'_{12}}{2} = (\sin^2 \frac{\Delta\beta}{2} + \tan|\beta_1| \sin \Delta\beta/2)^{\frac{1}{2}} \qquad (5.21)$$

$$\sin \frac{a'_{21}}{2} = (\sin^2 \frac{\Delta\beta}{2} + \tan|\beta_2| \sin \Delta\beta'/2)^{\frac{1}{2}} \qquad (5.22)$$

where

$$\Delta\beta = \beta_0 - |\beta_1|$$
$$\Delta\beta' = \beta_0 - |\beta_2|$$
$$a'_{12} = 90^\circ - a_{12}$$
$$a'_{21} = 90^\circ - a_{21} \quad .$$

It should be noted here that β_1 and β_2 can be taken as positive quantities in the formulas, (5.21) and (5.22), because the *cosine* functions of β_1 and β_2 are absolutely positive.

(iv) Similarly, the modification of the formula (4.12),

$$\cos\beta_0 = \sin a_{12} \cos\beta_1 \ ,$$

is as follows :

$$\sin\frac{\beta_0}{2} = [\cos^2\beta_1 \sin^2(45^\circ - \Delta\beta/2) + (\sin^2\beta_1 - \sin|\beta_1| \cos\beta_1 \cos\Delta\beta)/2\,]^{\frac{1}{2}} \quad (5.23)$$

where

$\Delta\beta = a_{12} - \beta_1$, and β_0 is a small angle.

Furthermore, the left–hand side of the equation (4.12) is absolutely positive, so the equation should be expressed as follows :

$$\cos\beta_0 = |\sin a_{12}| \cos\beta_1 \ . \qquad (5.24)$$

Consequently, the azimuth a_{12} in the formula (4.12) should be converted into an angle of the first quadrant, if the sine function of the azimuth a_{12} is negative.

(v) If the argument of the left–hand side of the equation (2.10) ,

$$\sin\beta_2 = \sin\beta_0 \sin(\sigma_1 + \sigma_t) \ ,$$

is nearly equal to 90° , the author devises the following modification :

$$\sin\frac{\beta_2}{2} = \left[\sin^2 \beta_0 \sin^2 \frac{\Delta\beta}{2} + (\cos^2 \beta_0 - \sin\beta_0 \cos\beta_0 \sin\Delta\beta)/2 \right]^{\frac{1}{2}} \quad (5.25)$$

where $\Delta\beta = \left| \sigma_1 + \sigma_t \right| - \beta_0$, by means of similar procedure as described above.

(vi) As for the formula (4.15) ,

$$\sin\lambda = \frac{\sin\sigma_t \sin a_{12}}{\cos\beta_2} ,$$

where the value of λ is nearly $90°$, the author modifies it as follows :

$$\sin(45° - \lambda/2) = \left[\left\{ \sin^2 a_{12} \sin^2 \frac{\Delta a}{2} - \sin^2 \frac{\beta_2}{2} + (\cos^2 a_{12} - \sin a_{12} \cos a_{12} \sin\Delta a)/2 \right\} / \cos\beta_2 \right]^{\frac{1}{2}} \quad (5.26)$$

where

$$\Delta a = \sigma_t - \left| a_{12} \right|$$

The applicability of these equations mentioned above depends upon the following inequality :

$$\left| 1 - f^2(x) \right| < 0.1 ,$$

where $f(x)$ represents the term of the left–hand side of these equations. The reason is that the form in the left–hand side of the inequality is used for the calculation of inverse trigonometric functions.

(vii) As for the integral formula (4.1),

$$\phi(\lambda) = \sigma_t \int_0^1 \frac{dz}{1 + \frac{b}{a}\sqrt{1 + k^2 \sin^2(\sigma_1 + \sigma_t z)}}$$

the value of $k^2 \sin^2(\sigma_1 + \sigma_t z)$ is obviously a small quantity.

Therefore, by putting such sequential equations as :

$$u = \arctan\left[k \sin(\sigma_1 + \sigma_t z) \right] \quad \text{and}$$
$$w = \arctan\left(\frac{b}{a}\right) \sec u ,$$

to numerically compute the value of the integrand in the integral formula (4.1). Then, the form of the integrand can be simplified as follows : $\cos^2 w$,

which is obviously free from the error of significance.

VI. – The estimation of the kind of a quadrant

Generally speaking, it is quite obvious that the inverse trigonometric functions are multi–valued ones. However, for the evaluation of geometrical quantities it can safely be defined that they are two–valued functions. Therefore, an additional condition is absolutely necessary to find a unique solution of the equation, which includes an inverse trigonometric function in the left–hand side of the equation.

The condition can intutively be derived from the construction of the figure shown in Fig, 2–1.

1. To begin with, σ_t should be fixed positive, while λ is defined as :

$$\lambda = \lambda_2 - \lambda_1 ,$$

where λ_2 and λ_1 denote the longitudes of the respective stations measured eastward from the Greenwich meridian.

Hence, it is clear that the sign of λ can not be fixed. Consequently, the solution of the equation (4.4) :

$$\cos \sigma_t = \sin \beta_1 \sin \beta_2 + \cos \beta_1 \cos \beta_2 \cos \lambda ,$$

can be divided into two categories as follows :

i. $\sigma_t = \arccos (\sin \beta_1 \sin \beta_2 + \cos \beta_1 \cos \beta_2 \cos \lambda)$, if $\cos \sigma_t > 0$

ii. $\sigma_t = 180^\circ - \arccos (\sin \beta_1 \sin \beta_2 + \cos \beta_1 \cos \beta_2 \cos \lambda)$, if $\cos \sigma_t < 0$

iii. $\sigma_t = 90^\circ$, if $\cos \sigma_t = 0$.

Therefore, the value of σ_t can be restricted as follows :

$$180^\circ \geq \sigma_t \geq 0^\circ$$

2. Since β_0 denotes the maximum reduced latitude of the point on the geodesic line or on its extension, it is clear that the value of β_0 can be restricted as follows :

$$90^\circ \geq \beta_0 \geq 0^\circ$$

Therefore, the equation (4.5)

$$\cos \beta_0 = \cos \beta_1 \cos \beta_2 \sin \lambda / \sin \sigma_t$$

can directly be solved with respect to β_0 as follows :

$$\beta_0 = \arccos (\cos \beta_1 \cos \beta_2 \sin \lambda / \sin \sigma_t)$$

3. Since we have a relationship between α and β_0 as $\alpha = 90^\circ - \beta_0$, we can rewrite the formula (4.5) as follows :

$$\sin \alpha = \cos \beta_1 \cos \beta_2 \sin \lambda / \sin \sigma_t \ , \qquad (6.1)$$

where α is the azimuth of the geodesic line at the point of the intersection between the extension of the geodesic line and the equator. However, it is clear that we can not estimate the sign of $\cos \alpha$ from the formula (6.1) only. Therefore, we should use together with the construction of the figure shown in Fig. 2–1 to determine the sign of the *cosine* function.

By examining the figural construction shown in Fig. 2–1, we can intuitively express the following inequalities :

i. Provided that $\beta_2 > \beta_1$, we have $\cos \alpha > 0$

ii. Provided that $\beta_2 < \beta_1$, we have $\cos \alpha < 0$

iii. Provided that $\beta_2 = \beta_1$, we have $\cos \alpha = 0$.

4. By manipulating simple trigonometric formulas, we can describe the following formulas :

$$\sin \sigma_2 + \sin \sigma_1 = 2 \sin \frac{\sigma_2 + \sigma_1}{2} \cos \frac{\sigma_2 - \sigma_1}{2} \qquad (6.2)$$

$$\sin \sigma_2 - \sin \sigma_1 = 2 \sin \frac{\sigma_2 - \sigma_1}{2} \cos \frac{\sigma_2 + \sigma_1}{2} \qquad (6.3)$$

By rewriting the formulas mentioned above by using the following notations :

$$\sigma_m = \frac{\sigma_2 + \sigma_1}{2} \quad \text{and} \quad \sigma_t = \sigma_2 - \sigma_1 \ ,$$

we obtain the following formulas :

$$\sin \sigma_2 + \sin \sigma_1 = 2 \sin \sigma_m \cos \frac{\sigma_t}{2} \tag{6.4}$$

$$\sin \sigma_2 - \sin \sigma_1 = 2 \cos \sigma_m \sin \frac{\sigma_t}{2} \tag{6.5}$$

By substituting (5.17) and (5.18) ,

$$\sin \sigma_1 = \frac{\sin \beta_1}{\cos a} \quad \text{and} \quad \sin \sigma_2 = \frac{\sin \beta_2}{\cos a} ,$$

for (6.4) and (6.5), respectively, we obtain the following formulas :

$$t_1 = (\sin \beta_2 + \sin \beta_1) / \cos a = 2 \sin \sigma_m \cos \frac{\sigma_t}{2} \tag{6.6}$$

$$t_2 = (\sin \beta_2 - \sin \beta_1) / \cos a = 2 \cos \sigma_m \sin \frac{\sigma_t}{2} \tag{6.7}$$

In the inverse problem for the computation of very long geodesic lines the reduced latitudes of the respective points, β_1 and β_2 , are exactly obtainable quantities by computation. Therefore, we can readily estimate the sign of the following terms :

$$\sin \beta_2 + \sin \beta_1 \quad \text{and} \quad \sin \beta_2 - \sin \beta_1 \ .$$

Furthermore, referring to the items in the sections **1.** and **3.**, it is obvious that we can express the following inequalities :

$$180° > \sigma_t > 0° \ , \quad \text{then} \quad 90° > \sigma_t / 2 > 0°$$

Hence, $\sigma_t/2$ obviously denotes an angle of the first quadrant, so that both $\sin \frac{\sigma_t}{2}$ and $\cos \frac{\sigma_t}{2}$ are absolutely positive. In addition, the sign of $\cos$ in (6.6) and (6.7) can be estimated in such way as described in the section **3.** Accordingly, it is clear that we can determine the signs of $\cos \sigma_m$ and $\sin \sigma_m$, respectively. On the other hand, we can compute the value of $\cos 2\sigma_m$ by using the formula (5.19) :

$$\cos 2\sigma_m = \cos \sigma_t - 2 \sin \beta_1 \sin \beta_2 / \cos^2 a$$

Putting $\cos 2\sigma_m = t$, we intuitively obtain the following solution of

$2\sigma_m$ by combining the signs of $sin\ \sigma_m$ and $cos\ \sigma_m$. Then, in case that $sin\ \sigma_m > 0$, we have the solution of the equation (5.19) with respect to $2\sigma_m$, which can be described as follows :

i. Provided that the following inequalities simultaneously hold :

$cos\ \sigma_m > 0$ and $cos\ 2\sigma_m > 0$, $2\sigma_m$ obviously denotes an angle of the first quadrant. Therefore, we obtain the following solution :

$$2\sigma_m = |arc\ cost| \tag{6.8}$$

ii. Similarly, provided that the following inequalities simultaneously hold :

$$cos\ \sigma_m < 0 \quad \text{and} \quad cos\ 2\sigma_m < 0 ,$$

we have the following solution :

$$2\sigma_m = 180° + |arc\ cost| \tag{6.9}$$

iii. Provided that the following inequalities simultaneously hold :

$$cos\ \sigma_m > 0 \quad \text{and} \quad cos\ 2\sigma_m < 0 ,$$

we have the following solution :

$$2\sigma_m = 180° - |arc\ cost| \tag{6.10}$$

iv. Provided that the following inequalities simultaneously hold :

$$cos\ \sigma_m < 0 \quad \text{and} \quad cos\ 2\sigma_m > 0 ,$$

we have the following solution :

$$2\sigma_m = 360° - |arc\ cost| \tag{6.11}$$

In addition, in case that $sin\ \sigma_m < 0$, all the values of σ_m mentioned above directly change their signs. As for σ_1 , we can obtain its value by using the following formula :

$$\sigma_1 = 2\sigma_m + \sigma_t .$$

5. The additional conditions to estimate the quadrants of α_{12} and α_{21} in the formulas (2.15) and (4.10),

$$\sin a_{12} = \cos\beta_0 / \cos\beta_1 \quad \text{and} \quad \sin a_{21} = \cos\beta_0 / \cos\beta_2 ,$$

are expressed as follows :

$$\tan a_{12} = \sin\lambda \cos\beta_2 / (\sin\beta_2 \cos\beta_1 - \cos\lambda \sin\beta_1 \cos\beta_2) \tag{6.12}$$

$$\tan a_{21} = \sin\lambda \cos\beta_1 / (\sin\beta_2 \cos\beta_1 \cos\lambda - \sin\beta_1 \cos\beta_2) \tag{6.13}$$

6. The additional condition to estimate the quadrant of σ_1 for the solution of the equation (4.13) with respect to σ_1 ,

$$\tan\sigma_1 = \tan\beta_1 / \cos a_{12} ,$$

can be derived by the following way.

Judging from the figural construction shown in Fig. 2–1, we can intuitively find that the quadrants, to which the angles of a and the azimuth a_{12} pertain, are the same. The reason is that the highest point and the intersection of the equator on the geodesic line are obviously placed at the counter–points on either side of the starting point of the geodesic line. Thus, it is clear that the sign of $\cos a$ is equal to that of $\cos a_{12}$. Consequently, the sign of the term in the right–hand side of the equation (5.17).

$$\sin\sigma_1 = \sin\beta_1 / \cos a ,$$

is obviously equal to that of the following :

$$\sin\beta_1 / \cos a_{12} .$$

Thus, we can estimate the sign of $\sin\sigma_1$ by using the known quantity a_{12} . As a result, we obtain the additional conditions to the formula (4.13), in order to estimate the quadrant of σ_1 . Then, we can express the solutions of σ_1 as follows :

i. Provided that the following inequalities simultaneously hold :

$$\tan\beta_1 / \cos a_{12} > 0 \quad \text{and} \quad \sin\beta_1 / \cos a_{12} > 0 ,$$

we have the following solution :

$$\sigma_1 = \arctan(\tan\beta_1 / \cos a_{12}) .$$

ii. Provided that the following inequalities simultaneously hold :

$$\tan\beta_1 / \cos a_{12} < 0 \quad \text{and} \quad \sin\beta_1 / \cos a_{12} > 0 ,$$

we have the following solution :

$$\sigma_1 = 180^\circ - \arctan(\tan\beta_1 / \cos\alpha_{12}) \ .$$

iii. Provided that the following inequalities simultaneously hold :

$$\tan\beta_1 / \cos\alpha_{12} > 0 \quad \text{and} \quad \sin\beta_1 / \cos\alpha_{12} < 0 \ ,$$

we have the following solution :

$$\sigma_1 = 180^\circ + \arctan(\tan\beta_1 / \cos\alpha_{12}) \ .$$

iv. Provided that the following inequalities simultaneously hold :

$$\tan\beta_1 / \cos\alpha_{12} < 0 \quad \text{and} \quad \sin\beta_1 / \cos\alpha_{12} < 0 \ ,$$

we have the following solution :

$$\sigma_1 = 360^\circ - \arctan(\tan\beta_1 / \cos\alpha_{12}) \ .$$

VII. – Anti–podal Problems

In case that the two points which are situated on or almost on the equator of the ellipsoid, are separated nearly **180°** of longitude apart, the whole procedures described above become invalid. The reason is as follows :

It is quite obvious that the difference of longitudes of the two points on the conformal sphere λ , which is one of the arguments in the right–hand side of the equation (4.4), should satisfy the following inequality :

$$|\lambda| \leq 180^\circ \ .$$

Otherwise, the projected arc of the geodesic line on the conformal sphere obviously moves on another side of the sphere due to the shortest line definition of the geodesic line. However, provided that λ is equal to **180°** of longitude, the azimuth of the projected arc of the geodesic line at its intersection of the equator of the sphere is obviously indeterminate. In other words, in case that both β_1 and β_2 are equal to zeroes, and λ is equal to **180°** in the formula (4.4), the value of σ_t should obviously be **180°** . Then, the values of the denominator and the numerator in the right–hand side of the equation (4.5) simultaneously become zeroes ; namely, the value of $\cos\beta_0$, that is, the value of the left–hand side of the equation (4.5) becomes also indeterminate.

Therefore, we should take different procedures from those previously described to determine the value of β_0 .

In order to eliminate λ from the equations (4.4) and (4.5), we take the following procedures :

To begin with, we make the following equation from the equation (4.5) :

$$\sin \sigma_t = \cos\beta_1 \cos\beta_2 \sin\lambda / \cos\beta_0 \tag{7.1}$$

By making the square sum of the equations (4.4) and (7.1), we obtain the following equations :

$$\begin{aligned} \cos^2 \sigma_t + \sin^2 \sigma_t &= (\sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos\lambda)^2 + \\ &\quad + (\cos\beta_1 \cos\beta_2 \sin\lambda)^2 / \cos^2\beta_0 \qquad (7.2) \\ &= 1 \ . \end{aligned}$$

By transforming the equation (7.2) to a quadratic equation with respect to $\cos\lambda$, we obtain the following equation :

$$\begin{aligned} &(\sin\beta_0 \cos\beta_1 \cos\beta_2)^2 \cos^2\lambda - 2\cos^2\beta_0 \sin\beta_1 \sin\beta_2 \cos\beta_1 \cos\beta_2 \cos\lambda + \\ &+ \cos^2\beta_0 (\cos^2\beta_1 + \cos^2\beta_2) - (\cos\beta_1 \cos\beta_2)^2 (1 + \cos^2\beta_0) = 0 \end{aligned} \tag{7.3}$$

By solving the quadratic equation (7.3) with respect to $\cos\lambda$ with some simplification, on condition that

$$\sin\beta_0 \cos\beta_1 \cos\beta_2 \neq 0 \ ,$$

we obtain the following dual solutions :

$$\cos\lambda = \frac{\cos^2\beta_0 \sin\beta_1 \sin\beta_2 \pm \sqrt{D}}{\sin^2\beta_0 \cos\beta_1 \cos\beta_2} \ , \tag{7.4}$$

where

$$D = (\cos^2\beta_1 - \cos^2\beta_0)(\cos^2\beta_2 - \cos^2\beta_0) \ .$$

By putting $\beta_1 = \beta_2 = \beta$ in the solution of the quadratic equation (7.4), we obtain the following :

$$\cos\lambda = \frac{\cos^2\beta_0 \sin^2\beta \pm \sqrt{D}}{\sin^2\beta_0 \cos^2\beta} , \tag{7.5}$$

where

$$D = (\cos^2\beta - \cos^2\beta_0)^2 \ .$$

Since β_0 denotes the highest latitude on the geodesic line projected on the conformal sphere, we obviously have the following inequality :

$$\cos^2\beta > \cos^2\beta_0 \;; \quad \text{that is}, \quad \cos^2\beta - \cos^2\beta_0 > 0 .$$

Therefore, the square–root of **D** in the equation (7.5) should be as follows :

$$\sqrt{D} = \cos^2\beta - \cos^2\beta_0$$

Then, the solution (7.5) can be transformed into the following forms :

$$\cos\lambda = 1, \text{ or } \cos\lambda = (\cos^2\beta_0 \sin^2\beta - \cos^2\beta + \cos^2\beta_0) / \sin^2\beta_0 \cos^2\beta \quad (7.6)$$

From the former solution in (7.6) we can easily derive the following :

$$\lambda = 0^\circ ,$$

which is obviously a trivial solution.

From the latter we can easily derive such a solution as :

$$\cos\lambda = -1 \text{, that is, } \lambda = 180^\circ, \text{ on condition that } \beta = 0 .$$

Accordingly, we can conclude as follows :

In case that $\beta_1 = \beta_2 = 0^\circ$ and $\beta_0 \neq 0$, the solution $\lambda = 180^\circ$

holds independently of the value of β_0 .

Provided that the value of β_0 is zero, the equation (7.3) simply becomes an equation of the first degree such as :

$$2\sin\beta_1 \sin\beta_2 \cos\beta_1 \cos\beta_2 \cos\lambda + 2\cos^2\beta_1 \cos^2\beta_2 - (\cos^2\beta_1 + \cos^2\beta_2) = 0$$

therefore, the solution of the equation described above can directly be obtained as follows :

$$\cos\lambda = \frac{\cos^2\beta_1 + \cos^2\beta_2 - 2\cos^2\beta_1 \cos^2\beta_2}{2\sin\beta_1 \sin\beta_2 \cos\beta_1 \cos\beta_2} \quad (7.7)$$

Accordingly, provided that $\beta_1 = \beta_2 = 0^\circ$ and $\beta_0 = 0^\circ$, the value of λ obviously becomes indeterminate ; in other words, the value of λ is equal to the difference of longitudes of the two points on the ellipsoid.

To take in reverse, let us at first assume that the value of β_0 is equal to zero or **90°** respectivelv in the equation (2.27) for the anti–podal points on the conformal sphere ; in other words, both the reduced latitudes of the two points are equal to zeroes, and the difference of the longitude is **180°** thereon.

Provided that the value of β_0 is equal to zero, we can readily derive the following equations :

$$\cos\beta_0 = 1, \quad \text{and} \quad k^2 = e'^2 \sin^2\beta_0 = 0 .$$

Thus, we can readily transform the right–hand side of the integral equation (2.27) as follows :

$$180° - L = \sigma_t e^2 \int_0^1 \frac{dz}{1+\frac{b}{a}} = \frac{\sigma_t e^2}{1+\frac{b}{a}} \tag{7.8}$$

By the brief manipulation of the indecies of the ellipsoid, we can describe the following equation :

$$\frac{e^2}{1+\frac{b}{a}} = f . \tag{7.9}$$

Here, since the value of λ is **180°** independently of the value of β_0 , it is clear that the value of σ_t is also equal to **180°**.

Thus, we can readily obtain the following solution with respect to **L** :

$$L = 180° (1 - f) . \tag{7.10}$$

Provided that the value of β_0 is equal to **90°** , or the value of $\cos\beta_0$ is equal to zero, then we can immediately obtain the solution of the integral equation (2.27) as follows :

$$L = 180° \tag{7.11}$$

Speaking conversely, the two points on the equator of the ellipsoid, whose angle of the longitudinal interval is between **180° (1 – f)** and **180°** , are projected as the anti-podal points on the equator of the conformal sphere, whose angle of the longitudinal interval is exactly **180°** .

Thus, in order to solve the anti–podal problem, the following procedures can be taken :

If the value of λ in the formula (2.28) becomes larger than **180°** in the process of the iteration, both the formulas (2.27) and (4.5) should be replaced by the following formula for computing the value of β_0 :

$$\cos\beta_0 = \frac{180^\circ - L}{e^2 \sigma_t \displaystyle\int_0^1 \frac{dz}{1 + \frac{b}{a}\sqrt{1 + k^2 \sin^2(\sigma_1 + \sigma_t z)}}} \tag{7.12}$$

The formula mentioned above can reasilu be derived from the equation (2.27) by plugging 180° into λ. The value of σ_t in the right–hand side of the formula (7.12) can be computed in the following way :

By substituting 180° for λ in the right–hand side of the formula (4.4), we obtain the following equations :

$$\cos\sigma_t = \sin\beta_1 \sin\beta_2 - \cos\beta_1 \cos\beta_2$$

$$= -\cos(\beta_1 + \beta_2)$$

Therefore, by taking into consideration of the kind of a quadrant, we obtain the solution of the equation described above with respect to σ_t as follows :

$$\sigma_t = 180^\circ - \beta_1 - \beta_2 \tag{7.13}$$

In addition, the value of σ_1 can also be computed by using the formula (2,10) as follows :

$$\sin\beta_2 = \sin(\sigma_1 + \sigma_t)\sin\beta_0 \ .$$

Since the terms of k^2 and σ_1 in the right–hand side of the equation (7.12) implicitly include those of $\cos\beta_0$, the equation (7.12) should be solved by the method of iteration.

Here, we take 45° as the first approximation of β_0 in the right – hand side of the equation (7.12). The reason is that the angle 45° is the mean value of all β_0 .

By plugging the first approximation of β_0 into the right–hand side of the equation (7.12), we can determine the value of the second approximation of β_0 from the left–hand side thereof. The iteration should be converged under the condition of the following inequality :

$$\left| \frac{d\phi}{d\beta_0} \right| < 1 \ ,$$

where ϕ represents the whole terms of the right–hand side of the equation (7.12).

If the values of σ_1 , σ_t , and β_0 , are determined by the iterative process described above, the length of the geodesic line and its azimuth can directly be computed by using the formulas (2.26) and (4.10).

As far as the direct problem is concerned, the condition for the solution of the anti–podal problem is to restrict the length of geodesic lines.

Through the iterative process to compute the value of σ_t in the formula (4.11), the value of σ_t should be kept less than or equal to **180°** . Otherwise, the given length of the geodesic line is so long that there should be another geodesic line between the points on another side of the ellipsoid. This is the violation against the shortest line definition of a geodesic line.

Consequently, we obtain the necessary and sufficient condition for the solution of the direct problem between the anti–podal points on the conformal sphere as follows :

$$\sigma_t = 180^\circ$$

If the value of σ_t becomes larger than **180°** during the iterative process to compute the value of σ_t in the formula (4.11), the given problem should obviously be rejected as a wrong one.

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