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DESIGN OF SURVEY SYSTEMS USING NONLINEAR PROGRAMMING METHODS

Abstract

This paper discusses theory and results of an attempt to use non–linear programming methods to arrive at an optimal, in the sense of least cost, solution to the design of a survey system to meet specified accuracy. In other words, the method determines the combination of various types of observations which will yield the required accuracy of control points for a minimum cost. The theoretical background of the procedure is discussed, and methods of extension to photo–grammetry and other sciences are presented. Much of the paper is concerned with discussing results of numerical solutions for the optimal design of several small, but typical, mapping problems.

It is believed that this research is original with the author, as extensive literature searches and correspondence has produced no knowledge of prior research into this application of nonlinear programming. The method at its current state of development appears to be capable of yielding significant improvements to the present concepts of survey network design.

Discussion of Problem

In experimental science a frequently recurring problem is to determine certain non–measurable parameters which are functionally related to measurable parameters. In addition, in most cases the relating functions are known. If one now knew the influence of cost upon the value of the variance–covariance matrix of the observable parameters, then, by the method of least squares he could arrive at the influence of cost upon the accuracy of the non–measurable parameters. One may then reverse the problem and ask what would be the minimum cost of attaining a specified accuracy for the non–observable variables. This is the problem toward which this paper is addressed. It should be noted at this point that the mathematics and method presented herein are applicable to all problems of optimization of experimental design as outlined above. At any point where a method is restricted to the specific problems of geodesy and photogrammetry, particularly horizontal control surveys, special mention will be made of this fact.

At this point a brief review of the history of attempts of solution of the above problem might be in order. The first such attempts consisted of generalized specifications. These specifications were generally rigid enough to insure that accuracy requirements were met, but often required far more work than actually required.

With electronic computers came the age of computer simulation. By choosing the set of measurements to be made before actually making them, the error could be propagated and the error for the non–measurable parameters found. One would then pick a near optimal solution simply by choosing the least costly set of observations from among those sets of observations yielding acceptable results for the unknowns. With the advent of the "Kalman Filtering" algorithms, which are nothing more than sequential least squares if no time variation is involved, the capability for easily adding and deleting observations was realized. This facilitated the search for least cost solutions. However, since for most real world problems, there is a practical infinity of possible observation sets, the distinct possibility still exists that the optimum (least cost) data set might never be tried. Thus, although a minimum cost for the set of solutions tried can be obtained, the solution giving true minimum cost may not be in the set tried, and therefore will not be discovered.

The method proposed herein alleviates this problem of omission of data sets, because all possible observations are considered, and the computer picks which ones to use to arrive at an optimum configuration. Thus, the solution of the design of experiments problem becomes less dependent upon human past experience, and more dependent upon analytical methods.

Summary of Results from Error Theory

Since results for minimum variance and maximum likelihood estimators for observation equations of the form $\mathbf{AX} = \mathbf{L} + \mathbf{V}$; where $\mathbf{A}$ is known coefficient matrix, $\mathbf{X}$ is vector of parameters, $\mathbf{L}$ is vector of observations with variance–covariance matrix Σ_{LL}, and $\mathbf{V}$ is vector of residuals with $\mathbf{E(V)} = \mathbf{0}$; are well documented in the literature [1], only a brief summary of pertinent formulae will be presented here.

If the mathematical model may be represented as $\mathbf{AX} = \mathbf{L} + \mathbf{V}$, then the variance–covariance matrix Σ_{XX} of the least squares estimates of $\mathbf{X}$ (minimum variance if $\mathbf{E(X)} = \mathbf{0}$ and variance of $\mathbf{X}$ finite, also maximum likelihood if $\mathbf{L}$ normally distributed) is $(\mathbf{A}^T \Sigma_{LL}^{-1} \mathbf{A})^{-1}$. Although this is all the statistical knowledge which is needed for the method, it would be wise for anyone not familiar with statistical methods to study some other references before continuing to read the remainder of this paper.

Discussion of Linear and Nonlinear Programming

A mathematical programming problem is defined as any problem requiring the maximization or minimization of a function subject to equality or inequality constraints. Symbolically the problem looks like

$$\max \text{ or } \min \quad f(X)$$

subject to

$$\begin{aligned} g_1(X) &\lessgtr b_1 \\ &\vdots \\ g_n(X) &\lessgtr b_n \end{aligned}$$

where $\lessgtr$ means that one of the symbols $\geqq, \leqq, =$ applies, and X is a vector. The g_i and f may be any functions of X, linear or nonlinear. If they are all linear, the problem is termed a linear programming problem. Otherwise it is a nonlinear programming problem.

A practical method for solving the linear programming problem was developed by George Dantzig in 1948. This algorithm, called the simplex algorithm, consists of an ordered search of the vertices of the feasible solution set (the set of all vectors X which satisfy the constraints). The search is ordered such that the next vertex looked at is always at least as good (from the viewpoint of maximizing or minimizing the function f) as the one before. Since for the linear programming problem, the boundaries of the feasible solution sets are hyperplanes in m space (if X is an m vector), then there are only a finite number of such vertices, and therefore the process must converge in a finite number of iterations, This still leaves open the possibility that, for large m and n, the number of iterations could become a very large finite number, but in practice it has been found that the algorithm will normally converge in a very reasonable number of iterations.

A mathematical description of the simplex algorithm follows. No proofs are given, or even hinted at. Proofs may be found in [1], or any good text on linear programming.

Given the linear programming problem

$$\min \ \Sigma c_j x_j \text{ or } c^T x$$

subject to

$$\Sigma g_{1j} X_j \leq b_1$$

$$\vdots \qquad \vdots$$

$$\Sigma g_{ij} X_j \leq b_i$$

$$\Sigma g_{i+1,j} X_j = b_{i+1}$$

$$\vdots \qquad \vdots$$

$$\Sigma g_{nj} X_j = b_n$$

We change the problem into so called standard form by adding so called slack variables to each of the first j constraints to transform them into equalities. The cost of these slack variables is, of course, zero, so the vector C will not be influenced.

We now have

$$\min C^T X$$

Subject to

$$\Sigma g_{1j} X_j + X_{m+1} = b_i$$

$$\vdots \qquad \vdots$$

$$\Sigma g_{ij} X_j + X_{m+i} = b_i$$

$$\Sigma g_{i+1,j} X_j + 0 = b_{i+1}$$

$$\vdots \qquad \vdots$$

$$\Sigma g_{nj} X_j + 0 = b_n$$

The above is called the linear programming problem in standard form.

Changing the notation slightly, we can write the above problem as

$$\min C^T X$$

subject to

$$GX = B$$

where G is a matrix.

For the problem above, we define a basic feasible solution as any solution $\mathbf{X}$ having $\mathbf{n}$ positive elements. (For any problem, either $\mathbf{n}$ will be less than $\mathbf{m}$ or one or more of the constraints will be redundant). This is not a precise definition, but is as close as one can come without assuming a detailed knowledge of convex set and linear manifold theory. The interested reader will find this elaborated upon in [1].

We will now define the simplex algorithm beginning with the assumption that we possess one basic feasible solution $\hat{\mathbf{X}}$. Define $\mathbf{C}^*$ to be the costs associated with the current basis vectors in order. Define the vector $\mathbf{Z} = \mathbf{C}^{*\mathrm{T}}\mathbf{G}$. Form $\mathbf{Z}^* = \mathbf{C}^{\mathrm{T}} - \mathbf{Z}$. There are three possible cases to consider.

1. If $\mathbf{Z}_k^* \geq \mathbf{0}$ for all $\mathbf{k}$, then $\hat{\mathbf{X}}$ is optimal.

2. If for some $\mathbf{k}$ (not corresponding to one of the positive elements in $\mathbf{X}$) $\mathbf{Z}^* < \mathbf{0}$ and $\mathbf{G}_{ik} > \mathbf{0}$ for some $\mathbf{i}$, then by replacement of a vector in the basic feasible solution, a better value of the objective function may be obtained. The vector to come into the basis is chosen by the minimum of all the $\mathbf{Z}^*$, $\mathbf{Z}_k^*$ satisfying the above condition, and the vector to leave the basis is chosen as the minimum over $\mathbf{j}$ or $\dfrac{\mathbf{b}_j}{\mathbf{g}_{jk}}$, subscript called $\mathbf{r}$.

This is accomplished by setting the new $\mathbf{g}'_{ij} = \mathbf{g}_{ij} - \dfrac{\mathbf{g}_{ik}}{\mathbf{g}_{rk}}\mathbf{g}_{rj}$ if $\mathbf{i} \neq \mathbf{r}$, and

$$\mathbf{g}'_{rj} = \frac{\mathbf{g}_{rj}}{\mathbf{g}_{rk}},\ \mathbf{b}'_{ij} = \mathbf{b}_{ij}. - \frac{\mathbf{g}_{ik}}{\mathbf{g}_{rk}}\mathbf{b}_{rj},\ \mathbf{i} \neq \mathbf{r} \text{ and } \mathbf{b}'_{rj} = \frac{\mathbf{b}_{rj}}{\mathbf{a}_{rk}}$$

One should note that this change of basis vectors is exactly the same as the process involved in Gaussian Elimination. In fact, Gaussian Elimination is simply a process of changing basis vectors, although it is seldom presented as such.

3. If for some $\mathbf{k}$ $\mathbf{Z}_k^* < \mathbf{0}$ and $\mathbf{g}_{ik} \leq \mathbf{0}$ for all $\mathbf{i}$, $\mathbf{i} = \mathbf{1}, \ldots, \mathbf{p}$

then there exists no lower bound for $\mathbf{Z}$ and the problem has no optimal solution.

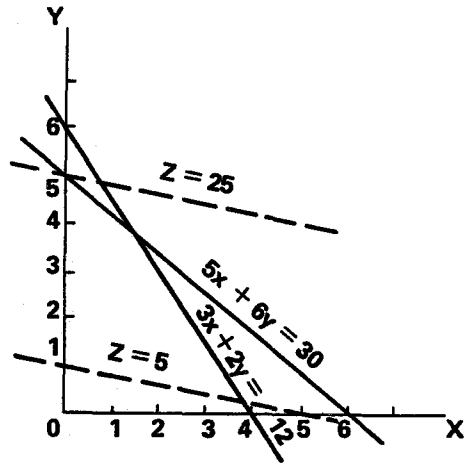
The iterative procedure is then continued with the new basis until either an optimum solution is found, or the non–existence of a minimum is determined.

We have assumed that the user has available an initial feasible solution. In very few problems will this be the case Therefore, the user will be faced with determining a starting point for the simplex algorithm. A method called the method of artificial variables has been developed to solve this problem. The method consists of appending to the right of the $\mathbf{G}$ matrix of the linear programming problem in standard form a unit matrix. The cost function is changed so that the

costs of the actual variables are zero, and the costs of the new variables introduced, called artificial variables, is high, say 1000. Since the real variables cost nothing, the simplex algorithm will attempt to drive the artificial variables (which initially constitute a basic feasible solution, obviously) out of the basis. Once they have been driven out, the cost function may be replaced with the real one, and with the artificial variables now deleted from the problem, the simplex algorithm will continue on to solve the original problem. Pitfalls which may arise with this method will not be discussed here. For a discussion see [1] .

A sample linear programming problem follows :

$$\begin{aligned} &\text{Maximize} && Z = x + 5y \\ &\text{Subject to} && 5x + 6y \leq 30 \\ & && 3x + 2y \leq 12 \\ & && x \geq 0 \\ & && y \geq 0 \end{aligned}$$



Discussion of Nonlinear Programming

Although on the surface a nonlinear programming problem would appear to be much the same as a linear programming problem, there are in reality many differences, each of which add to the complexity of the nonlinear problem.

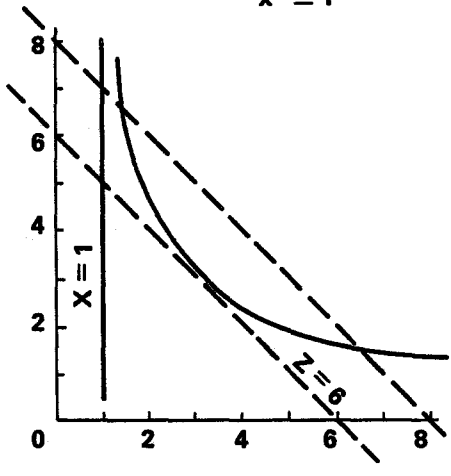
The first complexity is concerned with the convexity properties of the set of feasible solutions. Since convexity will not be covered here (see [1]), we will be satisfied to say that, in the linear case, convergence to a unique answer is guaranteed (if there is any optimum answer) by the convexity of the set of feasible solutions. This convexity is guaranteed by the fact that the constraints are linear inequalities and thus define hyperplanes which separate half spaces which are convex sets. Without the proper convexity properties, which is a common problem in nonlinear programming, solutions leading to local optima can occur, and therefore, the solution obtained may not be the global optimal solution. Fortunately, the problems addressed to date in this research do not have this difficulty, but it was thought pertinent at this point to mention that the problem could occur.

The other principal problem with nonlinear programming is that in many cases the optimum solution will occur not at a vertex of the feasible solution set, but along an edge. You will remember that the simplex algorithm depended upon searching vertices. It turns out that the guarantee of convergence in a finite number of iterations is dependent upon this fact, since there are only a finite number of vertices. However, if all possible boundary points are considered as possible solutions, then there are obviously uncountably infinite possible solutions, and a mere search, even though ordered, cannot be guaranteed to converge finitely. The following will illustrate the nonlinear problem :

Min $Z = x + y$

Subject to $xy \geq 9$

$x \geq 1$



Assuming that one is faced with solving a nonlinear programming problem which has a feasible solution set which is convex, i.e. a unique solution, one method for solving the problem is the method of separable programming. This method requires that each of the constraint functions, and the objective function, may be written as a sum of functions of one variable. Not all problems are of this type, however, by manipulation, most can be reformulated in such a way that they are separable (see [1]).

Given a separable programming problem, it may be reformulated as a linear programming problem in the following manner. For each variable, that variable axis is partitioned arbitrarily over the range of conceivable values for that variable. If one now defines λ_i such that $\mathbf{X} = \Sigma \mathbf{x}_i \lambda_i$, $\Sigma \lambda_i = \mathbf{1}$ and if more than one λ_i is positive, then there can be at most two and these must be adjacent. The λ_i are interpolative constants, and therefore, if $\mathbf{g}_i = \mathbf{g}(\mathbf{X}_i)$, the separate functions in the constraints may be written as $\Sigma \lambda_i \mathbf{g}_i$. Therefore, we are left with a linear programming problem with the λ_i as variables, subject to the added constraints that the λ_i for each original variable sum to one, and the condition that, for λ corresponding to the same original variable, no more than two λ_i may be positive, and these two must be adjacent. The method of artificial variables may be used to obtain an initial feasible solution, the same as in normal linear programming problems.

Applications to Least Squares Estimation Problems

Given a least squares estimation problem with a mathematical model $\mathbf{AX} = \mathbf{L} + \mathbf{V}$ (possibly the result of linearization of a more complicated model), we will now attempt to apply nonlinear programming to the cost optimization of the problem of finding a set of observations which will allow the values of the propagated errors to fall within certain bounds. Before dealing with specific problems, the problem will be discussed in general.

It has been stated previously that the variance–covariance matrix Σ_{XX} , for the adjusted coordinates is given by $(\mathbf{A}^T \Sigma_{LL} \mathbf{A})^{-1}$. The matrix $\mathbf{A}$ is, of course, constant. However, Σ_{LL} is a function of several parameters which influence the variance of each observation. Let the set of parameters which influence the accuracy of the i^{th} observation be denoted by a vector $\mathbf{M}_i$. Then one arrives at a nonlinear programming problem of the form

$$\min \ \mathbf{C}_i(\mathbf{M}_i)$$

subject to

$$(\mathbf{A}^T \Sigma_{LL}^{-1}(\mathbf{M}_1, \ldots, \mathbf{M}_n)\mathbf{A})^{-1} \leq \mathbf{B} .$$

Where $C_i(M_i)$ represents the cost of making the i^{th} observation, and $\Sigma_{LL}^{-1}(M_1, \ldots, M_n)$ represents the inverse of the variance–covariance matrix written as a function of $M_1, \ldots, M_n$. The reader will notice that, in order to arrive at the constraint equations, a matrix of functions must be inverted. As no straightforward method for doing this exists to the knowledge of the author, some sort of approximate method must be employed. One such method, which met with rather limited success, was employed by the author in [1] and in a paper given on this same topic at the 1969 ACSM–ASP Convention in Washington, D.C. [2].

The author's research efforts during the past year have been almost exclusively directed toward finding a better approximation to this inverse, and it is believed that the approximation which will be presented will alleviate all of the shortcomings of the previous method, with the added advantage that it is materially faster in convergence than the older method.

The new method is, strangely enough, based upon a common error committed in the propagation of variance–covariance matrices for least squares estimators involving nonlinear functions. This is to assume that the parameter values obtained by using only a rough estimate of the true variance–covariance matrix of the observations is correct and to propagate using this information. Thus it is assumed that

$$X = (A^T \hat{\Sigma}_{LL}^{-1} A)^{-1} A^T \hat{\Sigma}_{LL}^{-1} L$$

where $\hat{\Sigma}_{LL}$ is the approximation to the variance–covariance matrix. From statistical theory we know that, if $PQ = T$ and the variance–covariance matrix of Q is Σ_{qq} then

$$\Sigma_{TT} = P \Sigma_{qq} P^T .$$

Therefore,

$$\Sigma_{xx} = (A^T \hat{\Sigma}_{LL}^{-1} A)^{-1} A^T \hat{\Sigma}_{LL}^{-1} \Sigma_{LL} \hat{\Sigma}_{LL}^{-1} A (A^T \hat{\Sigma}_{LL}^{-1} A)^{-1}$$

Although the above form is theoretically incorrect (it is correct if $\hat{\Sigma}_{LL} = \Sigma_{LL}$) it turns out that, even for rather crude initial approximations Σ_{LL}, that the estimate of Σ_{xx} obtained by the above method is surprisingly good. For a small problem, this will be shown by the following example :

$$\mathbf{A} = \begin{matrix} 0 & 0 & -1 \\ -178.6 & 103.1 & -1 \\ .5 & .866 & 0 \end{matrix}$$

$$\hat{\Sigma}_{LL} = \begin{matrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & .01 \end{matrix}$$

$$\Sigma_{LL} = \begin{matrix} .75 & .25 & 0 \\ .25 & .75 & 0 \\ 0 & 0 & .0001 \end{matrix}$$

for these matrices

$$(A^T \hat{\Sigma}_{LL}^{-1} A)^{-1} A^T \hat{\Sigma}_{LL}^{-1} \Sigma_{LL} \hat{\Sigma}_{LL}^{-1} A (A^T \hat{\Sigma}_{LL}^{-1} A)^{-1} = (A^T \Sigma_{LL}^{-1} A)^{-1}$$

to four significant figures.

Consideration of the above phenomenon brings to mind a possible solution for the method to be used to approximate the inversion of the matrix $(A^T \Sigma_{LL}^{-1} A)$. First one makes a good guess as to what the optimal values of the variables will be (it turns out that convergence may be achieved for almost any guess, so the actual values of the guess are unimportant). Then one solves the nonlinear programming problem, using

$$(A^T \hat{\Sigma}_{LL}^{-1} A)^{-1} A^T \hat{\Sigma}_{LL}^{-1} \Sigma_{LL} \hat{\Sigma}_{LL}^{-1} A (A^T \hat{\Sigma}_{LL}^{-1} A)^{-1}$$

as the left hand side of the constraints (with Σ_{LL}, of course, written as a matrix of functions). The solution vector from this iteration is then used to form a new $\hat{\Sigma}_{LL}$ for the next iteration.

In practice it has been found that the algorithm obtains convergence for most problems within four to five iterations. The method has never failed to converge on a problem. A detailed discussion of how the method can actually be used on a rather small survey network will follow in the next section.

Application to Several Small Survey Problems

It was decided to try the method on several small horizontal control survey networks combining both direction and distance measurements. There is nothing particularly special about this problem; it simply presents a problem of small enough magnitude to be handled without sophisticated programming methods, and yet one with enough complexity so as to pose some challenge to the method.

Using regression analysis techniques, a model for the variance–covariance matrix for a station adjusted set of directions as a function of the number of pointings made on each point was derived. The variance of the average of **n** readings on a distance is σ^2/n, where σ^2 is the variance of one reading, so no approximate mathematical model was needed for distances.

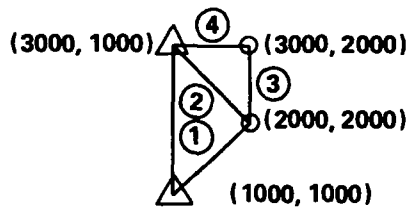
Using these models, one begins by giving the nonlinear programming program guesses at the number of times each measurement is made, a set of required variances for chosen adjusted quantities, a cost vector telling the cost of each repetition of each separate type of measurement, and an observation, or partial derivative, matrix for the least squares adjustment (referred to above as **A**).

The program then returns the number of times each observation should be made, and the true variance–covariance matrix if the observations are made that number of times.

At present there is no provision for using a nonlinear cost function in the program, although such provision could easily be made. For purposes of testing the method, such a cost function was deemed unnecessary.

Four problems were run, and each will be discussed in detail below :

Problem 1



$$\mathbf{A} = \begin{matrix} .71 & .71 & 0 & 0 \\ -.71 & .71 & 0 & 0 \\ -1. & 0. & 1 & 0 \\ 0. & 0. & 0 & 1 \end{matrix}$$

The coordinates are given in terms of a north, east coordinate system in meters. Distance observations numbered 1, 2, 3, and 4 were made. On all runs it was required that the variances of the north and east coordinates for both unknown points (the points on the right) be under **.5 meters**. It was assumed that the variance of a single distance observation was one meter^2.

With a cost vector **C**, $\mathbf{C}_1$ = cost of each individual observation on distance 1, etc.., equal to **(1, 1, 1, 1)** it was found that to meet accuracy requirements, distance 1 must be measured 4 times, two 5 times, three 5 times and four 4 times, with total cost 18.

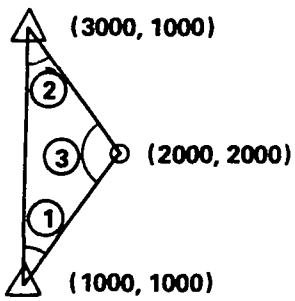
With **C = (1, 1, 10, 10)**, i.e., distances 3 and 4 being 10 times more costly

to observe than 1 and 2, we find the number of necessary observations to be 5, 5, 5, 4 with total cost 100.

With **C = 1, 1, 10, 1** we get **5, 5, 5, 4** with total cost 64 and with **C = 10, 10, 1, 1** we get **5, 3, 15, 4** with total cost 99.

The above numbers are, in all cases, rounded to the next highest even observation, and therefore, some minor changes do not appear. It is, however, evident that even for this very simple problem, the optimum would not be achieved by measuring all distances the same number of times. It should be noted that, since distance four is the only observation determining the east coordinate of the upper unknown point, that it must be made the same number of times no matter what its cost. This turns out to be true in the results. Also, it should be noted that as the cost of the first two observations increase, the number of times they are to be measured decreases. Why this does not apply to the third observation to the same extent is not known ; however, small changes which are masked by the rounding up of the answers did appear.

Problem 2 was a very simple triangulation problem. It consisted of two known stations and one unknown station, with three angles observed,

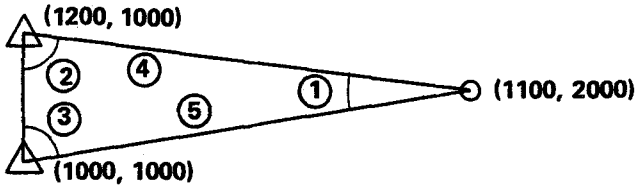


$$\mathbf{A} = \begin{matrix} 0 & 0 & -1 & 0 & 0 \\ -.71 \times 10^{-3} & .71 \times 10^{-3} & -1 & 0 & 0 \\ -.71 \times 10^{-3} & -.71 \times 10^{-3} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -.71 \times 10^{-3} & .71 \times 10^{-3} & 0 & 0 & -1 \\ -.71 \times 10^{-3} & -.71 \times 10^{-3} & 0 & 0 & -1 \end{matrix}$$

with the last three columns coming from the unknown orientation of the sets of directions. With the requirement that the variances of the coordinates of the unknown point be less than $.5 \times 10^{-4}$ meters^2 and **C = 1, 10, 5** with σ^2 for a single observation equal to **30** *secs*^{2} we get the number of observations to be **50, 16, 15** respectively with total cost 285 With **C = 1, 1, 10** we get 36, 25, 1

with total cost 71. With **C = 1, 100, 10** we get 50, 15, 25 with total cost 1800. With **C = 1, 1, 1** we get 25, 37, 3, 5, total cost 65.3. The effect of changing the relative costs upon the optimum solution is easily seen here. It must be remembered that all of the above combinations of measurements meet the accuracy requirements, but that the only differences were in the relative costs associated with each measurement. It is easily seen that measurement costs can greatly influence the configuration of the optimum solution.

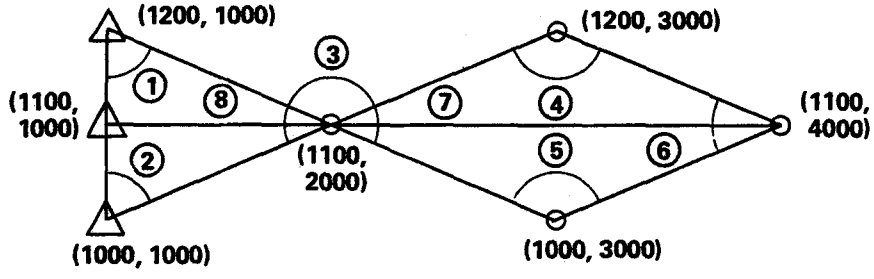
Problem 3 combined direction and distance observations.



The variance of a single direction observation was **3** sec^2 and the variance of a single distance observation was **1** $meter^2$. The variances on north and east of the unknown point $.1 \times 10^{-2}$.

With **C = 1, 1, 1, 1, 1** the optimum solution was 15, 1, 1, 1, 1 with total cost 19. With **C = 10, 5, 5, 1, 1** the solution was 15, 1, 21, 1, 1 with total cost 167. With **C = 200, 100, 1001, 1** the solution was 14, 1, 1, 25, 25 with total cost 3050. It is obvious from these results that the directions at the unknown point are the most critical observations. Even when they cost 200 times as much as the distances, they must still be made 14 times for an optimal solution. Also, it should be noted that, especially for this problem the direction observations lend much more strength to the network than the distance observations.

Problem 4 was a somewhat more complex combined angle and distance measurement problem.



The only condition enforced was that the variances of the north and east coordinates of the far right hand point be less than **.001** $meters^2$. The variance for a single direction was taken as **3** $seconds^2$. With **C = 1, 1, 1, 1, 1, 1, 1, 1** we get 5, 5, 5, 1, 1, 1, 5, 17 with total cost 40. With **C = 1, 1, 1, 1, 1, 1, 100, 100** we get

5, 5, 50, 1, 1, 1, 1, 13 with total cost 1464. With **C = 1, 1, 4, 1, 1, 1, 100, 100** we get 15, 15, 15, 1, 1, 1, 1, 14 with total cost 1593. The influence of the cost vector upon the optimal solution is obvious in this case.

The reader will notice that the total cost is increased rather significantly by the addition of cost constraints. This is logical, since in most cases, the cost of the most critical measurement was increased, and, since this measurement was required essentially without respect to cost, the total cost increases.

It should be mentioned here that we did not allow the values of the variables to go to zero, since we incorporated no provision for handling zero weights (or infinite variances). Also, since our polynominal approximation ended at 50, no value could exceed 50. In only one case does the restriction of 50 enter in ; however, the restriction that the variables could not be less than 1 entered into several problems. A possible solution to this is discussed in the conclusions.

As a comparison, for problem 4 the accuracy specified was about one part in 100.000. If we assumed **C = 1, 1, 1, 1, 1, 1, 1, 1** and used a set of specifications telling us to measure each direction 16 times (8 sets of both direct and reverse) and to measure each distance 4 times, the total cost would be 104, vs a total cost of 40 for the optimal solution, but worse than that, the results of the survey would not meet required accuracy. This example should make the advantages of this method obvious to the reader.

Discussion of Applications to More General Problems

The method can be extended rather simply to general problems. For problems involving trade offs between various possible methods, one could simply set up the problem as if all observations possible were being made and, if the variance functions are picked so that if an observation is not made, it will have zero weight in the adjustment, then a least cost solution would allow some measurements not to be made at all. Thus the method could be used as a decision tool as to which measurements should be made, or where control points should be placed, etc.. Ideally, in this mode, integer programming techniques should be used, so that a binary (0,1) possible range of answers would be possible. This would complicate the programming solution somewhat, but it is certainly feasible,

Another possibility is that the method could be applied separately to each of two or more competitive systems to find out the minimum cost of attaining a given accuracy with each. It would then be simple to see which system would be least costly for accomplishing a given purpose. This application could save considerable expense by eliminating the complete development of two computing systems when one is found to be considerably less cost effective than the other.

It should be realized at this point that the computer programs used to apply this method to these larger problems will be rather large and involved. Therefore, it is not envisioned at this time that this method will be used to obtain an optimum design for each individual project, but only to arrive at an optimum

design for a typical project of a class. The other projects in this class could be accomplished in a near optimum manner using the experience gained from the test project.

Conclusion

As can be seen from the sample problem in the last section, the method works relatively well for small problems. The difficulties encountered in completely deleting measurements are being eliminated by using integer programming techniques.

The method has been shown to be feasible for ground survey problems. The author believes it to be feasible for larger and more general classes of problems both in the mapping field, and in other areas of interest. Of course, the actual application of the method may have to be tailored to the individual problem ; however, the general concepts will apply.

It is hoped that, in the next year, larger problems can be attacked and the feasibility of the method for totally different problems can be studied. The author would like to apply the method to such things as optimal location of control for photogrammetric block adjustments, configuration of navigational satellite systems, and other problems.

If the author can be of any assistance to anyone else interested in working in this area, he would be more than willing to discuss the matter further.

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