M. K. PAUL
Gravity Division, Earth Physics Branch,
Department of Energy, Mines and Resources,
Ottawa, Canada.

A METHOD OF EVALUATING THE TRUNCATION ERROR COEFFICIENTS FOR GEOIDAL HEIGHT

Abstract

Neglecting distant zones in the computation of geoidal height using Stokes' formula gives rise to some truncation error. This truncation error is expressible as a weighted summation of the zonal harmonic components of the gravity anomaly. Making use of the well–known properties of Legendre polynomials, a compact method of computing these theoretical coefficients has been developed in this paper.

Introduction

The computation of geoidal height from Stokes' formula involves the integration of the gravity anomaly weighted by Stokes' function over the entire surface of the spherical earth. With usual notation, the formula for such computation is

$$N = \frac{R}{4\pi G} \int_0^{2\pi} d\alpha \int_0^{\pi} \Delta g(\alpha, \psi)\, S(\cos\psi) \sin\psi \, d\psi \tag{1}$$

where $S(\cos\psi)$ is the Stokes' function, defined by

$$S(\cos\psi) = 1 + \operatorname{cosec}\psi/2 - 6\sin\psi/2 - \cos\psi\,\{5 + 3\log(\sin\psi/2 + \sin^2\psi/2)\} \tag{2}$$

In most of the practical cases, the integration in (1) is carried out numerically up to a chosen angular distance ψ_0 around the point of computation. This gives rise to a truncation error for geoidal height which is given by

$$\delta N = \frac{R}{4\pi G} \int_0^{2\pi} d\alpha \int_{\psi_0}^{\pi} \Delta g(\alpha, \psi) S(\cos\psi) \sin\psi \, d\psi \tag{3}$$

Harmonic expansion of this truncation error is available (Molodenskii et al., 1962) as

$$\delta N = \frac{R}{2G} \sum_{n=2}^{\infty} Q_n(\cos\psi_0)\, \Delta g_n \tag{3'}$$

where

$$Q_n(\cos\psi_0) = \int_{\psi_0}^{\pi} S(\cos\psi)\, P_n(\cos\psi) \sin\psi \, d\psi \tag{4}$$

Δg_n = nth order zonal harmonic component of Δg at the point of computation,

and P_n = Legendre polynomial of order n .

Separation of the gravity anomaly into its harmonic components is rather a standard procedure, using any established method of harmonic analysis. Consequently, the computation of the truncation error of geoidal height using (3′) rests mainly upon the computation of $Q_n(\cos\psi_0)$ from (4). The present paper is concerned with the method of computation of $Q_n(\cos\psi_0)$.

Several methods for computation of $Q_n(\cos\psi_0)$ are available in the literature. With $\sin\psi_0/2 = t$, Molodenskii et al. (1962) developed for $Q_n(t)$ (up to $n = 8$) some power series in t which also involved $\log(1+t)$ and $\log t(1+t)$. The first few of them are

$$Q_0(t) = -4t + 5t^2 + 6t^3 - 7t^4 + (6t^2 - 6t^4) \log t(1+t)$$

$$Q_1(t) = -2t + 4t^2 + \frac{28}{3} t^3 - 14t^4 - 8t^5 + \frac{32}{3} t^6 + (6t^2 - 12t^4 + 8t^6) \log t(1+t) - 2 \log(1+t)$$

$$Q_2(t) = 2 - 4t + 5t^2 + 14t^3 - \frac{53}{2} t^4 - 30t^5 + 47t^6 + 18t^7 - \frac{51}{2} t^8 + (6t^2 - 24t^4 + 36t^6 - 18t^8) \log t(1+t)$$

On the other hand, de Witte (1967) integrated the differential equation corresponding to (4) numerically with the initial condition

$$Q_n(\cos\psi_0) = 0 \text{ at } \psi_0 = 0$$

Recently Hagiwara (1973) offered another series expansion of $Q_n(t)$:

$$Q_n(t) = -4 \sum_{k=0}^{[n/2]} T_{n,k}\, I_{n-2k}(t)$$

where

$$T_{n,k} = -\frac{(n-2k+1)(n-2k+2)}{2k(2n-2k+1)} T_{n,k-1} \quad k \geqslant 1$$

$$T_{n,0} = \frac{(2n)!}{2^n (n!)^2}$$

$$I_m(t) = -2J_m(t) + 3J_{m+1}(t) + K_m(t) - 5K_{m+1}(t) - 3L_{m+1}(t)$$

$$J_m(t) = \sum_{k=0}^{m} \frac{(-2)^k}{2k+1} \binom{m}{k} (t^{2k+1} - 1)$$

$$K_m(t) = \frac{1}{4(m+1)} \{(-1)^{m+1} - (1-2t^2)^{m+1}\}$$

$$L_m(t) = -\frac{1}{4(m+1)} \{(1-2t^2)^{m+1} - 1\} \log t(1+t) + \{1+(-1)^m\}$$

$$\log(1+t) + 2\sum_{k=0}^{m} \{1+(-1)^{m-k}\} K_k(t) - 2\sum_{k=0}^{m} (-1)^{m-k} J_k(t)$$

and

$$[n/2] = \text{the integer part of } n/2 .$$

From the point of practical computation, it is important to point out some of the limitations of the above methods. As n increases, the number of terms in Molodenskii's expansion for Q_n also increases and the derivation of the corresponding expansion for Q_n becomes more and more involved. As a result, the computations may become rather formidable for sufficiently large values of n.

The increase in the number of terms also arises in Hagiwara's formula for Q_n and in his auxiliary functions J_m and L_m, and hence the method confronts similar limitations as with Molodenskii's expansion. On the other hand, de Witte's method gradually accumulates more and more error as ψ_0 increases. This usually occurs with numerical solution of any differential equation.

A new method for computation of Q_n is described below. It will be seen that this method is relatively free from the limitations as mentioned above.

Theory

Let us first of all build up some of the mathematical requirements of our solution for Q_n. Making use of the well–known properties of Legendre polynomials, it can be easily shown that

$$R_{n,k}(t) \equiv \int_{-1}^{t} P_n(z) P_k(z)\, dz$$

$$= \frac{\frac{n(n+1)}{2n+1} P_k(t)\{P_{n+1}(t) - P_{n-1}(t)\} - \frac{k(k+1)}{2k+1} P_n(t)\{P_{k+1}(t) - P_{k-1}(t)\}}{(n-k)(n+k+1)}$$

$$k \neq n \qquad (5)$$

and

$$R_{n,n}(t) \equiv \int_{-1}^{t} P_n^2(z)\, dz$$

$$= \frac{(n+1)(2n-1)}{n(2n+1)} R_{n+1,\,n-1}(t) - \frac{n-1}{n} R_{n,n-2}(t)$$

$$+ \frac{2n-1}{2n+1} R_{n-1,\,n-1}(t) \qquad (6)$$

With the initial values,

$$P_0(t) = 1$$

$$P_1(t) = t$$

$$R_{0,0}(t) = t + 1$$

$$R_{1,1}(t) = (t^3 + 1)/3$$

and the recurrence relation

$$P_n(t) = \frac{2n-1}{n} t P_{n-1}(t) - \frac{n-1}{n} P_{n-2}(t) \tag{8}$$

$R_{n,k}(t)$ $(k \neq n)$ and $R_{n,n}(t)$ can be computed in sequence from (5) and (6) respectively.

Now, substituting $\cos\psi = z$ in (4) as well as in the expansion of the Stokes' function in Legendre polynomial (Heiskanen and Moritz, 1967) we have

$$Q_n(t) = \int_{-1}^{t} S(z) P_n(z) dz \tag{9}$$

where

$$S(z) = \sum_{k=2}^{\infty} \frac{2k+1}{k-1} P_k(z) \tag{10}$$

and

$$t = \cos\psi_0 \tag{11}$$

Again, if $S(z)$ in (9) is replaced by the right hand side of (10), the order of summation and integration in the subsequent equation is interchanged and, then, the definitions in (5) and (6) are made use of, we have

$$Q_n(t) = \sum_{\substack{k=2 \\ k\neq n}}^{\infty} \frac{2k+1}{k-1} R_{n,k}(t) + \frac{2n+1}{n-1} R_{n,n}(t) \tag{12}$$

The convergence of the infinite summation in the above equation is very slow. As a result, the evaluation of $Q_n(t)$ directly from (12) is rather formidable. This difficulty can be removed by modification of (12) in the following way :

Substituting from (5) for $R_{n,k}(t)$ in (12) and then expanding the coefficients of Legendre functions in terms of partial fractions of the form $\frac{1}{k \pm \nu}$, we have

$$Q_n(t) = \frac{n(n+1)}{(2n+1)(n-1)(n+2)} \left[P_n(t) \sum_{\substack{k=1 \\ k\neq n-1}}^{\infty} P_k(t) \right.$$

$$\left\{\frac{2(2n+1)}{n(n+1)}\,\frac{1}{k}-\frac{n+2}{k-n+1}-\frac{n-1}{k+n+2}\right\}$$

$$+\ \{P_{n+1}(t)-P_{n-1}(t)\}\sum_{\substack{k=2\\k\neq n}}^{\infty} P_k(t)\left\{\frac{3}{k-1}-\frac{n+2}{k-n}+\frac{n-1}{k+n+1}\right\}$$

$$-P_n(t)\sum_{\substack{k=3\\k\neq n+1}}^{\infty} P_k(t)\left\{\frac{2(2n+1)}{n(n+1)}\,\frac{1}{k-2}-\frac{n+2}{k-n-1}-\frac{n-1}{k+n}\right\}\Bigg]$$

$$+\frac{2n+1}{n-1}\,R_{n,n}(t) \tag{13}$$

If we now assume

$$U_n(t,h) = \sum_{\substack{k=0\\k\neq n-1}}^{\infty} \frac{P_k(t)\,h^{k-n+1}}{k-n+1}\ ,\ h < 1 \tag{14}$$

$$n = 0, 1, 2, \ldots .$$

then with the well–known relation

$$\frac{1}{\sqrt{1-2th+h^2}} = \sum_{k=0}^{\infty} P_k(t)\,h^k\ ,\ h < 1 \tag{15}$$

we can easily obtain

$$\int_\epsilon^1 \frac{dh}{h^n\sqrt{1-2th+h^2}} = U_n(t,1)-U_n(t,\epsilon)-P_{n-1}(t)\,\ln\epsilon \tag{16}$$

$$0 < \epsilon < 1\ .$$

Further, integrating $\int_\epsilon^1 \frac{dh}{h^{n-2}\sqrt{1-2th+h^2}}$ by parts and performing necessary algebraic simplification, we also obtain

$$(n-1)\int_{\epsilon}^{1}\frac{dh}{h^{n}\sqrt{1-2th+h^{2}}}-(2n-3)t\int_{\epsilon}^{1}\frac{dh}{h^{n-1}\sqrt{1-2th+h^{2}}}$$

$$+(n-2)\int_{\epsilon}^{1}\frac{dh}{h^{n-2}\sqrt{1-2th+h^{2}}}+\sqrt{2-2t}-\frac{\sqrt{1-2\epsilon t+\epsilon^{2}}}{\epsilon^{n-1}}=0 \quad (17)$$

Substitution of (16) and making use of (8) reduces (17) to

$$(n-1)U_n(t,1)-(2n-3)t\,U_{n-1}(t,1)+(n-2)\,U_{n-2}(t,1)+\sqrt{2-2t}$$

$$=(n-1)U_n(t,\epsilon)-(2n-3)t\,U_{n-1}(t,\epsilon)+(n-2)U_{n-2}(t,\epsilon)+$$

$$\frac{\sqrt{1-2\epsilon t+\epsilon^{2}}}{\epsilon^{n-1}} \quad (18)$$

If we then expand the right hand side of (18) in Legendre functions using (14) and the relation

$$\sqrt{1-2\epsilon t+\epsilon^{2}}=1-\epsilon t+\sum_{k=1}^{\infty}\frac{P_{k-1}(t)-P_{k+1}(t)}{2k+1}\epsilon^{k+1} \quad (19)$$

and repeatedly apply (8), we obtain a greatly simplified recurrence relation for $U_n(t,1)$ which is free from ϵ,

$$U_n(t,1)=\left[(2n-3)t\,U_{n-1}(t,1)-(n-2)\,U_{n-2}(t,1)-\sqrt{2-2t}\right.$$

$$\left.+\frac{P_{n-3}(t)-P_{n-1}(t)}{2n-3}\right]/(n-1) \quad (20)$$

Using (20) and the initial conditions

$$\left.\begin{aligned} U_0(t,1) &= \log\left(1+\frac{2}{\sqrt{2-2t}}\right), & t\neq 0\\ &=0 & t=0\\ \text{and}\quad U_1(t,1) &= \log\frac{2}{1-t+\sqrt{2-2t}}, & t\neq 0\\ &=0 & t=0 \end{aligned}\right\}, \quad (21)$$

the values of $U_n(t,1)$ can be easily computed.

Defining similarly

$$V_n(t,h) = \sum_{k=0}^{\infty} \frac{P_k h^{k+n+1}}{k+n+1}, \; h \leqslant 1 \tag{22}$$

$$n = 0, 1, \ldots$$

and following similar steps as above, we obtain the corresponding recurrence relation

$$V_n(t,1) = [(2n-1)t\, V_{n-1}(t,1) - (n-1)\, V_{n-2}(t,1) + \sqrt{2-2t}\,]/n \tag{23}$$

Using (23) and the initial conditions

$$\left.\begin{aligned} V_0(t,1) &= \ell n\left(1 + \frac{2}{\sqrt{2-2t}}\right), \quad t \neq 0 \\ &= 0 \qquad\qquad\qquad\qquad\;\;, \quad t = 0 \\ \text{and} \\ V_1(t,1) &= t\, V_0(t,1) + \sqrt{2-2t} - 1 \end{aligned}\right\}, \tag{24}$$

the values of $V_n(t,1)$ also can be easily computed.

The equation (13) can now be re-written in terms of $U_n(t,1)$ and $V_n(t,1)$ as

$$\begin{aligned} Q_n(t) = \frac{n(n+1)}{(2n+1)(n-1)(n+2)} \Bigg[& P_n(t) \left\{ \frac{2(2n+1)}{n(n+1)} (U_1^*(t) - U_3^*(t)) \right. \\ & \left. - (n+2)(U_n^*(t) - U_{n+2}^*(t) - (n-1)(V_{n+1}^*(t) - V_{n-1}^*(t)) \right\} \\ & + \{P_{n+1}(t) - P_{n-1}(t)\} \{3U_2^*(t) - (n+2)U_{n+1}^*(t) + (n-1)V_n^*(t)\} \Bigg] \\ & - \frac{2n^2+2n+1}{(n-1)(2n+1)^2} P_n(t) \{P_{n+1}(t) - P_{n-1}(t)\} + \frac{2n+1}{n-1} R_{n,n}(t), \; n \geqslant 2 \end{aligned} \tag{25}$$

where

$$
\begin{aligned}
U_1^*(t) &= U_1(t,1) \\
U_2^*(t) &= U_2(t,1)+1 \\
U_3^*(t) &= U_3(t,1)+\frac{1}{2}+t \\
U_n^*(t) &= U_n(t,1)+\frac{1}{n-1} \\
U_{n+1}^*(t) &= U_{n+1}(t,1)+\frac{1}{n}+\frac{t}{n-1} \\
U_{n+2}^*(t) &= U_{n+2}(t,1)+\frac{1}{n+1}+\frac{t}{n}+\frac{3t^2-1}{2(n-1)} \\
V_{n-1}^*(t) &= V_{n-1}(t,1)-\frac{1}{n}-\frac{t}{n+1}-\frac{3t^2-1}{2(n+2)} \\
V_n^*(t) &= V_n(t,1)-\frac{1}{n+1}-\frac{t}{n+2} \\
V_{n+1}^*(t) &= V_{n+1}(t,1)-\frac{1}{n+2}
\end{aligned}
\tag{26}
$$

We have thus obtained a formula for $Q_n(t)$ which involves a finite fixed number of terms and this is achieved even without resort to any kind of approximation or any loss of analytical rigour. The involved functions in the present formulation, viz, $P_n(t)$, $R_{n,n}(t)$, $U_n(t,1)$ and $V_n(t,1)$ are also representable by a small finite number of terms, as are evident from the recurrence relations (8), (6), (20) and (23), respectively. A fixed finite term representation of the present formula forms its main advantage over those due to Molodenskii (1962) and Hagiwara (1973).

Computation

From a computational standpoint, the worth of a theoretical formula depends, perhaps, on the extent to which it satisfies the following major conditions : (1) the ease at which the formula can be handled or programmed for a digital computer ; (2) the accuracy of the results that can be achieved with it ; and (3) optimal computation time.

Our formula fulfils these requirements very favourably. A compact program write–up of this formula is of no problem ; our version of the program consists of fifty–five Fortran instructions. Since our formula is exact, any inaccuracy in the results is necessarily linked with the "rounding–off" characteristics of a computer. However, as our formula involves a fixed finite number of terms, the cumulative round–off error is expected to be smaller in our cases than in others where the number of terms increases with n , the index of the coefficient. That is why – as we believe – our computed values in Table 1 differ from those of Hagiwara (1973), for large values of n (viz, $n = 16, 17$ and 18). Our computation time is also found to be reasonably small ; total time of computation in a CDC 6400 computer for 950 coefficients corresponding to $\psi_0 = 0^\circ\ (10^\circ)\ 180^\circ$ and $n = 0, 1, \ldots 49$, is 1.1 seconds only.

The results of our computation are shown below in Table 1.

As a check on our computed Q_n–values, we have attempted to reproduce the piecewise continuous function

$$\bar{S}(\cos\psi, \psi_0) = 0 \qquad , 0 \leqslant \psi \leqslant \psi_0$$
$$= S(\cos\psi), \ \psi_0 \leqslant \psi \leqslant \pi \tag{27}$$

from its well known series representation

$$\bar{S}(\cos\psi, \psi_0) = \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n(\cos\psi_0) P_n(\cos\psi) \tag{28}$$

It has been found that in order to obtain some agreement to an order of 10^{-3} between computed values from (27) and (28), the summation in (28) has to be carried out for at least 500 terms when ψ and ψ_0 is separated by more than 15° . For a smaller distance between ψ and ψ_0 , even higher number of terms are necessary. This besides checking our computation procedure of $Q_n(\cos\psi_0)$, also demonstrates the role of Q_n–values for large n . It is expected however, that the computation of δN from (3) will hardly require more than fifty terms in the summation, This is because δN , unlike $\bar{S}(\cos\psi, \psi_0)$, is a continuous function over earth's surface.

Some preliminary practical computations have also been carried out with these Q_n–values. Using 1969 SAO Geopotential Coefficients to provide the gravity anomaly, the truncation errors of geoidal height have been computed at selected stations over Canada for different values of ψ_0 . Some trivial checks on these computations are that at $\psi_0 = 0$, the truncation error should be equal to the geoidal height, directly computable from the geopotential coefficients ; then as ψ_0 increases, the truncation error should continuously decrease and finally should vanish at $\psi_0 = \pi$. Needless to say our computations satisfy these checks fully.

o

o o

Table 1

ϕ_0 (in degree)	$Q_n(\cos\phi_0)$ (n = 0 to 49)								
0.0	0.0000	0.0000	2.0000	1.0000	.6667	.5000	.4000	.3333	.2857	.2500
	.2222	.2000	.1818	.1667	.1538	.1429	.1333	.1250	.1176	.1111
	.1053	.1000	.0952	.0909	.0870	.0833	.0800	.0769	.0741	.0714
	.0690	.0667	.0645	.0625	.0606	.0588	.0571	.0556	.0541	.0526
	.0513	.0500	.0488	.0476	.0465	.0455	.0444	.0435	.0426	.0417
10.0	-.4137	-.4115	1.5928	.5992	.2742	.1177	.0297	-.0235	-.0561	-.0757
	-.0864	-.0907	-.0904	-.0868	-.0807	-.0729	-.0639	-.0543	-.0443	-.0343
	-.0247	-.0155	-.0071	.0006	.0073	.0129	.0176	.0212	.0238	.0254
	.0260	.0257	.0247	.0231	.0208	.0181	.0151	.0118	.0084	.0050
	.0017	-.0014	-.0042	-.0067	-.0089	-.0106	-.0119	-.0128	-.0132	-.0132
20.0	-.7979	-.7825	1.2474	.2903	.0107	-.0942	-.1272	-.1249	-.1045	-.0759
	-.0453	-.0168	.0069	.0245	.0353	.0397	.0384	.0326	.0238	.0135
	.0030	-.0063	-.0137	-.0185	-.0205	-.0198	-.0169	-.0123	-.0066	-.0008
	.0045	.0089	.0117	.0129	.0125	.0107	.0077	.0041	.0002	-.0034
	-.0063	-.0083	-.0091	-.0088	-.0075	-.0054	-.0028	.0000	.0026	.0048
30.0	-1.0483	-1.0102	1.0620	.1606	-.0575	-.1032	-.0868	-.0501	-.0133	.0140
	.0284	.0304	.0234	.0116	-.0008	-.0104	-.0153	-.0152	-.0111	-.0046
	.0021	.0072	.0098	.0094	.0066	.0024	-.0020	-.0053	-.0069	-.0065
	-.0045	-.0014	.0017	.0041	.0052	.0049	.0032	.0009	-.0015	-.0033
	-.0041	-.0038	-.0025	-.0006	.0013	.0027	.0034	.0031	.0020	.0005
40.0	-1.1215	-1.0717	1.0212	.1444	-.0508	-.0803	-.0573	-.0238	.0024	.0158
	.0174	.0114	.0031	-.0038	-.0070	-.0065	-.0035	.0002	.0029	.0038
	.0029	.0011	-.0009	-.0021	-.0022	-.0014	-.0001	.0010	.0015	.0013
	.0006	-.0003	-.0009	-.0011	-.0008	-.0002	.0004	.0008	.0008	.0004
	-.0001	-.0005	-.0006	-.0005	-.0002	.0002	.0005	.0005	.0003	.0000

Table 1 (cont'd)

ϕ_0 (in degree)	$Q_n(\cos\phi_0)$ (n=0 to 49)								
50.0	-1.0157	-.9987	1.0439	.1223	-.0942	-.1164	-.0670	-.0060	.0334	.0401
	.0218	-.0040	-.0208	-.0209	-.0082	.0071	.0150	.0119	.0018	-.0079
	-.0110	-.0065	.0017	.0077	.0079	.0028	-.0035	-.0068	-.0052	-.0002
	.0044	.0056	.0029	-.0015	-.0045	-.0042	-.0011	.0025	.0041	.0028
	-.0004	-.0031	-.0034	-.0014	.0014	.0031	.0026	.0003	-.0021	-.0028
60.0	-.7611	-.8543	1.0400	.0236	-.1885	-.1344	-.0091	.0690	.0628	.0072
	-.0381	-.0387	-.0056	.0250	.0270	.0045	-.0180	-.0203	-.0037	.0137
	.0159	.0031	-.0189	-.0130	-.0026	.0090	.0108	.0023	-.0075	-.0092
	-.0020	.0065	.0080	.0017	-.0056	-.0070	-.0016	.0049	.0062	.0014
	-.0044	-.0055	-.0013	.0039	.0050	.0011	-.0035	-.0045	-.0011	.0032
70.0	-.4106	-.7072	.9584	-.1300	-.2403	-.0531	.0946	.0811	-.0151	-.0636
	-.0264	.0312	.0389	-.0006	-.0312	-.0188	.0139	.0243	.0036	-.0183
	-.0147	.0064	.0169	.0052	-.0114	-.0120	.0024	.0123	.0058	-.0073
	-.0100	.0001	.0091	.0059	-.0045	-.0083	-.0014	.0068	.0057	-.0025
	-.0069	-.0022	.0050	.0054	-.0011	-.0057	-.0028	.0035	.0050	-.0000
80.0	-.0275	-.6082	.8067	-.2602	-.1870	.0733	.1106	-.0161	-.0748	-.0095
	.0506	.0217	-.0320	-.0265	.0174	.0268	-.0059	-.0241	-.0026	.0196
	.0085	-.0142	-.0119	.0085	.0132	-.0032	-.0128	-.0013	.0110	.0048
	-.0084	-.0071	.0053	.0082	-.0021	-.0082	-.0008	.0073	.0032	-.0057
	-.0049	.0037	.0057	-.0015	-.0059	-.0006	.0053	.0023	-.0042	-.0036
90.0	.3252	-.5766	.6358	-.3064	-.0683	.1283	.0233	-.0760	-.0112	.0520
	.0064	-.0385	-.0041	.0300	.0028	-.0243	-.0020	.0202	.0015	-.0171
	-.0012	.0148	.0009	-.0129	-.0007	.0114	.0006	-.0102	-.0005	.0091
	.0004	-.0083	-.0004	.0075	.0003	-.0069	-.0003	.0064	.0002	-.0059
	-.0002	.0055	.0002	-.0051	-.0002	.0048	.0001	-.0045	-.0001	.0042

Table 1 (cont'd)

ϕ_0 (in degree)	$Q_n (\cos \phi_0)$ (n=0 to 49)								
100.0	.5963	-.5988	.5040	-.2740	.0243	.0896	-.0460	-.0336	.0408	.0081
	-.0315	.0049	.0221	-.0113	-.0137	.0137	.0065	-.0136	-.0009	.0118
	-.0032	-.0091	.0058	.0060	-.0071	-.0029	.0073	.0001	-.0066	.0021
	.0052	-.0036	-.0035	.0045	.0017	-.0047	.0000	.0043	-.0015	-.0035
	.0026	.0024	-.0032	-.0011	.0033	-.0001	-.0031	.0011	.0025	-.0019
110.0	.7530	-.6375	.4486	-.2224	.0489	.0384	-.0431	.0076	.0196	-.0174
	-.0017	.0132	-.0074	-.0051	.0091	-.0020	-.0060	.0056	.0013	-.0055
	.0026	.0030	-.0042	.0002	.0035	-.0025	-.0015	.0032	-.0008	-.0023
	.0023	.0006	-.0024	.0011	.0015	-.0020	-.0001	.0019	-.0012	-.0009
	.0017	-.0003	-.0014	.0012	.0005	-.0015	.0005	.0010	-.0012	-.0002
120.0	.7855	-.6496	.4311	-.2084	.0469	.0285	-.0346	.0102	.0103	-.0130
	.0035	.0055	-.0066	.0015	.0035	-.0039	.0006	.0025	-.0026	.0002
	.0019	-.0018	.0001	.0015	-.0013	-.0000	.0012	-.0010	-.0001	.0010
	-.0008	-.0001	.0008	-.0007	-.0001	.0007	-.0005	-.0001	.0006	-.0005
	-.0001	.0006	-.0004	-.0001	.0005	-.0003	-.0001	.0004	-.0003	-.0001
130.0	.7068	-.6034	.4295	-.2373	.0776	.0189	-.0494	.0341	-.0032	-.0188
	.0215	-.0092	-.0059	.0136	-.0105	.0010	.0073	-.0093	.0046	.0024
	-.0068	.0059	-.0012	-.0038	.0055	-.0033	-.0009	.0040	-.0041	.0013
	.0021	-.0037	.0026	.0002	-.0026	.0030	-.0013	-.0011	.0026	-.0022
	.0003	.0017	-.0024	.0013	.0006	-.0019	.0019	-.0005	-.0011	.0019
140.0	.5489	-.4913	.3888	-.2632	.1395	-.0392	-.0248	.0504	-.0451	.0226
	.0029	-.0203	.0247	-.0174	.0042	.0083	-.0149	.0138	-.0068	-.0021
	.0088	-.0106	.0076	-.0015	-.0045	.0078	-.0072	.0035	.0014	-.0051
	.0063	-.0045	.0008	.0029	-.0049	.0046	-.0022	-.0010	.0035	-.0042
	.0030	-.0005	-.0020	.0035	-.0033	.0016	.0007	-.0026	.0031	-.0022

Table 1 (cont'd)

ϕ_0 (in degree)	$Q_n (\cos \phi_0)$ (n=0 to 49)								
150.0	.3556	-.3330	.2908	-.2344	.1708	-.1073	.0508	-.0063	-.0233	.0377
	-.0388	.0301	-.0162	.0014	.0107	-.0178	.0192	-.0155	.0086	-.0005
	-.0065	.0109	-.0119	.0098	-.0055	.0003	.0044	-.0075	.0083	-.0070
	.0039	-.0002	-.0033	.0056	-.0063	.0052	-.0030	.0001	.0026	-.0044
	.0049	-.0041	.0023	-.0001	-.0021	.0035	-.0040	.0034	-.0019	.0001
160.0	.1741	-.1689	.1590	-.1448	.1272	-.1072	.0859	-.0644	.0439	-.0252
	.0092	.0036	-.0129	.0187	-.0213	.0210	-.0184	.0142	-.0090	.0037
	.0013	-.0055	.0085	-.0101	.0104	-.0095	.0076	-.0050	.0021	.0007
	-.0032	.0051	-.0062	.0065	-.0060	.0049	-.0033	.0014	.0005	-.0022
	.0035	-.0043	.0045	-.0042	.0035	-.0023	.0010	.0004	-.0016	.0026
170.0	.0460	-.0457	.0450	-.0440	.0426	-.0410	.0391	-.0369	.0346	-.0320
	.0294	-.0266	.0237	-.0208	.0180	-.0151	.0124	-.0098	.0073	-.0050
	.0029	-.0010	-.0007	.0022	-.0034	.0044	-.0051	.0056	-.0059	.0060
	-.0059	.0057	-.0053	.0047	-.0041	.0035	-.0027	.0020	-.0013	.0005
	.0002	-.0008	.0014	-.0019	.0023	-.0026	.0028	-.0029	.0029	-.0029
180.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

REFERENCES

[1] L. de WITTE : Truncation Errors in the Stokes, and Vening Meinesz Formulae for Different Order Spherical Harmonic Gravity Terms. Geop. Jour. Roy. Astr. Soc. 12, 449, 1967.

[2] W.A. HEISKANEN and H. MORITZ : Physical Geodesy, Freeman, 1967.

[3] M.S. MOLODENSKII, V.F. EREMEEV and M.I. URKINA : Methods for Study of the External Gravitational Field and Figure of the Earth. Translated From the Russian by the Israel Prog. for Sc. Transl. Jerusalem, 1962.

[4] Y. HAGIWARA : Truncation Error Formulas for Geoidal Height and the Deflection of the Vertical. Bull. Geodesique, 106, 453, 1973.