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RECURRENCE RELATIONS FOR INTEGRALS OF ASSOCIATED LEGENDRE FUNCTIONS *

Abstract

Recurrence relations for the evaluation of the integrals of associated Legendre functions over an arbitrary interval within $(0^\circ, 90^\circ)$ *have been derived which yield sufficiently accurate results throughout the entire range of their possible applications. These recurrence relations have been used to compute integrals up to degree* 100 *and similar computations can be carried out without any difficulty up to a degree as high as the memory in a computer permits. The computed values have been tested with independent check formulae, also derived in this work ; the corresponding relative errors never exceed* 10^{-23} *in magnitude.*

Introduction

In recent times, the use of spherical harmonic techniques is gaining increasing importance to represent and analyse observed data in the fields of physical and satellite geodesy. Dependable results for higher degree harmonics are now obtainable from data collected by various geodetically oriented artificial satellites set into orbits around the earth. With continuous improvements in computing facilities with respect to speed and storage capacity, spherical harmonic analysis of surface gravity data up to a degree as high as 200 has also been recently reported (Nagy, 1977).

At some stage, all such analyses require the evaluation of integrals of the form

$$\begin{matrix} C_{nm} \\ S_{nm} \end{matrix} = \int_\sigma f(\theta, \lambda)\, P_{nm}(\cos\theta) \begin{matrix} \cos \\ \sin \end{matrix} (m\lambda)\, d\sigma \,, \tag{1}$$

$$m = 0, 1, \ldots n, \quad n = 0, 1, \ldots N,$$

where $f(\theta, \lambda)$ is an observable quantity (such as gravity, elevation, etc.) everywhere over a unit sphere, σ, representing the model earth, $P_{nm}(\cos\theta)$ is the associated Legendre function of degree n and order m, $d\sigma \equiv \sin\theta\, d\theta\, d\lambda$ is the differential area corresponding to the colatitude θ and longitude λ of the point of observation and N is the highest degree of the harmonic functions considered in a particular analysis.

When the area of the unit sphere is subdivided into a number of elementary blocks of reasonable size (viz. $5^\circ \times 5^\circ$, $1^\circ \times 1^\circ$) and $f(\theta, \lambda)$ is assumed constant over

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any such block from practical considerations, the equation (1) can be re-written as

$$\begin{matrix} C_{nm} \\ S_{nm} \end{matrix} = \sum_{k=1}^{K} f_k \int_{\sigma_k} P_{nm}(\cos\theta) \begin{matrix} \cos \\ \sin \end{matrix} (m\lambda)\, d\sigma \tag{2}$$

$$m = 0, 1, \ldots n, \quad n = 0, 1, \ldots N,$$

with

$$\sigma = \sum_{k=1}^{K} \sigma_k \tag{3}$$

where f_k is the constant value for $f(\theta, \lambda)$ over σ_k and K is the total number of blocks.

Again, when σ_k is reasonably small and N is not too large (not greater than 30, say), the variation of $P_{nm}(\cos\theta) . [\cos(m\lambda)$ or $\sin(m\lambda)]$ over σ_k may be negligible enough to replace it by a constant value $P_{nm}(\cos\theta_k) . [\cos(m\lambda_k)$ or $\sin(m\lambda_k)]$ where (θ_k, λ_k) is an appropriate point over σ_k – possibly, its center. Equation (2) then reduces to

$$\begin{matrix} C_{nm} \\ S_{nm} \end{matrix} = \sum_{k=1}^{K} f_k P_{nm}(\cos\theta_k) \begin{matrix} \cos \\ \sin \end{matrix} (m\lambda_k)\, \sigma_k , \tag{4}$$

$$m = 0, 1, \ldots n, \quad n = 0, 1, \ldots N .$$

Until recently, most of the practical computations seldom exceeded 30 for N and, as such, the use of equation (4) in such computations should be adequate.

On the other hand, with the present trend of carrying out similar computations for higher values of N, it is essential to work out exact, or at least more accurate, methods of evaluating

$$\begin{matrix} C_{knm} \\ S_{knm} \end{matrix} = \int_{\sigma_k} P_{nm}(\cos\theta) \begin{matrix} \cos \\ \sin \end{matrix} (m\lambda)\, d\sigma \tag{5}$$

$$k = 1, 2, \ldots K, \quad m = 0, 1 \ldots n, \quad n = 0, 1, \ldots N .$$

When σ_k is bounded by parallels $(\theta = \theta_{1k}$ and $\theta = \theta_{2k})$ and meridians $(\lambda = \lambda_{1k}$ and $\lambda = \lambda_{2k})$, equation (5) can be factorized as

$$\begin{matrix} C_{knm} \\ S_{knm} \end{matrix} = - I_{knm} \cdot \begin{matrix} J_{km} \\ K_{km} \end{matrix} \tag{6}$$

where

$$I_{knm} = - \int_{\theta_{1k}}^{\theta_{2k}} P_{nm}(\cos\theta) \sin\theta \, d\theta , \tag{7}$$

$$\begin{matrix} J_{km} \\ K_{km} \end{matrix} = \int_{\lambda_{1k}}^{\lambda_{2k}} \begin{matrix} cos \\ sin \end{matrix} (m\lambda)\, d\lambda \, , \tag{8}$$

As equation (8) can be readily evaluated, the only problem is the evaluation of (7) which, after suppression of the index k and substitutions of $t = cos\theta$, $t_1 = cos\theta_1$ and $t_2 = cos\,\theta_2$, assumes the form

$$I_{nm}(t_1, t_2) = \int_{t_1}^{t_2} P_{nm}(t)\, dt \tag{9}$$

Thus, we recognize the necessity of the evaluation of (7) in carrying out a high order spherical harmonic analysis of any kind of observed data.

Some works on the evaluation of such integrals have been published previously in the literature. The method considered by Young (1970) has some problem of instability around polar regions. Gaussian quadrature methods of Christodoulidis and Katsambalos (1977) have disadvantages with respect to accuracy and computation time as N becomes much larger than 60 – the highest value they have used. Their alternative method, which involves the evaluation of the integrals of the form $\int sin^n \theta \, d\theta$ using some series in which the number of terms increases with n , is computationally inefficient and subject to increasing round-off errors.

The recurrence relations developed below are free from such problems. The number of terms in these relations remains unchanged as n increases to higher and higher values. The instability around polar regions has been eliminated by a special technique and the question of error in numerical integration does not arise.

The Proposed Method

We start by listing below some of the well-known relationships involving Legendre polynomials and associated Legendre functions which we are going to use repeatedly. They can be found in any standard text book dealing with the subject (viz. Hobson (1955), Heiskanen and Moritz (1967)).

$$P_{nm}(t) = y^m \frac{d^m P_n(t)}{dt^m} \tag{10}$$

$$P_{n-1,m}(t) - P_{n-3,m}(t) = (2n-3)\, y\, P_{n-2,m-1}(t) \tag{11}$$

$$P_{n,m+1}(t) + (n-m+1)(n+m) P_{n,m-1}(t) = 2m \frac{t}{y} P_{nm}(t) \tag{12}$$

$$(n-m) P_{n-1,m-1}(t) + (n+m-3) P_{n-3,m-1}(t) = (2n-3)\, t\, P_{n-2,m-1}(t) \tag{13}$$

with

$$y^2 = 1 - t^2 \tag{14}$$

First we define a set of integrals $J_{nm}(t_1, t_2)$ and relate them to the other set of integrals $I_{nm}(t_1, t_2)$ as below :

$$(m+1) J_{nm}(t_1, t_2) \equiv (m+1) \int_{t_1}^{t_2} \frac{t}{y} P_{nm}(t) dt$$

$$= -(m+1) \int_{y=y_1}^{y_2} y^m \frac{d^m P_n(t)}{dt^m} dy$$

$$= -y P_{nm}(t) \Big|_{t_1}^{t_2} + I_{n,m+1}(t_1, t_2) \qquad (15)$$

Again, integrating (12) with respect to t from t_1 to t_2, we can write

$$2m J_{nm}(t_1, t_2) = I_{n,m+1}(t_1, t_2) + (n-m+1)(n+m) I_{n,m-1}(t_1, t_2) \qquad (16)$$

Hence, eliminating $I_{n,m+1}(t_1, t_2)$ from (15) and (16) we have

$$(m-1) J_{nm}(t_1, t_2) = (n-m+1)(n+m) I_{n,m-1}(t_1, t_2) + y P_{nm}(t) \Big|_{t_1}^{t_2} \qquad (17)$$

Now, substituting (10) and (11) in (13) we can write,

$$(n-m) P_{n-1,m-1}(t) + (n+m-3) P_{n-3,m-1}(t)$$
$$= \frac{t}{y} [P_{n-1,m}(t) - P_{n-3,m}(t)] \qquad (18)$$

which, after integration similar to the above, gives

$$(n-m) I_{n-1,m-1}(t_1, t_2) + (n+m-3) I_{n-3,m-1}(t_1, t_2)$$
$$= J_{n-1,m}(t_1, t_2) - J_{n-3,m}(t_1, t_2) \qquad (19)$$

Substitution of (17) in (19) then gives

$$I_{n-1,m-1}(t_1, t_2) = \frac{(n-3)(n+m-3)}{n(n-m)} I_{n-3,m-1}(t_1, t_2)$$
$$- \frac{(2n-3)}{n(n-m)} y^2 P_{n-2,m-1}(t) \Big|_{t_1}^{t_2} \qquad (20)$$

Because of the singularity at $m = n$, the above relation can not be used to compute $I_{n-1,n-1}(t_1, t_2)$. As such, a recurrence relation for $I_{n-1,n-1}(t_1, t_2)$ has been derived separately from the relation

$$P_{n-1,n-1}(t) = (2n-3)y\,P_{n-2,n-2}(t) \qquad (21)$$

by performing on it integration similar to the above, when we get

$$I_{n-1,n-1}(t_1,t_2) = \frac{(n-1)(2n-3)(2n-5)}{n} I_{n-3,n-3}(t_1,t_2) + \frac{2n-3}{n} y^2 P_{n-2,n-3}(t) \Big|_{t_1}^{t_2} \qquad (22)$$

Let us now examine some of the computational properties of equations (20) and (22) in polar regions, when y is very small, Then, the magnitudes of I_{nm} and P_{nm} are of the orders of y^m and y^{m-2}, respectively. While this does not pose any problem in dealing with (20), computations with (22) result in addition and subtraction of numbers of the same order to produce a number smaller than them by an order of y^2. This will, naturally, cause loss of accuracy in the results if (22) is applied repeatedly as a recurrence relation. Accordingly, an alternative formula has been developed for polar regions which is free from such inaccuracy. To obtain this, we consider a second formula for $P_{n-1,n-1}(t)$ in terms of y (y being related to t through (14)) as

$$P_{n-1,n-1}(t) = (2n-3)(2n-5)\ldots 3\, y^{n-1} \qquad (23)$$

Therefore, with

$$\left.\begin{aligned} y_1^2 &= 1-t_1^2 \\ y_2^2 &= 1-t_2^2 \end{aligned}\right] \qquad (14a)$$

we can write,

$$I_{n-1,n-1}(t_1,t_2) = (2n-3)(2n-5)\ldots 3 \int_{t_1}^{t_2} y^{n-1}\,dt$$

$$= -(2n-3)(2n-5)\ldots 3 \int_{y_1}^{y_2} y^n (1-y^2)^{-\frac{1}{2}}\,dy \qquad (24)$$

Then, expanding the integrand in the right hand side of (24) in a power series of y and performing term by term integration, we obtain

$$I_{n-1,n-1}(t_1,t_2) \qquad (25)$$

$$= -(2n-3)(2n-5)\ldots 3\, y^{n+1} \left[\frac{1}{n+1} + \frac{1}{2}\frac{y^2}{n+3} + \frac{1.3}{2.4}\frac{y^4}{n+5} + \frac{1.3.5}{2.4.6}\frac{y^6}{n+7} + \ldots\right] \Big|_{y_1}^{y_2}$$

which converges rapidly over its proposed range of application.

Thus equations (13), (20), (21), (22) and (25) along with the following initial values will enable us to compute values of $P_{nm}(t)$ and $I_{nm}(t_1,t_2)$ up to any required degree and for all values of θ :

$$\left.\begin{aligned}
&P_{00}(t) = 1, P_{10}(t) = t, \ P_{11}(t) = y, \ P_{20}(t) = \frac{3t^2-1}{2} \\
&P_{21}(t) = 3ty, \ P_{22}(t) = 3y^2, \ I_{00}(t_1, t_2) = t_2 - t_1, \\
&I_{10}(t_1, t_2) = \frac{t_2^2 - t_1^2}{2}, \ I_{11}(t_1, t_2) = \frac{t_2 y_2 - \theta_2 - t_1 y_1 + \theta_1}{2}, \\
&I_{20}(t_1, t_2) = \frac{t_1 y_1^2 - t_2 y_2^2}{2}, \ I_{21}(t_1, t_2) = y_1^3 - y_2^3, \\
&I_{22}(t_1, t_2) = 3t_2 - t_2^3 - 3t_1 + t_1^3
\end{aligned}\right] \quad (26)$$

Fully Normalised Version

Fully normalised associated Legendre functions and their integrals are related to the corresponding non-normalised ones by

$$\left.\begin{aligned}
&\bar{P}_{n0}(t) = H_{n0} P_{n0}(t), \ \bar{I}_{n0}(t_1, t_2) = H_{n0} I_{n0}(t_1, t_2), \\
&\bar{P}_{nm}(t) = H_{nm} P_{nm}(t), \ \bar{I}_{nm}(t_1, t_2) = H_{nm} I_{nm}(t_1, t_2), \ m \neq 0
\end{aligned}\right] \quad (27)$$

where

$$\left.\begin{aligned}
&H_{n0} = \sqrt{2n+1}, \\
&H_{nm} = \sqrt{2(2n+1)(n-m)!\,/\,(n+m)!}, \ m \neq 0
\end{aligned}\right] \quad (28)$$

Modifying the results of the previous section according to the above relationships, we arrive at

$$\bar{P}_{n-1,m-1}(t) = \left[\frac{(2n-1)(2n-3)}{(n-m)(n+m-2)}\right]^{\frac{1}{2}} t\,\bar{P}_{n-2,m-1}(t) - \left[\frac{(2n-1)(n+m-3)(n-m-1)}{(2n-5)(n+m-2)(n-m)}\right]^{\frac{1}{2}} \bar{P}_{n-3,m-1}(t), \ m \neq n, \quad (13a)$$

$$\bar{P}_{n-1,n-1}(t) = \sqrt{(2n-1)/(2n-2)}\ y\ \bar{P}_{n-2,n-2}(t), \quad (21a)$$

$$\bar{I}_{n-1,m-1}(t_1, t_2) = -\frac{1}{n}\left[\frac{(2n-1)(2n-3)}{(n-m)(n+m-2)}\right]^{\frac{1}{2}} y^2\,\bar{P}_{n-2,m-1}(t)\Big|_{t_1}^{t_2} + \frac{n-3}{n}\left[\frac{(2n-1)(n+m-3)(n-m-1)}{(2n-5)(n+m-2)(n-m)}\right]^{\frac{1}{2}} \bar{I}_{n-3,m-1}(t_1, t_2), \ m \neq n. \quad (20a)$$

$$\bar{I}_{n-1,n-1}(t_1,t_2) = \frac{1}{2n}\left[\frac{2n-1}{(n-1)(n-2)}\right]^{\frac{1}{2}} y^2\,\bar{P}_{n-2,n-3}(t)\Big|_{t_1}^{t_2}$$

$$+\frac{1}{2n}\left[\frac{(n-1)(2n-1)(2n-3)}{n-2}\right]^{\frac{1}{2}}\bar{I}_{n-3,n-3}(t_1,t_2), \qquad (22a)$$

(applicable when θ is not very small)

$$\bar{I}_{n-1,n-1}(t_1,t_2) = -\left[\frac{(2n-1)(2n-3)\ldots 3}{(2n-2)(2n-4)\ldots 4}\right]^{\frac{1}{2}} y^{n+1}$$

$$\cdot\left[\frac{1}{n+1}+\frac{1}{2}\,\frac{y^2}{n+3}+\frac{1.3}{2.4}\,\frac{y^4}{n+5}+\frac{1.3.5}{2.4.6}\,\frac{y^6}{n+7}+\ldots\right]\Bigg|_{y_1}^{y_2} \qquad (25a)$$

(applicable when θ is very small)

and

$$\left.\begin{aligned}
&\bar{P}_{00}(t) = 1,\ \bar{P}_{10}(t) = \sqrt{3}\,t,\ \bar{P}_{11}(t) = \sqrt{3}\,y,\ \bar{P}_{20}(t) = \frac{\sqrt{5}}{2}(3t^2-1)\\
&\bar{P}_{21}(t) = \sqrt{15}\,t\,y,\ \bar{P}_{22}(t) = \frac{\sqrt{15}}{2}\,y^2,\ \bar{I}_{00}(t_1,t_2) = t_2 - t_1\\
&\bar{I}_{10}(t_1,t_2) = \frac{\sqrt{3}}{2}(t_2^2 - t_1^2),\ \bar{I}_{11}(t_1,t_2) = \frac{\sqrt{3}}{2}(t_2 y_2 - \theta_2 - t_1 y_1 + \theta_1)\\
&\bar{I}_{20}(t_1,t_2) = \frac{\sqrt{5}}{2}(t_1 y_1^2 - t_2 y_2^2),\ \bar{I}_{21}(t_1,t_2) = \sqrt{5/3}\,(y_1^3 - y_2^3)\\
&\bar{I}_{22}(t_1,t_2) = \sqrt{5/12}\,(3t_2 - t_2^3 - 3t_1 + t_1^3)
\end{aligned}\right] \qquad (26a)$$

Check Formulae

It was felt necessary to devise some formulae to test the accuracy of computed $\bar{I}_{nm}$ values, especially when m and n are very large. Accordingly, the following four check formulae have been worked out :

$$\left.\begin{aligned}
t' &= \cos[(\theta_1+\theta_2)/2],\\
T_1 &= \bar{I}_{nm}(t_1,t') + \bar{I}_{nm}(t',t_2),\\
\delta T_1 &\equiv T_1 - \bar{I}_{nm}(t_1,t_2) = 0, \quad m=0,1,\ldots n, \quad n=0,1\ldots N.
\end{aligned}\right] \qquad (27)$$

$$\left.\begin{aligned} t' &= \cos(\theta_2 - \theta_1), \quad y' = \sin(\theta_2 - \theta_1), \\ T_2 &= \sqrt{2n+1}\,[y_1 \{ \sqrt{n(n+1)/2}\, \bar{I}_{n1}(1,t') - y' \bar{P}_{no}(t') \} + t_1 \bar{I}_{no}(1,t')], \\ \delta T_2 &= T_2 - \sum_{m=0}^{n} \bar{P}_{nm}(t_1) \bar{I}_{nm}(t_1, t_2) = 0, \quad n = 0, 1, \ldots N, \end{aligned}\right] \quad (28)$$

$$\left.\begin{aligned} t' &= \cos(\theta_2 - \theta_1), \quad y' = \sin(\theta_2 - \theta_1), \\ T_3 &= \sqrt{2n+1}\,[y_2 \{ \sqrt{n(n+1)/2}\, \bar{I}_{n1}(1,t') - y' \bar{P}_{no}(t') \} - t_2 \bar{I}_{no}(1,t')], \\ \delta T_3 &= T_3 - \sum_{m=0}^{n} \bar{P}_{nm}(t_2) \bar{I}_{nm}(t_1, t_2) = 0, \quad n = 0, 1, \ldots N, \end{aligned}\right] \quad (29)$$

and

$$\left.\begin{aligned} t' &= \cos[(\theta_2 - \theta_1)/2], \quad y' = \sin[(\theta_2 - \theta_1)/2], \\ t_m &= \cos[(\theta_2 + \theta_1)/2], \quad y_m = \sin[(\theta_2 + \theta_1)/2], \\ T_4 &= \sqrt{2n+1}\; y_m [\sqrt{2n(n+1)}\, \bar{I}_{n1}(1,t') - 2 y' \bar{P}_{no}(t')], \\ \delta T_4 &\equiv T_4 - \sum_{m=0}^{n} \bar{P}_{nm}(t_m) \bar{I}_{nm}(t_1, t_2) = 0, \quad n = 0, 1, \ldots N. \end{aligned}\right] \quad (30)$$

Of these four sets of equations, (27) follows readily from the relation that the sum of integrals over two halves of an interval is equal to the integral over the entire interval, but the derivation of (28), (29) and (30) is rather involved and, hence, has been reported separately in the Appendix.

It is important to observe some characteristics of these equations. Equations (27) relate the integrals for a particular value of m, while such relation is true for any integral besides $\bar{I}_{nm}$. In contrast, equations (28), (29) and (30) hold only for these special integrals, $\bar{I}_{nm}$, involving at a time the entire subset corresponding to $m = 0$ to n. As such, from a strict standpoint, these checks are not conclusive on individual $\bar{I}_{nm}$ values, although it is highly improbable that an $\bar{I}_{nm}$ value containing significant error can satisfy all these tests simultaneously,

The following pair of supplementary formulae which can be readily derived from the addition theorem of spherical harmonic functions, has been applied to test the accuracy of computed $\bar{P}_{nm}$ values :

$$\left.\begin{aligned} T_5 &= 2n+1, \\ \delta T_5 &= T_5 - \sum_{m=0}^{n} \bar{P}^2_{nm}(t) = 0 \end{aligned}\right] \quad (31)$$

Computations

A Fortran sub-program has been developed for the present method and computations of $\overline{I}_{nm}$ values have been carried out with a Control Data CYBER 74 computer over intervals $(0°, 5°)$, $(0°.03, 5°)$, $(45°, 50°)$ and $(85°, 90°)$ of θ for $N = 100$. For economy of space, results for only a few values of m corresponding to $n = 60$ and 100 have been given in *Table 1.* The interval $(0°.03, 5°)$ has been selected following Christodoulidis and Katsambalos (1977) who considered this instead of the interval $(0°, 5°)$ where they confronted computational instability. Our method does not have such problem and our results in *Table 1* clearly indicate significant differences $\overline{I}_{nm}$ $(\cos 0°, \cos 5°)$ and $\overline{I}_{nm}$ $(\cos 0°.03, \cos 5°)$ values when m is small.

The sub-program also provides options to apply the check formulae (27), (28), (29), (30) and (31). Our $\overline{I}_{nm}$ and $\overline{P}_{nm}$ values computed by the present method were thoroughly subjected to these tests and the results are very satisfactory. Relative errors $(\delta T_i/T_i,\ i = 1, 2, \ldots 5)$ never exceed 10^{-23} in magnitude. T_i and δT_i values $(i = 1, 2, .. 5)$ for some of the cases have been tabulated in *Table 2.*

It is interesting to compare our results with those of Christodoulidis and Katsambalos (1977). Of their two sets of $\overline{I}_{nm}$ values corresponding to their "analytical" and "numerical" methods respectively, the first one has been found to agree better (up to 16 significant digits) with our results over the interval of $(45°, 50°)$, while the other agrees up to 8 to 14 significant digits only. On the other hand, the nature of such agreement is just the reverse over the interval of $(0°.03, 5°)$. Then our results agree better (up to 12 to 13 significant digits) with their "numerical" results, while the agreement with their other set is sometimes not even a single significant digit. Over both the intervals, the agreement between both of their sets is worse than that of one of them with ours. In the light of established accuracy of our computations through different checks applied to them, we may infer that the "numerical" method of Christodoulidis and Katsambalos (1977) is applicable only over polar regions while their other formula works well over non-polar regions. *Table 3* compares these results for some specific values of m and $n = 60$.

Discussion

Our recurrence relations for both $\overline{I}_{nm}(t_1, t_2)$ and $\overline{P}_{nm}(t)$ are constructed in such a manner that neither t nor $(1 - t^2)^{\frac{1}{2}}$ appears in the denominators of any of the terms which is necessary for the stability of computation throughout the interval $(0, 1)$ of t.

With respect to m and n, we use the same scheme of computation for $\overline{I}_{nm}(t_1, t_2)$ and $\overline{P}_{nm}(t)$. For both of them, we keep n fixed and vary m from 0 to n and then repeat n from 0 to N. This reduces the storage requirements in the computer for $\overline{P}_{nm}(t_1)$ and $\overline{P}_{nm}(t_2)$ to six vectors corresponding to the present and the earlier two values of n for each of them — those required in the recurrence relations for $\overline{P}_{nm}(t_1)$, $\overline{P}_{nm}(t_2)$ and $\overline{I}_{nm}(t_1, t_2)$.

A numerical quadrature procedure like that of Christodoulidis and Katsambalos (1977) requires many more values of $\overline{P}_{nm}(t)$ over an interval (t_1, t_2) than what is needed in our computations with the recurrence relations viz. $\overline{P}_{nm}(t_1)$ and $\overline{P}_{nm}(t_2)$. From the standpoint of computation time and storage requirements, this is also another advantage of the method proposed here.

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Table 1

$\bar{I}_{60,m}(\cos\theta_1, \cos\theta_2)$ and $\bar{I}_{100,m}(\cos\theta_1, \cos\theta_2)$ for different values of (θ_1, θ_2) and m as computed by the present method

n	m	(θ_1, θ_2)			
		$(0^\circ, 5^\circ)$	$(0^\circ.03, 5^\circ)$	$(45^\circ, 50^\circ)$	$(85^\circ, 90^\circ)$
60	0	.548137850049 E- 02	.548288616756 E- 02	−.101502934228 E−01	.157068165998 E−01
	1	−.474760732301 E- 02	−.474758480863 E- 02	.163370227936 E−01	.122629596218 E−01
	2	−.169633306239 E- 01	−.169633304902 E- 01	.149333427060 E−01	−.222719409762 E−01
	25	−.364719758524 E- 17	−.364719758524 E- 17	.412161664812 E−01	.275021039367 E−01
	30	−.134466017166 E- 22	−.134466017166 E- 22	.633025657142 E−01	−.323176953329 E−01
	35	−.167975273604 E- 28	−.167975273604 E- 28	−.237920622310 E−01	−.500052606799 E−01
	58	−.152684163609 E- 62	−.152684163609 E- 62	−.428357998853 E−06	.173039024915 E+00
	59	−.171769999157 E- 64	−.171769999157 E- 64	−.650253457088 E−07	−.156284019900 E+00
	60	−.134956775310 E- 66	−.134956775310 E- 66	−.682281709959 E−08	−.339511858572 E+00
100	0	−.327290603108 E- 02	−.327096328992 E- 02	.120791752063 E−01	−.682505938196 E−02
	1	−.305799103935 E- 02	−.305794284145 E- 02	.194196687389 E−01	.284470608830 E−01
	2	.535137868013 E- 03	.535138343533 E- 03	−.167217779233 E−01	.967685387302 E−02
	45	−.290616724692 E- 30	−.290616724692 E- 30	−.128076596202 E−01	.185080586288 E−01
	50	−.927131041231 E- 36	−.927131041231 E- 36	−.186983132913 E−02	.190453242146 E−01
	55	−.153166302422 E- 41	−.153166302423 E- 41	−.408706067914 E−02	−.104771636444 E−01
	98	−.712449302422 E−105	−.712449302422 E−105	−.115091070674 E−10	.142228559339 E+00
	99	−.621791227936 E−107	−.621791227936 E−107	−.135914144133 E−11	−.212985064231 E+00
	100	−.380864609779 E−109	−.380864609779 E−109	−.111926685751 E−12	−.367683384688 E+00

Table 2

Some results for the errors, $\delta T_1, \delta T_2, \ldots \delta T_5$ along with the corresponding $T_1, T_2, \ldots T_5$ for selected values of θ_1, θ_2, n and m. These reflect on the accuracies of $\overline{I}_{nm}$ and $\overline{P}_{nm}$ values achieved by the present method.

θ_1	θ_2	n	m	T_1	δT_1	n	T_2	δT_2	T_3	δT_3	T_4	δT_4	T_5	δT_5
0°	5°	60	0	−.5481 E−02	−.8 E−30	0	−.0038	.3 E−30	−.0038	0	−.0038	.4 E−30	1	0
			30	−.1344 E−22	.3 E−51	30	−.0791	−.2 E−26	−.1745	.1 E−27	−.2001	−.3 E−26	61	.4 E−23
			60	−.1350 E−66	0	60	.0603	.7 E−26	−.1786	−.8 E−26	−.2541	.4 E−26	121	.2 E−22
		100	75	−.1704 E−67	.5 E−96	75	.0224	.4 E−26	−.1679	−.2 E−25	−.2265	.2 E−25	151	.4 E−22
			100	−.381 E−109	0	100	−.0464	−.1 E−25	−.1744	−.2 E−25	−.1536	.2 E−25	201	.5 E−22
45°	50°	60	0	−.1015 E−01	.5 E−29	0	−.0643	.3 E−28	−.0643	.2 E−29	−.0643	.3 E−28	1	0
			30	.6330 E−01	.6 E−29	30	−2.111	.6 E−25	−2.176	.6 E−25	−3.382	.8 E−25	61	.3 E−24
			60	.6823 E−01	.4 E−30	60	−.9190	.5 E−25	−1.081	.6 E−25	−4.295	.5 E−24	121	.1 E−23
		100	75	−.1166 E+00	.3 E−29	75	−1.166	−.2 E−25	−1.294	−.5 E−25	−3.828	−.8 E−24	151	.2 E−23
			100	−.1119 E−12	.1 E−41	100	−1.822	.2 E−25	−1.909	.2 E−25	−2.596	.9 E−24	201	.3 E−23
85°	90°	60	0	.1571 E−01	.2 E−29	0	−.0872	−.2 E−27	−.0872	−.2 E−27	−.0872	−.2 E−27	1	0
			30	−.3232 E−01	.9 E−29	30	−2.902	.9 E−25	−2.906	.8 E−25	−4.582	.1 E−24	61	.4 E−24
			60	−.3395 E+00	.1 E−27	60	−1.350	.8 E−25	−1.360	.8 E−25	−5.820	.7 E−24	121	.1 E−23
		100	75	−.2798 E−02	−.3 E−28	75	−1.663	−.6 E−25	−1.671	−.6 E−25	−5.187	.1 E−23	151	.2 E−23
			100	−.3677 E+00	.1 E−27	100	−2.526	.3 E−25	−2.531	.3 E−25	−3.518	.1 E−23	201	.3 E−23

Table 3

Comparison of our $\overline{I}_{60,m}\,(\cos\theta_1\,,\cos\theta_2)$ values with those of Christodoulidis and Katsambalos for different values of $(\theta_1\,,\theta_2)$ and m

θ_1	θ_2	m	our results	their "analytical" results	their "numerical" results
0°.03	5°	0	.5482886167562783 E–02	.5482886167562783 E–02	.5482886167563951 E–02
		1	–.4747584808625611 E–02	–.1429691150909562 E–02	–.4747584808623913 E–02
		2	–.1696333049016903 E–01	–.1042650523110295 E–01	–.1696333049016750 E–01
		15	–.6527295117140670 E–08	–.6399896886412206 E–08	–.6527295117141022 E–08
		30	–.1344660171661714 E–22	–.1338502451142383 E–22	–.1344660171661876 E–22
		45	–.1053006541694950 E–41	–.1051721773727472 E–41	–.1053006541695136 E–41
		58	–.1526841636090619 E–62	–.1526749001416031 E–62	–.1526841636090957 E–62
		59	–.1717699991570431 E–64	–.1717699991570431 E–64	–.1717699991570818 E–64
		60	–.1349567753100739 E–66	–.3094914604131854 E–34	–.1349567753100917 E–66
45°	50°	0	–.1015029342279937 E–01	–.1015029342279937 E–01	–.1015029342279950 E–01
		1	.1633702279363149 E–01	.1633702279363149 E–01	.1633702279363132 E–01
		2	.1493334270600661 E–01	.1493334270600661 E–01	.1493334270600641 E–01
		15	.2308394482087358 E–01	.2308394482087358 E–01	.2308394419855036 E–01
		30	.6330256571416836 E–01	.6330256571416836 E–01	.6330256536972070 E–01
		45	–.1191690925767232 E–00	–.1191690925767232 E–00	–.1191690925767225 E–00
		58	–.4283579988526653 E–06	–.4283579988526653 E–06	–.4283579988526660 E–06
		59	–.6502534570876073 E–07	–.6502534570876073 E–07	–.6502534570876096 E–07
		60	–.6822817099586357 E–08	–.6822817099586357 E–08	–.6822817099586383 E–08

APPENDIX

From the addition theorem of spherical harmonic functions for two points in the same meridian, we have

$$\sum_{m=0}^{n} \overline{P}_{nm}(\cos\theta_c)\,\overline{P}_{nm}(\cos\theta) = (2n+1)\,P_n[\cos(\theta-\theta_c)] \quad \text{(A1)}$$

Multiplying both sides of (A1) by $(-\sin\theta)$ and integrating with respect to θ from θ_1 to θ_2, we get

$$\sum_{m=0}^{n} \overline{P}_{nm}(t_c)\,\overline{I}_{nm}(t_1,t_2) = (2n+1)[t_c I_{no}(t_3,t_4) - y_c \int_{t=t_3}^{t_4} P_n(t)\,dy] \quad \text{(A2)}$$

where $t=\cos\theta$, $y=\sin\theta$, $t_1=\cos\theta_1$, $y_1=\sin\theta_1$, $t_2=\cos\theta_2$, $y_2=\sin\theta_2$,

$t_c=\cos\theta_c$, $y_c=\sin\theta_c$, $t_3=\cos(\theta_1-\theta_c)$ and $t_4=\cos(\theta_2-\theta_c)$.

Now,

$$\int_{t=t_3}^{t_4} P_n(t)\,dy = y\,P_n(t)\Big|_{t_3}^{t_4} - \int_{t_3}^{t_4} y\,\frac{dP_n(t)}{dt}\,dt$$

$$= y\,P_n(t)\Big|_{t_3}^{t_4} - I_{n1}(t_3,t_4) \quad \text{(A3)}$$

When (A3) is substituted in (A2) and the right hand side is reduced to its fully normalised form, we obtain

$$\sum_{m=0}^{n} \overline{P}_{nm}(t_c)\,\overline{I}_{nm}(t_1,t_2)$$

$$= \sqrt{2n+1}\,[y_c\{\sqrt{n(n+1)/2}\,\overline{I}_{n1}(t_3,t_4) - y\,\overline{P}_{no}(t)\Big|_{t_3}^{t_4}\} + t_c\,\overline{I}_{no}(t_3,t_4)] \quad \text{(A4)}$$

Equation (A4) reduces after necessary simplifications to (28), (29) and (30) when θ_c is substituted by θ_1, θ_2 and $(\theta_1+\theta_2)/2$ respectively. Similar substitution of θ for θ_c in (A1) gives equation (31).

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