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HEIGHT DATUM DEFINITION, HEIGHT DATUM CONNECTION AND THE ROLE OF THE GEODETIC BOUNDARY VALUE PROBLEM

Abstract

Vertical datum definition is identical with the choice of a potential (or height) value for the fundamental bench mark. Also the connection of two adjacent vertical datums poses no principal problem as long as the potential (or height) value of two bench marks of the two systems is known and they can be connected by levelling. Only the unification of large vertical datums and the connection of vertical datums separated by an ocean remains difficult.

Two vertical datums can be connected indirectly by means of a combination of precise geocentric positions of two points, as derived by space techniques, their potential (or height) value in the respective height datum and their geoid height difference. The latter requires the solution of the linear geodetic boundary value problem under the assumption that potential and gravity anomalies refer to a variety of height datums. The unknown off-sets between the various datums appear in the solution inside and outside the Stokes integral and can be estimated in a least squares adjustment, if geocentric positions, levelled heights and adequate gravity material are available for all datum zones. The problem can in principle also be solved involving only two datums, in case a precise global gravity field becomes available purely from satellite methods.

1. Introduction

The discussion about vertical datum definition, the connection of vertical datums, and their relations to mean sea level (MSL) and its variations started in the seventies. Examples of the early discussions are (Balazs, 1973), (Sturges, 1974), or (Fischer, 1978). Serious steps toward clearer definitions of all involved quantities and solution strategies of the geodetic part of the problem were taken in (Mather, 1973), (Lelgemann, 1977) and (Colombo, 1980). In recent years in several documents the various aspects of the establishment of a world vertical datum, the role played by space techniques in this context and the connection to global monitoring of MSL were discussed, see e.g. (Rapp, 1983), (Rapp et al., 1984) or (Colombo, 1985). The need of a worldwide reference system has been underlined by the oceanographic community in (IAPSO, 1985). The purpose of the present contribution is to continue the analysis of the geodetic part of the problem. First the definition of vertical datum and its relation to MSL shall be discussed. We shall then turn to the datum definition for

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levelling networks of continental size. In the second part the worldwide vertical datum connection is treated, with emphasis on the role of the geodetic boundary value problem. It is tried to properly identify potential difference (or height) values and gravity anomalies, as they are applied in practice, and to use these quantities without modification in the solution.

2. Vertical Datum Definition

In principle the definition of a vertical datum is straightforward. Each country has its levelling network. Measured are height increments. In combination with measured gravity potential differences, orthometric, dynamic or normal heights can be computed, compare e.g. (Heiskanen & Moritz, 1967; ch. 4). Orthometric, dynamic or normal heights are determined by adding small gravity dependent corrections to the levelled height increments. In flat or mildly undulated terrain these corrections are often negligible and are either neglected or the measured gravity required for their computation is replaced by normal gravity.

Since it would be inconvenient in daily practice to deal with the individual height differences between points, a certain fixed potential (or height) value, $W_P^{(a)}$, is assigned to one fundamental bench mark at P. This definition is called the ***choice of a vertical datum or height system*** and is indicated by the (a) of $W_P^{(a)}$. After such a choice is taken the potential (or height) values of all other points are referred to the fundamental point. In terms of potential values, the potential at point P_i becomes

$$W_i^{(a)} = C_{PP_i} + W_P^{(a)} \quad , \tag{1}$$

where C_{PP_i} is the observed or adjusted potential difference between points P and P_i. Naturally any point of the network could be chosen as datum point and any potential value could be assigned to it. The potentials expressed in one vertical datum can be transformed to another datum by an S-transformation. Suppose the potential values of a levelling network are expressed in datum (a) with reference bench mark Q. In datum (b) with reference point P we find

$$W_i^{(b)} = W_i^{(a)} - W_Q^{(a)} + W_Q^{(b)} = W_i^{(a)} - W_Q^{(a)} + C_{PQ} + W_P^{(b)} \quad . \tag{2}$$

Hence the transformation or ***connection of vertical datums*** is feasible, whenever the potential difference C_{PQ} between the fundamental bench marks and the assigned values $W_Q^{(a)}$ and $W_P^{(b)}$ are known. This implies, for example, that in case the reference potential (or heights) values would be defined to be zero with the fundamental bench marks all situated on one equipotential surface, e.g. the geoid, all potential (or height) values would be in one common datum This comes close to the actual situation, where national or continental vertical datums are referred to reference points close to mean sea level. Since the local mean sea level in various parts of the world deviates from one common equipotential surface typically by less than 1 m, we conclude that at this precision level a unified world vertical datum can be realized without any problem, even in case the reference points cannot be connected directly be levelling. The situation is displayed in ***Figure 1.***

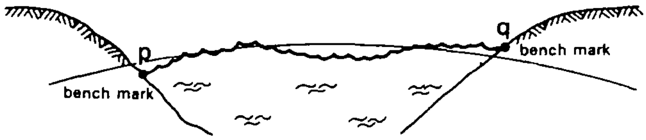

Fig. 1 — Deviation of fundamental bench marks from geoid.

In view of ourdays precision requirements, e.g. for the worldwide monitoring of changes in relative mean sea level (IAPSO, 1985), which are at least one order of magnitude higher, the whole situation has to be revisited, for now the deviations of mean sea level (MSL) from the geoid become relevant or are the quantity to be determined. What is the relationship between vertical datum definition and mean sea level ? Historically seen, there exists a close connection. Take for example the reference point of the United European Levelling Network (UELN), the Normaal Amsterdams Peil (NAP). It can be traced back till 1682 and was originally defined by eight bench marks. The marks were placed at eight locks in the city of Amsterdam. They were referenced to regular water level measurements at the locks for the purpose of storm surge forcasting. Only later they were used as vertical datum for levelling. The original water level records since January 1, 1700 are still existing. NAP still refers to these bench marks, although meanwhile a new fundamental bench mark in Amsterdam has replaced the original marks, which got lost or inaccessible in the course of this century. For a description of the history of the NAP system we refer to (Waalewijn, 1986). Naturally the NAP system does not coincide with MSL anymore. Would it be meaningful to adjust the vertical datum to MSL at regular intervals ? The answer is negative. As an example, we show in ***Figure 2*** a record of MSL over the past century of eight different tide gauges along the Dutch coast, all expressed in the NAP system. The coast length is only about 300 km. Levelling errors over this distance are much smaller than the displayed variation in MSL. Two observations can be made. First, there exist significant changes in the yearly MSL of the order of 10mm with a general trend of 150 to 200 mm/century. Thus, adaption of the vertical datum to MSL every year or decade would be an undesirable major undertaking. Second, there is also a considerable variation of MSL among the eight gauges, which means that the deviations of MSL from one common equipotential surface can be rather large even over small distances and vary in addition from year to year. This effect is shown in ***Figure 3*** where the changes in MSL of the eight gauges over the period of 1975–1979 are expressed as a function of the distance between the stations.

In short, the adaption of MSL of one tide gauge of one particular year as vertical reference remains as arbitrary as any other choice and offers no particular advantage. It only meets our intuitive experience, when the zero value of a vertical datum comes close to MSL. Periodical adaption of a vertical datum to MSL is undesirable. The definition of a vertical datum through the simultaneous choice of the potential (or height) reference at more than one bench mark — e.g. at the eight gauges of ***Figures 1*** and ***2*** — with an unknown small deviation from one common equipotential surface is equivalent to a forced deformation of the basic reference surface of the levelling network, see also (Laskowski, 1983). In adjustment terms it corresponds with an overconstraint.

The same conclusions remain valid when we consider the establishment of

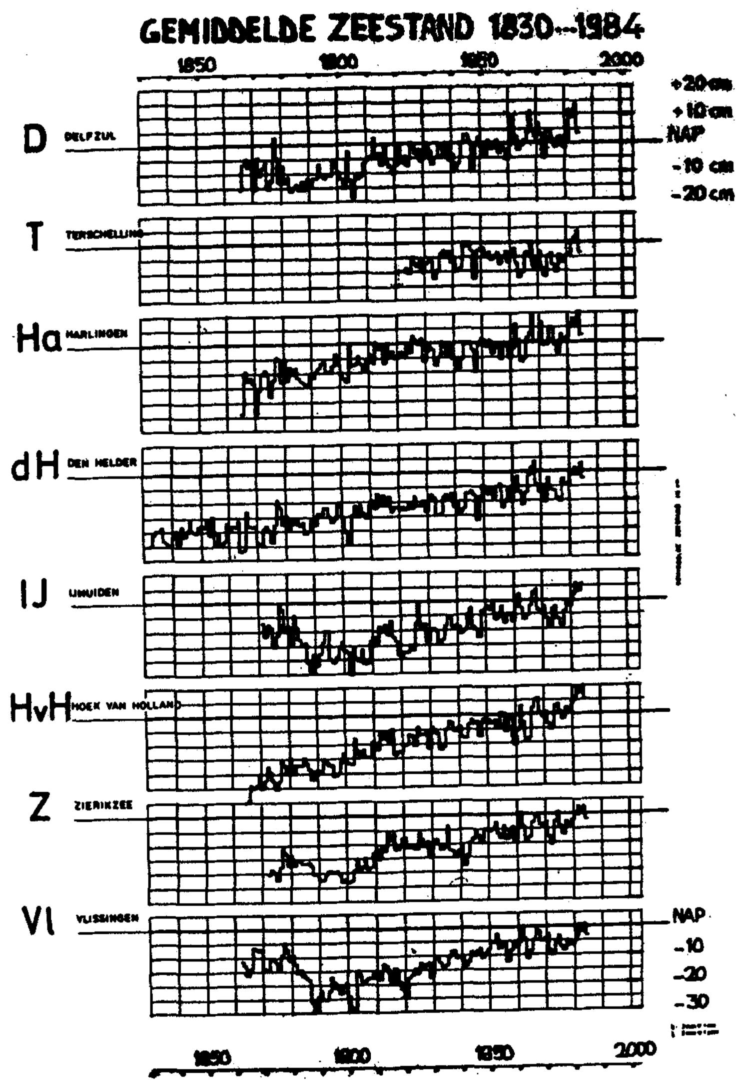

Fig. 2 — Yearly mean sea level over the past century at eight tide gauges along the coast of The Netherlands.

unified height systems on a continental scale, such as the UELN or the North American Vertical Datum. The relative precision of these networks is very high. For the UELN, consisting only of first order levelling points, the formal precision reported in (Kok, 1985) between Amsterdam and Sicily is about 35 mm, between Amsterdam and Finland 45 mm. The problem is, however, that the (internal and external) reliability is rather poor, compare the discussion in (Alberda, 1963) or (Kok, ibid.). In essence gross errors of varying sign could be present inside the large loops of such a network without being detectable. The only remedy for an improvement of the reliability of the free network would be its densification. The importance of densification must be underlined, if wrong conclusions in the second stage of the adjustment are to be

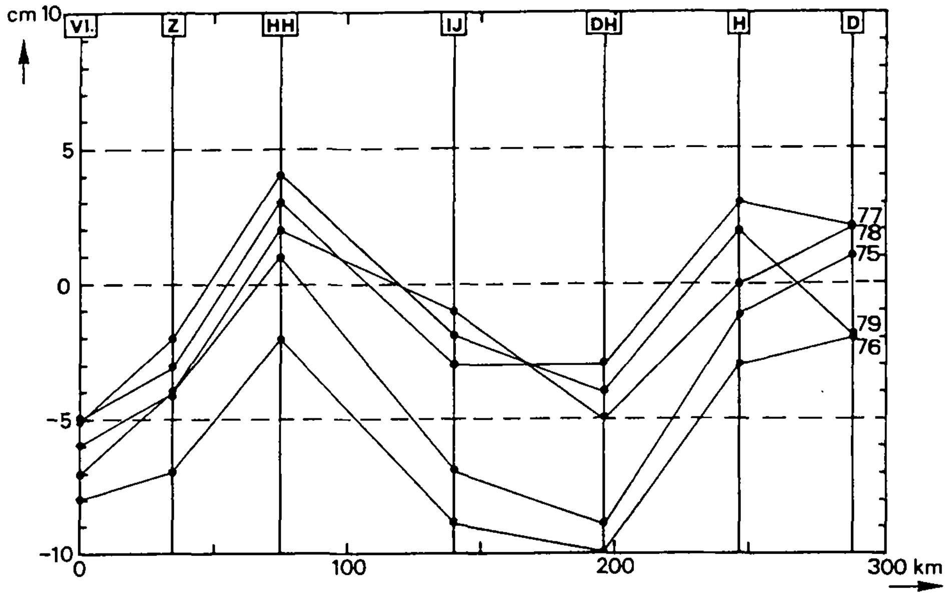

Fig. 3 – Yearly change in mean sea level film 1975–1979 along the coast of The Netherlands as derived from seven tide gauges.

avoided, where levelling and sea level results are compared. Besides of gross errors various sources of systematic errors could seriously affect the quality of continental networks. Take for example the use of normal orthometric corrections instead of the correct ones based upon measured gravity. Zeger (1985) reports expected changes of between − 10 mm and + 416 mm for 32 representative points of the Austrian first order levelling net. The numbers consist in this case of a constant part coming from the datum change from the old reference point in Trieste/Italy to NAP , and of a variable part caused by the application of real orthometric corrections. As a consequence of the poor reliability some external control is required, should the networks be useful for the purposes they were set-up for. One possible type of external control could come from MSL data from a number of selected sites, corrected for the best currently available models of sea surface topography. Depending on the quality of the potential (or height) differences between these sites, as derived from MSL records and oceanographic models, the continental levelling network could either be tied to the corrected MSL of these sites, or the oceanographically determined potential (or heights) differences could be included as a new set of observables in a second adjustment step. Since – as explained above – the first option could result in systematic distortions of the reference equipotential surface throughout the entire network, this procedure is only recommended in cases where the oceanographic information is definitely superior in terms of precision and reliability to the potential (or height) values that resulted from the adjustment (step one) of the levelling network. A second type of external control would be the incorporation of satellite derived geometric positions of surface points in combination with geoid information.

This second approach, in the case of large continental networks considered solely for the purpose of external control, is probably the only means to connect vertical datums of different continents on a worldwide scale at the quoted precision level of below 10 cm . It shall be discussed in the sequel. A third option would be the use of precise sea surface heights near coast lines as derived from satellite altimetry. However, since we expect major complications when applying this method at a sub-cm level this approach is not discussed here.

3. Vertical Datum Connection

The problem to be solved is best understood by considering *Figure 1* and formula (2). Two height systems, (a) and (b) located at two different continents are to be connected. All potential differences (or heights) of system (a) refer to bench mark Q , those of system (b) to bench mark P . Let us assume the same reference potential (or height) values, $W_Q^{(a)}$ and $W_P^{(b)}$, are chosen in (a) and (b) , respectively :

$$W_Q^{(a)} \overset{\text{def.}}{=} W_P^{(b)} \quad , \tag{3}$$

e.g. a height reference value of zero at the two origins. Then, as can be seen from (2), the only quantity to be determined is C_{PQ} , the potential difference between bench marks P and Q . Since P and Q are separated by ocean, C_{PQ} cannot be determined directly by levelling and gravity measurements. Alternatively C_{PQ} could be determined by oceanographic methods, but not with the required precision of better than $0.1\,\text{kgal}\cdot\text{m}$, equivalent to 10 cm in terms of heights, which corresponds to about 10^{-8} relative to the earth's radius. In addition, one of the main objectives of worldwide vertical datum connection is aiming exactly into the opposite direction, namely the establishment of control for oceanographic modelling. A second alternative is the indirect determination of C_{PQ} , incorporating geometric space techniques and geoid computation. The principle is schematically shown in *Figure 4.* This alternative shall be discussed in the sequel.

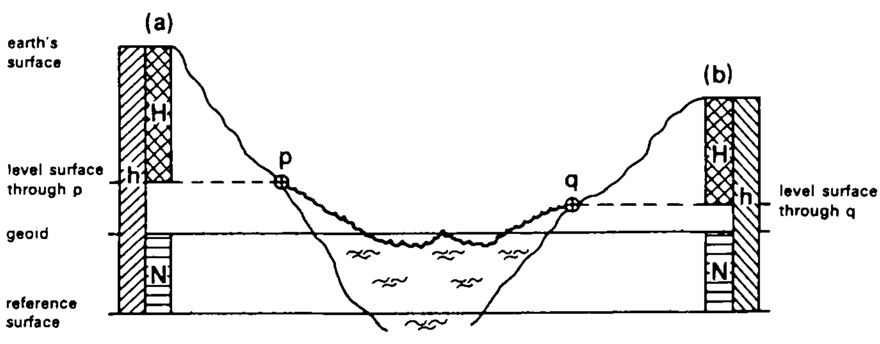

Fig. 4 — Derivation of C_{PQ} *from a combination of geometric height* h *, orthometric height* H*, and geoid height* N *in vertical datums* (a) *and* (b).

We assume that worldwide station coordinates in a geocentric coordinate system of a number of fundamental stations can be determined with the required precision. The computation therefrom of geometric heights h above a chosen reference surface is straightforward. Furtheron it is assumed that precise orthometric or normal heights, or potential differences are available for all points in system (a) and (b), respectively. We shall therefore concentrate on the geoid part or more explicitely on the solution of the geodetic boundary value problem and its presentation in the context of the determination of C_{PQ}.

In principle the same problem has been treated in (Mather, 1973) and (Colombo, 1980). However, Mather's article from 1973, to be seen in the context of the requirements and possibilities at that time, was less specific and followed a somewhat different line. Colombo's approach, on the other hand, starts from some special set-up with a number of fundamental stations, located in-land and with all data referring to them. It is the intention of our approach to show that the problem can also be solved on the basis of the traditional definitions of vertical datums, gravity anomalies and all related quantities.

Observables and Linear Model

The first observable considered here are ***potential differences.*** They are derived from levelled height increments, Δn, and gravity values, g, along the levelling line. Instead of P, or Q, the vertical datum reference point is for the moment denoted 0 with P being an arbitrary second point. It is, compare (Heiskanen & Moritz, 1967)

$$C_{P0} = C(P,0) = W(0) - W(P)$$

$$= \int_0^P g\, dH \approx \sum_i g_i \Delta n_i \quad . \tag{4}$$

The gravity potential is defined as the sum of gravitational potential V and centrifugal potential Z

$$W(P) = V(P) + Z(P) \quad . \tag{5}$$

The gravitational potential V, being harmonic in the space outside attracting masses, can be expressed as

$$V(P) = \frac{GM}{R} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \left(\frac{R}{r_P}\right)^{n+1} C_{nm\alpha} Y_{nm\alpha}(\varphi_P, \lambda_P) \tag{6}$$

with GM, gravitational constant times mean mass of the earth, and the following convention for the fully normalized, dimensionless potential coefficients

$$C_{nm\alpha} = \begin{cases} \bar{C}_{nm} & \text{for} \quad \alpha = 0 \\ \bar{S}_{nm} & \text{for} \quad \alpha = 1 \end{cases} \tag{7}$$

and for the associated Legendre functions

$$Y_{nm\alpha}(\varphi_P, \lambda_P) = \bar{P}_{nm}(\sin\varphi_P) \begin{cases} \cos m\lambda_P & \text{for} \quad \alpha = 0 \\ \sin m\lambda_P & \text{for} \quad \alpha = 1 \end{cases} \tag{8}$$

where φ_P, λ_P and r_P are the geocentric coordinates of P.

The centrifugal potential $Z(P) = \frac{1}{2}\omega^2 r_P^2 \cos^2\varphi_P$, where the earth's spin rate ω is known very accurately. Inserting (5) and (6) into (4) gives

$$C_{P0} = \sum_i g_i \Delta n_i$$

$$= W(0) - Z(P) - \frac{GM}{R} \sum_{nm\alpha} \left(\frac{R}{r_P}\right)^{n+1} C_{nm\alpha} Y_{nm\alpha}(P) \ . \tag{9}$$

From eq. (9) it is seen that a priori $W(0)$, the coefficients $C_{nm\alpha}$ and the coordinates of P are unknown and that the equation is non-linear in the coordinates of P.

The second observable is ***scalar gravity,*** g. In (4) it is only an auxiliary quantity with no stringent precision requirements associated with it. It is, with (5)

$$g(P) = + \left|\frac{\partial W}{\partial n}\right|_P = - \left|\frac{\partial Z}{\partial n} + \frac{\partial V}{\partial n}\right|_P \ , \tag{10}$$

with n the vertical direction in P. Inserting for V expression (6) the non-linear character of the problem becomes apparent again.

Linearization of eqs. (9) and (10) requires the availability of an approximate – or normal – gravity field and of approximate coordinates of all points. We define the normal gravity field as

$$U(P) = Z(P) + \frac{GM'}{R} \sum_{nm\alpha} \left(\frac{R}{r_P}\right)^{n+1} C'_{nm\alpha} Y_{nm\alpha}(\varphi_P, \lambda_P) \tag{11}$$

where GM' is the adopted or approximate value of GM and the coefficients $C'_{nm\alpha}$ are those either of an internationally adopted reference field (e.g. the GRS67 or GRS80) or of models, such as GEM10, GRIM3 or the OSU81 field. Various aspects of the linearization process are treated in (Rummel, 1984). The purpose of the derivations here is to keep things simple without loss of generality. Therefore (1) only the vertical components of the linear model shall be considered, (2) the Stokes approach shall be taken, although it can be easily shown that a completely analogous result is obtained from the Molodenskii approach, and (3) spherical as well as constant radius approximation shall be applied, although in reality – in view of the required precision – atmospheric, ellipsoidal and topographic effects have to be taken into account, see (Moritz, 1974).

Points on the same plumb line as P located on the chosen normal surface and on the level (equipotential) surface passing through 0 will be denoted by P'_0 and P_0, respectively, see ***Figure 5.***

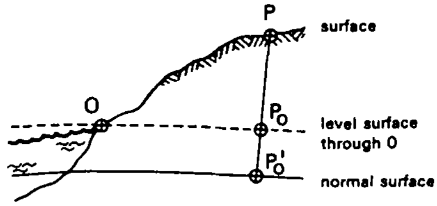

Fig. 5 – Definition of P , P_0 *and* P_0' .

Free air reduction of g(P) to sea level yields :

$$g(P_0) = g(P) - \frac{\partial \gamma}{\partial n} H$$

or when employing the definition of orthometric heights $H = \frac{C_{P0}}{\bar{g}}$ with $\bar{g}$ the mean gravity along the plumb line (see Heiskanen & Moritz, 1967 ; ch. 4)

$$g(P_0) = g(P) - \frac{\partial \gamma}{\partial n} \frac{1}{\bar{g}} C_{P0} \quad . \tag{12}$$

Linearization of $g(P_0)$ with respect to U and the approximate point P_0' gives

$$g(P_0) = \gamma(P_0') + \frac{\partial \gamma}{\partial n} N + \sum_{n,m,\alpha} \frac{\partial \gamma}{\partial C_{nm\alpha}} \Delta C_{nm\alpha} \tag{13}$$

In terms of a gravity anomaly (observable minus approximate value) eqs. (12) and (13) become

$$\begin{aligned} \Delta g &= g(P_0) - \gamma(P_0') \\ &= g(P) - \frac{\partial \gamma}{\partial n} \frac{1}{\bar{g}} C_{P0} - \gamma(P_0') \\ &= \frac{\partial \gamma}{\partial n} N + \sum_{n,m,\alpha} \frac{\partial \gamma}{\partial C_{nm\alpha}} \Delta C_{nm\alpha} \end{aligned} \tag{14}$$

We observe that both observables, g(P) and C_{P0} , enter into the gravity anomaly, an important fact when defining the a priori variances and covariances of the latter.

The usually unknown potential value of the adopted zero level surface, $W_{P_0} = W_0$, can be expressed in linearized form as

$$W_{P_0} = U_{P_0'} + \frac{\partial U}{\partial n} N + \sum_{n,m,\alpha} \frac{\partial U}{\partial C_{nm\alpha}} \Delta C_{nm\alpha}$$

or as anomaly with $\frac{\partial U}{\partial n} = -\gamma$

$$\Delta W_0 = -\gamma N + \sum_{n,m,a} \frac{\partial U}{\partial C_{nma}} \Delta C_{nma} \tag{15}$$

The sum on the right side gives the anomalous potential T :

$$T(P_0') = \sum_{n,m,a} \frac{\partial U}{\partial C_{nma}} \Delta C_{nma}$$

$$= \frac{\delta(GM)}{r} + \frac{GM'}{R} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \sum_{a=0} \left(\frac{R}{r}\right)^{n+1} \Delta C_{nma} Y_{nma}(\varphi_{P_0'}, \lambda_{P_0'}) , \tag{16}$$

where $\delta(GM) = GM - GM'$. In (16 $\frac{\delta(GM)}{r}$ is the zero-degree term. In general the summation over n would have to start with $n = 1$, where from the first-degree coefficients the shift of the geo-center from the chosen coordinate origin can be computed, compare (Heiskanen & Moritz, 1967; ch. 2.6). The definition $\Delta C_{10} = \Delta C_{11} = \Delta S_{11} = 0$ corresponds to a choice of the reference coordinate system in the geo-center.

With (16) the linear equations (14) and (15) attain the well-known form

$$\Delta W_0 = -\gamma N + T \tag{17}$$

and

$$\Delta g = g(P) - \frac{\partial \gamma}{\partial n} \frac{1}{g} C_{P0} - \gamma(P_0')$$

$$= \frac{\partial \gamma}{\partial n} N - \frac{\partial T}{\partial n} . \tag{18}$$

Elimination of N from (17) yields Bruns' formula

$$N = \frac{T - \Delta W_0}{\gamma} \tag{19}$$

insertion into (18) the fundamental equation of physical geodesy

$$\Delta g = g(P) - \frac{\partial \gamma}{\partial n} \frac{1}{\gamma} C_{P0} - \gamma(P_0')$$

$$= - \frac{\partial \gamma}{\partial n} \frac{1}{\gamma} \Delta W_0 + \frac{\partial \gamma}{\partial n} \frac{1}{\gamma} T - \frac{\partial T}{\partial n} \tag{20}$$

It can be shown that the operator $\left(\frac{\partial \gamma}{\partial n} \frac{1}{\gamma} - \frac{\partial}{\partial n}\right)$ in (20) applied to T , is invariant with respect to the first degree harmonics, compare (Krarup, 1971). In terms of linear algebra this means that the first degree harmonics span the null space of this operator. Our choice $\Delta C_{10} = \Delta C_{11} = \Delta S_{11} = 0$ transforms the singular problem of determining T into a regular one.

The above linearization is not the most general one, in the sense that the approximate coordinates were not chosen completely arbitrarily, as e.g. in (Rummel & Teunissen, 1982). Instead, in accordance with general practice, the observed potential differences (or heights) were not treated as a second independent observable but were employed for the reduction of gravity from P to P_0, with $U = const.$ being the approximate surface common to all points.

Solution of the Geodetic Boundary Value Problem

We now turn to the solution of the geodetic boundary value problem i.e. the determination of ΔW_0, N and T or equivalently $\Delta C_{nm\alpha}$ from eqs. (19) and (20). In a first step ΔW_0 is assumed to be constant all over the earth, or in other words, there exists one unified vertical datum.

In spherical approximation it is $\frac{\partial \gamma}{\partial n} \approx -\frac{2\gamma}{r}$ and $\frac{\partial T}{\partial n} \approx \frac{\partial T}{\partial r}$. Hence (20) becomes

$$\Delta g = \frac{2}{r}\Delta W_0 - \left(\frac{2}{r} + \frac{\partial}{\partial r}\right)T \tag{21}$$

and we find the well-known solution for

$$T(P) = -\frac{\delta(GM)}{R} + \frac{R}{4\pi}\int_\sigma St(\psi_{PQ})\,\Delta g(Q)\,d\sigma_Q \tag{22}$$

and for

$$N(P) = N_0 + \frac{R}{4\pi\gamma}\int_\sigma St(\psi_{PQ})\,\Delta g(Q)\,d\sigma_Q \quad . \tag{23}$$

where $St(\psi)$ is the Stokes integral function. The constant N_0 is called zero-order term. It is defined as

$$\begin{aligned} N_0 &= -\frac{\Delta W_0}{\gamma} + \frac{\delta(GM)}{R\gamma} \\ &= -\frac{R}{\gamma}\langle\Delta g\rangle + \frac{\Delta W_0}{\gamma} \\ &= -\frac{R}{2\gamma}\langle\Delta g\rangle + \frac{\delta(GM)}{2\gamma R} \quad . \end{aligned} \tag{24}$$

with $\langle\Delta g\rangle = \frac{1}{4\pi}\int_\sigma \Delta g\,d\sigma$, the global gravity anomaly average. The zero-order term is discussed in chapter 2–19 of (Heiskanen & Moritz, 1967), where it is also pointed out, that the measurement of one additional distance allows the determination of N_0, in principle. Hence together with a precise knowledge of GM, ΔW_0 is estimable.

We turn now to the actual situation in vertical datum connection. Assume there exist worldwide $(I + 1)$ vertical datums. Assume further that assumption of eq. (3) still holds, i.e. the initial potential (or height) value of each datum zone is the same, for example zero. The potential differences between the fundamental bench mark, 0, of one arbitrarily chosen vertical datum (a), e.g. NAP, and the bench marks, Q_i, of all other datums are denoted

$$C_{Q_i 0} = W(0) - W(Q_i) \qquad i = 1, \ldots, I \; . \tag{25}$$

While eq. (15) remains still valid for datum zone 0 the corresponding equation for the other I zones becomes

$$\Delta W_0 - C_{Q_i 0} = -\gamma N + \sum_{n,m,a} \frac{\partial U}{\partial C_{nma}} \Delta C_{nma} \; . \tag{26}$$

It is needless to say that the height anomaly N represents the vertical distance from the adopted reference surface to the equipotential surface passing through datum point Q_i. Besides the unknowns ΔW_0, N, and ΔC_{nma}, discussed above, the $C_{Q_i 0}$ enter new into the problem. Due to this change ***Bruns' equation*** becomes

$$N = \frac{T - \Delta W_0 + C_{Q_i 0}}{\gamma} \tag{27}$$

and accordingly the ***fundamental equation***

$$\begin{aligned} \Delta g^{(i)} &= g(P) - \frac{\partial \gamma}{\partial n} \frac{1}{g} C_{PQ_i} - \gamma (P'_0) \\ &= \frac{2}{r} \Delta W_0 - \frac{2}{r} C_{Q_i 0} - \left(\frac{2}{r} + \frac{\partial}{\partial r}\right) T \end{aligned} \tag{28}$$

where the upperindex (i) of Δg should point out the fact that the observed gravity is reduced to the level surface passing through Q_i. $C_{Q_i 0}$ in (28) represents a global step function with constant function value inside each datum zone. Inside the reference zone (a) the function value is zero by definition ($C_{Q_{i=0} 0} = 0$). The relationship between Δg and $\Delta g^{(i)}$ is

$$\Delta g = \Delta g^{(i)} + \frac{2}{r} C_{Q_i 0} \; , \tag{29}$$

where $\frac{2}{r} C_{Q_i 0} = \frac{2}{r} \bar{g} H_{Q_i 0} \approx 0.3806 \frac{\text{mgal}}{\text{m}} H_{Q_i 0}$ [m] can be interpreted as free air reduction from the level surface passing through Q_i to the one through 0, compare (Rummel & Teunissen, 1982). Only the $\Delta g^{(i)}$ are observable. With eqs. (27), (28), and (29) the general solution of the geodetic boundary value becomes

$$T(P) = \frac{\delta(GM)}{R} + \frac{R}{4\pi} \int_\sigma St(\psi_{PQ}) \left[\Delta g^{(j)} + \frac{2}{R} C_{Q_j 0}\right]_Q d\sigma_Q \; . \tag{30}$$

and

$$N^{(i)}(P) = \frac{\delta(GM)}{R\gamma} - \frac{\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma} + \frac{R}{4\pi\gamma}\int_\sigma St(\psi_{PQ})\left[\Delta g^{(j)} + \frac{2}{R}C_{Q_j0}\right]_Q d\sigma_Q , \quad (31)$$

where the upper index (i) of N again expresses the fact that P lies in the datum zone with datum point Q_i. The constant N_0, eq. (24), is to be replaced by (compare (31))

$$\begin{aligned} N_{Q_i} &= -\frac{\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma} + \frac{\delta(GM)}{R\gamma} \\ &= -\frac{R}{\gamma}<\Delta g> + \frac{\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma} \\ &= -\frac{R}{\gamma}<\Delta g^{(j)}> - \frac{2}{\gamma}<C_{Q_j0}> + \frac{\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma}. \end{aligned} \quad (32)$$

However GM' is known rather well. In a recent paper Smith et al. (1986) give a GM' value of 398 600.441 ± 0.005 km^3/s^2, derived from one decade of LAGEOS tracking. This implies a relative precision of $1.2 \cdot 10^{-8}$, approximately the relative precision we are aiming for. Therefore GM shall be regarded as known in the sequel ($\delta(GM) = 0$).

Denoting now $N_0 = -\frac{\Delta W_0}{\gamma}$ and $N_{Q_i0} = \frac{C_{Q_i0}}{\gamma}$, so that $N_{Q_i} = N_0 + N_{Q_i0}$, and furtheron indicating Stokes integral $\frac{1}{4\pi}\int St(\psi)[.]d\sigma$ by the operator notation $S(.)$ the bias in T and N caused by neglecting ΔW_0 and C_{Q_i0} can be expressed as

$$b(T) = 2S(C_{Q_i0}) \quad (33)$$

and

$$b(N) = N_0 + N_{Q_i0} + \frac{1}{\gamma}b(T) \quad (34)$$

An impression of the order of magnitude of b(N) can be obtained by the following experiment. Since the reference points of the vertical datums are in general close to MSL – compare the discussion of chapter 2 – the global step function $-\Delta W_0 + C_{Q_i0}$ can to some extent be approximated by a global model of sea surface topography. Engelis & Rapp (1983) computed spherical harmonic coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$ of the sea surface topography heights H_s (assuming zero elevation on land) based on oceanographic data compiled by Levitus

$$H_s(P) = R\sum_{n=0}^{6}\sum_{m=0}^{n}(\bar{a}_{nm}\cos m\lambda_P + \bar{b}_{nm}\sin m\lambda_P)\bar{P}_{nm}(\sin\varphi_P) . \quad (35)$$

We assume that at any arbitrary point $-\Delta W_0 + C_{Q_i0} \approx \gamma H_s$. Then with (35) the ***direct effect*** $N_0 + N_{Q_i0}$ in eq. (34) can be modelled by

$$N_0 + N_{Q_i0} = -\frac{1}{\gamma}(\Delta W_0 - C_{Q_i0}) \approx H_s \; . \tag{36}$$

and the *indirect effect* $\frac{1}{\gamma} b(T)$ by

$$\frac{1}{\gamma} b(T) = \frac{2}{\gamma} S(C_{Q_i0}) \approx R \sum_{n=2}^{6} \sum_{m=0}^{n} \frac{2}{n-1} (\bar{a}_{nm} \cos m\lambda_P + \bar{b}_{nm} \sin m\lambda_P) \bar{P}_{nm} (\sin\varphi) \; . \tag{37}$$

In ***Figures (6a-c)*** global maps of $N_0 + N_{Q_i0}$, $\frac{1}{\gamma} b(T)$ and of the total bias $b(N)$ are given. The total effect ranges from $+2.1$ m to -1.8 m.

Determination of ΔW_0 and C_{Q_i0}

We return to the objective of this chapter, the determination of the potential differences C_{Q_i0} and of ΔW_0. The following assumptions should hold : (1) For at least one station in each of the $(I+1)$ vertical datum zones geocentric coordinates are available, derived by space methods, with the required precision of better than 10^{-8}. The computation therefrom of the geometric height h above the adopted global reference surface is straightforward. (2) For all stations precise orthometric (or normal) heights or potential differences are available, referring to the respective datum point. (3) Gravity anomalies $\Delta g^{(i)}$ are given globally.

At each of the available stations k, $k = 1, 2, \ldots, K$ the height anomaly $N^{(i)}$ can be computed indirectly from the well-known relation :

$$N^{(i)} = h - H^{(i)} \tag{38}$$

The upper index (i) indicates again that N and H belong to datum zone i, i.e. they refer to the level surface passing through Q_i. $N^{(i)}$ of (38), derived from the orthometric height in combination with h from space methods, can now be compared with the corresponding gravimetrically determined $N^{(i)}$ of eq. (31). At each of the K stations the following relationship can be established with (31) and (38)

$$N^{(i)} = h - H^{(i)} = \frac{\delta(GM)}{R\gamma} - \frac{\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma} + \frac{R}{4\pi\gamma} \int_\sigma St(\psi) \left[\Delta g^{(j)} + \frac{2}{R} C_{Q_j0} \right] d\sigma \; . \tag{39}$$

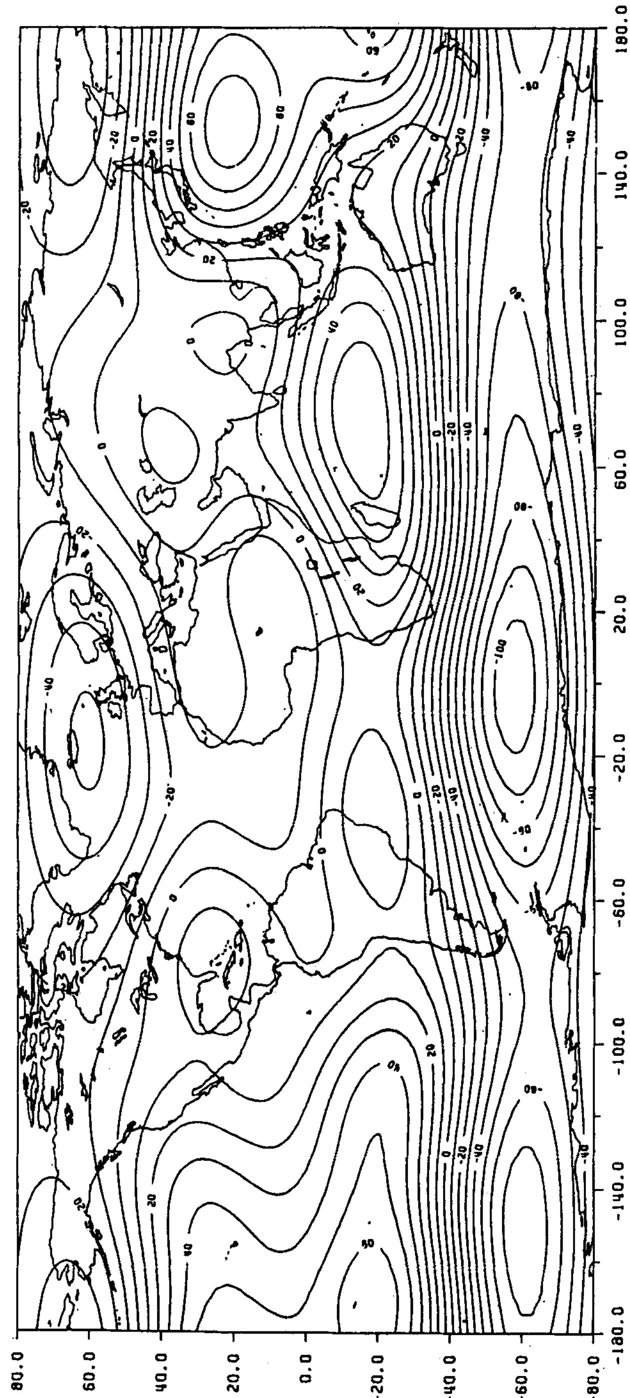

Fig. 6a — Global direct effect $N_0 + N_{Q_i0}$ *as derived from a spherical harmonic expansion of sea surface topography up to degree and order six.*

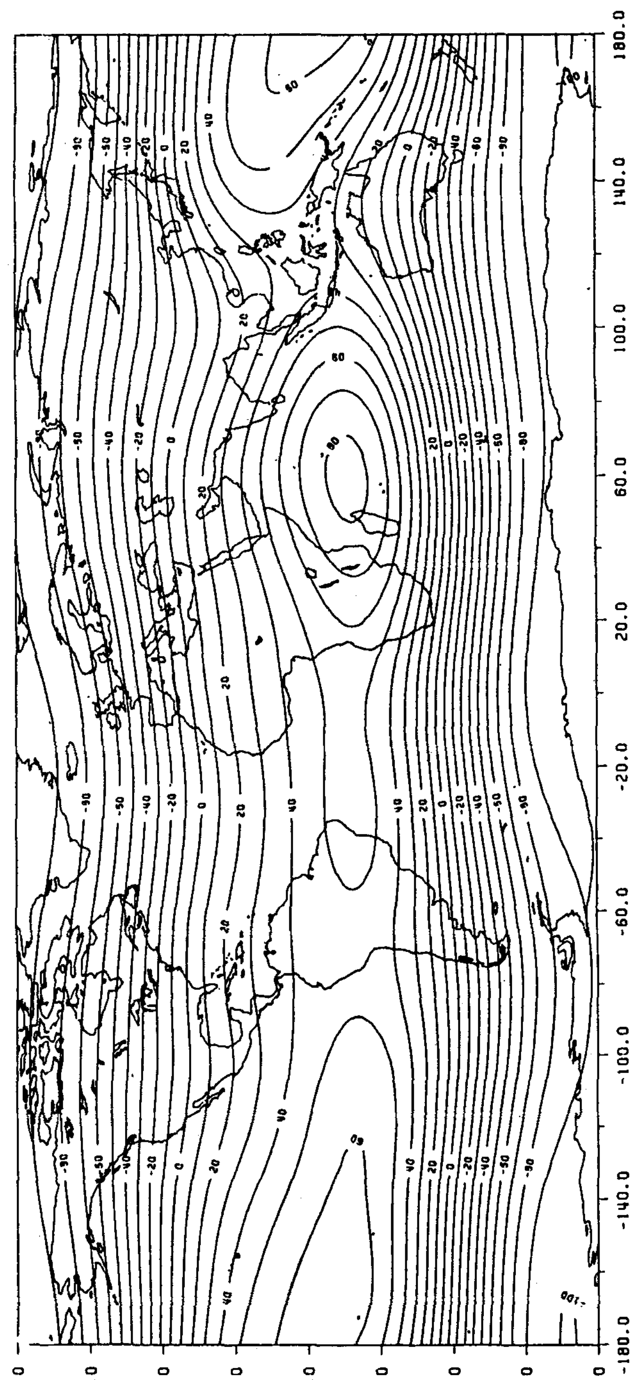

Fig. 6b – Global indirect effect $\frac{1}{\gamma}\,b\,(T)$ *from the same model as in Figure 6a.*

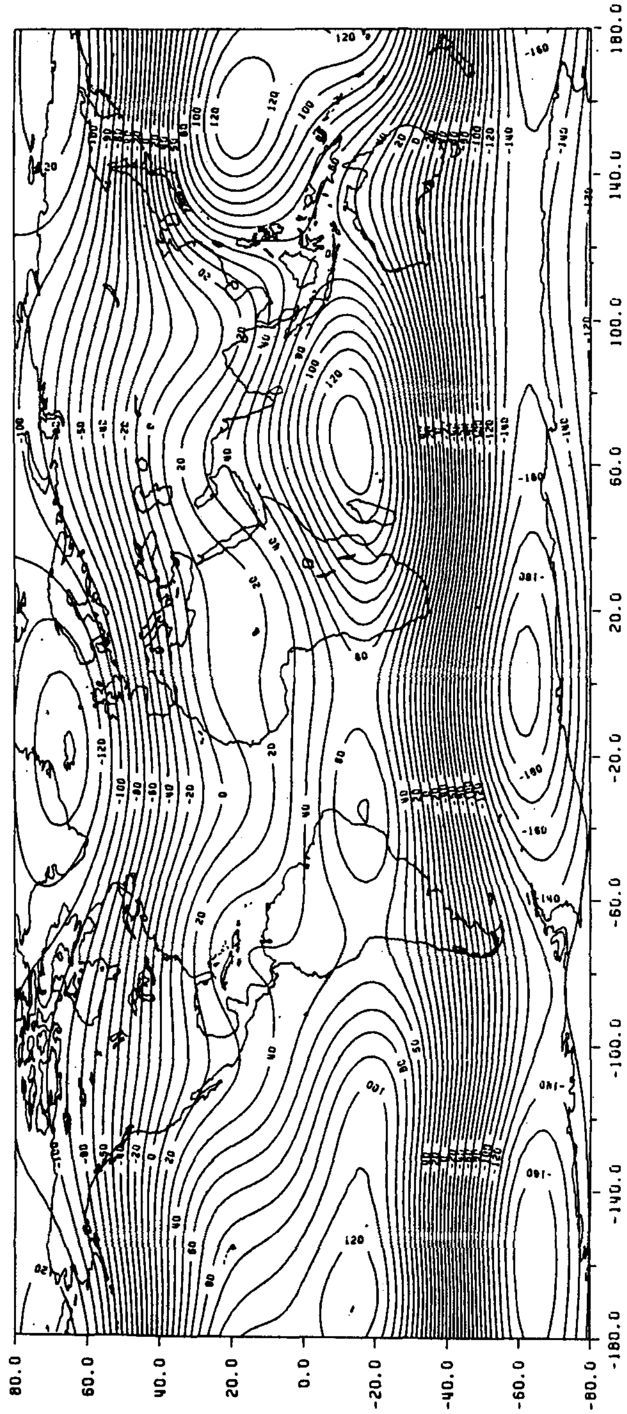

Fig. 6c – Total bias b (N) *from the same model as in Figure 6a, ranging from* + 1.2 *to* – 1.8 m.

Combining the terms with the three observable quantities h, $H^{(i)}$ and $\Delta g^{(j)}$ (again employing the operator notation for the Stokes integral) gives

$$y = h - H^{(i)} - \frac{R}{\gamma} S(\Delta g^{(j)}) \quad . \tag{40}$$

Now the following estimation model for ΔW_0 and the $C_{Q_i 0}$ can be established ($\delta(GM) \overset{\text{def.}}{=} 0$) :

$$y = -\frac{\Delta W_0}{\gamma} + \frac{C_{Q_i 0}}{\gamma} + \frac{2}{\gamma} S(C_{Q_j 0}) \tag{41}$$

We take a closer look at the Stokes integral in (41). Each unknown $C_{Q_j 0}$ takes a constant value inside the respective datum zone. If the area on the unit sphere σ covered by each datum zone is denoted $\Delta\sigma_i$, we obtain

$$\begin{aligned} \frac{2}{\gamma} S(C_{Q_j 0}) &= \frac{2}{4\pi\gamma} \int_\sigma St(\psi_{PQ}) C_{Q_j 0}(Q) \, d\sigma_Q \\ &= \frac{2}{\gamma} \sum_{j=1}^{I} C_{Q_j 0} \left\{ \frac{1}{4\pi} \int_{\Delta\sigma_j} St(\psi_{PQ}) \, d(\Delta\sigma_j) \right\} \\ &= \frac{2}{\gamma} \sum_{j=1}^{I} C_{Q_j 0} \, IS_{PQ_j} \end{aligned} \tag{42}$$

The value IS of the integral over the Stokes function depends on the position of P relative to $\Delta\sigma_i$. Now a complete linear adjustment model can be formulated for each of the K space stations P_k :

$$y_{P_k} = -\frac{\Delta W_0}{\gamma} + (1 + 2IS_{P_k Q_i}) \frac{C_{Q_i 0}}{\gamma} + 2 \sum_{j=1 (j \neq 1)}^{I} IS_{P_k Q_j} \frac{C_{Q_j 0}}{\gamma} \tag{43}$$

With $K \geq I + 1$ and at least one station in each datum zone all unknown $C_{Q_i 0}$ and ΔW_0 are estimable from (43). The structure of the coefficients matrix is shown in ***Figure 7.***

This is the solution to our problem. It is interesting to note, that in the strict sense the problem can only be treated worldwide – including all $C_{Q_i 0}$ – and not solely as a datum connection problem between vertical datums (a) and (b). This is caused by the term $2 \sum_{j=1 (j \neq i)}^{I} IS_{P_k Q_j} C_{Q_j 0}$ in (43), or in other words by the vertical datum effect in the computed gravity anomalies $\Delta g^{(i)}$, compare (29) and (31). It is the indirect contribution $\frac{1}{\gamma} b(T)$ displayed in ***Figure 6b.*** This effect could be completely eliminated, in case the gravity anomalies were computed solely from satellite derived potential coefficients. Gravity anomalies computed this way refer all to one level surface.

		ΔW_0	C_{10}	C_{20}	C_{30}	C_{40}	C_{50}	C_{60}	C_{70}	C_{80}
y_1	0	a	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	c_{18}
y_2	0	a	c_{21}	c_{22}	c_{23}	c_{24}	c_{25}	c_{26}	c_{27}	c_{28}
y_3	1	a	b_{31}	c_{32}	c_{33}	c_{34}	c_{35}	c_{36}	c_{37}	c_{38}
y_4	2	a	⋮	b_{42}	⋮	⋮	⋮	⋮	⋮	⋮
y_5	3	a		⋮	b_{53}					
y_6	3	a			b_{63}					
y_7	4	a			⋮	b_{74}				
y_8	5	a				⋮	b_{85}			
y_9	6	a					⋮	b_{96}		
y_{10}	7	a						⋮	$b_{10\ 7}$	
y_{11}	8	a							⋮	$b_{11\ 8}$

Fig. 7 — Structure of the coefficient matrix for an example with 11 *stations and* 9 *vertical datums. The types of elements appearing in the coefficient matrix are (compare eq. (43))* $a = -1$, $b_{ki} = (1 + 2IS_{P_k Q_i})$ *and* $c_{kj} = 2IS_{P_k Q_j}$.

In this case a mixture of situation A "one vertical datum", eqs. (19) and (21), and situation B "several vertical datums", eqs. (27) and (28), applies, where fundamental equation (21) is to be combined with Bruns' equation (27). This matter is discussed from a different angle in (Laskowski, 1983).

The results are based on the assumption that the K stations are indeed known with the quoted precision in a geocentric system. Ourdays results from precise laser tracking to LAGEOS suggest that this assumption can be made, compare (Smith et al., 1985). If one would have indications that the origins of the gravimetric and of the satellite systems do not coincide at this level, additional origin shift parameters could be introduced into (43), analogously to (Rapp & Rummel, 1976). In principle the shift parameters are estimable, too. Whether they can actually be separated from the other unknowns depends on the number and global distribution of the stations.

The feasibility of a global vertical datum connection considering presentdays data quality and distribution has been analyzed by Hajela (1983) using Colombo's (1980) approach. It showed that global vertical datum connection should become feasible in the coming decade. A similar numerical test of the model presented here has not yet been done.

4. Summary and Conclusions

Vertical datum definition is straightforward. Strict coupling of the potential (or height) value of the fundamental bench mark to mean sea level is neither required nor – from the operational point of view – desirable. Also the connection of two neighbouring vertical datums poses no particular problem, as long as the reference values of the fundamental bench marks are known and the potential (or height) difference between them can be measured. Problems start with the unification of vertical datums on a continental scale, for large networks consisting of first-order levelling traverses are of comparably poor reliability and are sensitive to systematic errors. Independent control could come from mean sea level records at selected tide gauge stations, integrated into the levelling network, in combination with oceanographic models of sea surface topography. However this type of control makes only sense if the oceanographically determined potential differences are of superior quality than the levelled ones.

Another, indirect, independent control can be established from a combination of levelled potential (or height) differences, geoid computation, and precise positions as derived from space methods. This method can also be employed for worldwide datum connection. The core of this approach forms the solution of the geodetic boundary problem under the realistic assumption that observable potential (or height) differences and gravity anomalies refer to different vertical datums with unknown level differences among them. In the geoid formula, which represents the solution of the geodetic boundary value problem, the unknown level differences appear at two places, compare eq. (31). One time they enter outside the Stokes integral as a constant off-set for all computation points inside one particular datum and a second time inside the Stokes integral. In the latter case they can be interpreted a sort of free-air reduction of the gravity anomalies to one common level but with unknown reduction height. Both the off-set and the indirect effect entering via the Stokes integral reach ourdays an order of magnitude of 0.5 to 1 m and do not cancel out. If the geoid heights can be determined independently by a combination of precise satellite positioning and the orthometric (or normal) heights of the respective datum, the potential differences can be solved for as unknowns in a least-squares adjustment problem. This global vertical datum connection can be performed with the required precision of better than 10^{-8} relative to the earth's radius, if in each datum zone at least one space station is present and if satellite derived coordinates and, globally, levelled heights and gravity are available with adequate accuracy. The problem could be solved restricted to only two datums, only if the indirect effect of the level differences would not be part of the model. This can only be achieved with gravity anomalies as derived from a satellite mission, for example from satellite gradiometry.

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