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ON DISCRETE α-UNIMODALITY

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Abstract. A continuous composition semigroup of probability generating functions $F := (F_t, t \geq 0)$ and the corresponding multiplication $\odot_F$ of van Harn *et al.* (1982, *Z. Wahrsch. Verw. Gebiete*, **61**, 97–118) are used to introduce the concept of $[F;\alpha]$-unimodality which generalizes the discrete α-unimodality due to Abouammoh (1987, *Statist. Neerlandica*, **41**, 239–244) and Alamatsaz (1993, *Statist. Neerlandica*, **47**, 245–252). We offer various characterizations and other properties of $[F;\alpha]$-unimodality. Notably, several convolution results are presented. Moreover, we explore the relationship between $[F;\alpha]$-unimodality and the concepts of discrete self-decomposability and stability. Finally, lower bounds for variances of $[F;\alpha]$-monotone and $[F;\alpha]$-unimodal random variables are derived and some examples are also mentioned.

Key words and phrases: Lattice distribution, semigroup, monotonicity, generating function, mixture, convolution, variance bounds.

1. Introduction

A real-valued random variable (rv) X is said to have an α-unimodal distribution about 0 for some $\alpha > 0$ if X has the following representation:

$$X \stackrel{d}{=} W^{1/\alpha} Y, \tag{1.1}$$

where W and Y are independent rv's and W is uniform $(0, 1)$. This definition is due to Olshen and Savage (1970) who examined properties and characterizations of such distributions. If Y in (1.1) is $\boldsymbol{R}_+$-valued ($\boldsymbol{R}_+ := [0, \infty)$), then the distribution of X is said to be α-monotone. Among other characterizations, a distribution on the real line is α-unimodal about 0 if and only if its restrictions to the positive half-line and to the negative half-line are α-monotone (for example, see Hansen (1990), p. 47).

It is clear that the α-unimodality of (1.1) does not apply to lattice distributions, i.e., distributions over the set of integers $\boldsymbol{Z} := \{0, \pm 1, \pm 2, \dots\}$. Abouammoh (1987, 1988) introduced a concept of discrete α-unimodality as follows: a lattice distribution $(p_n, n \in \boldsymbol{Z})$ is said to be α-unimodal (about 0) if

$$\begin{aligned} (n+1)p_{n+1} &\leq (n+\alpha)p_n, \qquad n \geq 0 \\ (1-n)p_{n-1} &\leq (\alpha - n)p_n, \qquad n \leq 0. \end{aligned} \tag{1.2}$$

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Discrete α-unimodality was further studied by Alamatsaz (1993), Bertin and Theodorescu (1995), and Wu and Dharmadhikari (1999). Using the binomial thinning operator $\circ$ of Steutel and van Harn (1979), Steutel (1988) defined α-monotonicity of a $\boldsymbol{Z}_+$-valued ($\boldsymbol{Z}_+ = \{0, 1, 2, \ldots\}$) rv X similar to (1.1):

$$X \stackrel{d}{=} W^{1/\alpha} \circ Y, \tag{1.3}$$

where W and Y are as in (1.1) and Y is $\boldsymbol{Z}_+$-valued. Steutel (1988) showed that if $(p_n, n \geq 0)$ is the probability distribution function (pdf) of X then (1.3) is equivalent to the top inequality in (1.2). Hence, as in the non-lattice case of (1.1), discrete α-unimodality about 0 is equivalent to α-monotonicity on both sides of 0 (see Alamatsaz (1993)).

Aly and Bouzar (2002) used the multiplication $\odot_F$ of van Harn *et al.* (1982) in lieu of $\circ$ in (1.3) to introduce a generalized notion of discrete monotonicity called $[F;\alpha]$-monotonicity. They derived the following characterization which we state as an alternative definition.

DEFINITION 1.1. A pdf $(p_n, n \geq 0)$ is $[F;\alpha]$-monotone if and only if for any $n \geq 0$,

$$\sum_{i=1}^{n+1} i p_i h_{n-i+1} \leq (n + \alpha a^{-1}) p_n, \tag{1.4}$$

where $(h_n, n \geq 0)$, a pdf (with $h_1 = 0$), and $a > 0$ are characteristics of the continuous composition semigroup of probability generating functions (pgf's) $F := (F_t, t \geq 0)$ from which $\odot_F$ stems (see definitions below).

The purpose of this paper is to present a generalized notion of discrete unimodality for lattice distributions called $[F;\alpha]$-unimodality. In Section 2 we propose a two-sided version of (1.4) as a definition that will contain (1.2) as a special case. We offer various characterizations and other properties of $[F;\alpha]$-unimodality thus generalizing the work of several authors. Moreover, we explore the relationship between $[F;\alpha]$-unimodality and the concepts of (discrete) F-self-decomposabilty and F-stability of van Harn *et al.* (1982). In Section 3 we establish several convolution properties. As a consequence, we give sufficient conditions for the $[F;\alpha]$-unimodality of an F-stable lattice distribution and that of a generalized two-sided version of (1.3), as defined by Pakes (1995). Finally, in Section 4 lower bounds for variances of $[F;\alpha]$-monotone and $[F;\alpha]$-unimodal rv's are derived. In the case of $[F;\alpha]$-unimodality, we obtain a sharper lower bound than the one obtained by Abouammoh *et al.* (1994) by relaxing at the same time their assumption $\alpha \geq 1$. Some examples are also mentioned. This paper is to be seen as a follow-up to the aforementioned article by Aly and Bouzar (2002).

In the rest of this section we briefly recall some definitions and results that are needed in the sequel. For proofs and further details we refer to Athreya and Ney ((1972), Chapter 3), van Harn *et al.* (1982) and van Harn and Steutel (1993). $F := (F_t; t \geq 0)$ is a continuous composition semigroup of pgf's such that $F_t \not\equiv 1$ and $\delta_F = -\ln F_1'(1) \geq 0$. We denote by U_F the infinitesimal generator of the semigroup F. The related A-function is defined by

$$A_F(z) = \exp\left\{-\int_0^z (U_F(x))^{-1} dx\right\}, \qquad z \in [0, 1). \tag{1.5}$$

Moreover, there exists a constant $a > 0$ and a pgf $H(z)$ given by

$$H(z) = \sum_{n \geq 0} h_n z^n, \tag{1.6}$$

with $h_1 = 0$ such that

$$U_F(z) = a\{H(z) - z\}, \qquad |z| \leq 1, \tag{1.7}$$

and $H'(1) = \sum_{i=1}^{\infty} i h_i \leq 1$. Finally, we recall that for a Z_+-valued rv X and $\nu \in (0,1)$, the generalized multiplication $\nu \odot_F X$ is defined by

$$\nu \odot_F X \stackrel{d}{=} \sum_{i=1}^{X} Y_i, \tag{1.8}$$

where $(Y_i, i \geq 1)$ is a sequence of iid rv's independent of X, with common pgf F_t, $t = -\ln \nu$.

2. A generalized notion of discrete unimodality

DEFINITION 2.1. Let $\alpha > 0$. A lattice distribution $(p_n, n \in \boldsymbol{Z})$ is said to be $[F; \alpha]$-unimodal (about 0) if

$$\begin{aligned} &\sum_{i=1}^{n+1} i p_i h_{n-i+1} \leq (\alpha a^{-1} + n) p_n, \qquad n \geq 0 \\ &\sum_{i=1}^{-n+1} i p_{-i} h_{-n-i+1} \leq (\alpha a^{-1} - n) p_n, \qquad n \leq 0, \end{aligned} \tag{2.1}$$

where $a > 0$ and $(h_n, n \geq 0)$ are as in (1.6)–(1.7) above.

A $\boldsymbol{Z}$-valued rv X is said to be $[F; \alpha]$-unimodal if its distribution is $[F; \alpha]$-unimodal. The definitions of $[F; \alpha]$-monotonicity (see Definition 1.1 above) and $[F; \alpha]$-unimodality can be extended to sequences of nonnegative real numbers. This extended definition will at times be used in the remainder of the paper without any further reference.

We start out by gathering a number of elementary properties, the proofs of which follow directly from (1.4) and (2.1) and are therefore omitted.

PROPOSITION 2.1. (i) *A lattice distribution $(p_n, n \in \boldsymbol{Z})$ is $[F; \alpha]$-unimodal if and only if $(p_n, n \geq 0)$ and $(p_{-n}, n \geq 0)$ are $[F; \alpha]$-monotone.*

(ii) *A $\boldsymbol{Z}$-valued rv X is $[F; \alpha]$-unimodal if and only if X^+ and X^- are $[F; \alpha]$-monotone and* $\max(p_1 h_0, p_{-1} h_0) \leq \alpha a^{-1} p_0$.

(iii) *If a lattice distribution $(p_n, n \in \boldsymbol{Z})$ is $[F; \alpha]$-unimodal, then it is $[F; \beta]$-unimodal for any $\beta \geq \alpha$.*

(iv) *A $\boldsymbol{Z}$-valued rv X with a symmetric distribution $(p_n, n \in \boldsymbol{Z})$ is $[F; \alpha]$-unimodal if and only if $|X|$ is $[F; \alpha]$-monotone and $2 p_1 h_0 \leq \alpha a^{-1} p_0$.*

We next present the main characterization of $[F; \alpha]$-unimodality.

PROPOSITION 2.2. *A lattice distribution* $(p_n, n \in \boldsymbol{Z})$ *is* $[F;\alpha]$*-unimodal if and only if* $p_1 h_0 \le \alpha a^{-1} p_0$ *and the sequence* $(q_n, n \in \boldsymbol{Z})$ *defined by*

$$(2.2)\qquad q_n = \begin{cases} \left((\alpha a^{-1} + n) p_n - \sum_{i=1}^{n+1} i p_i h_{n-i+1} \right) / (\alpha a^{-1} + p_1 h_0), & \text{if } n > 0 \\ \left((\alpha a^{-1} - n) p_n - \sum_{i=1}^{-n+1} i p_{-i} h_{-n-i+1} \right) / (\alpha a^{-1} + p_1 h_0), & \text{if } n \le 0 \end{cases}$$

is a lattice distribution.

PROOF. Suppose $(p_n, n \in \boldsymbol{Z})$ is $[F;\alpha]$-unimodal. Then by (2.1), $q_n \ge 0$ for any $n \in \boldsymbol{Z}$. Let M and N be integers such that $M < 0$ and $N > 0$. Then straightforward manipulations yield

$$(2.3)\qquad (\alpha a^{-1} + p_1 h_0) \sum_{n=M}^{0} q_n = \alpha a^{-1} \sum_{n=M}^{0} p_n + (1-M) b_{M-1} - \sum_{i=1}^{-M} i K_i p_{M-1+i},$$

and

$$(2.4)\qquad (\alpha a^{-1} + p_1 h_0) \sum_{n=1}^{N} q_n = \alpha a^{-1} \sum_{n=1}^{N} p_n + (N+1) c_{N+1} - \sum_{i=1}^{N} i K_i p_{N+1-i} + p_1 h_0,$$

where $K_i = \sum_{j=i+1}^{\infty} h_j$, $b_{M-1} = \sum_{i=1}^{-M} K_i p_{M-1+i} - p_{M-1} h_0$, and $c_{N+1} = \sum_{i=1}^{N} K_i p_{N+1-i} - p_{N+1} h_0$. Since $H'(1) < \infty$, we have $\lim_{i\to\infty} i K_i = 0$ and $\sum_{i\ge 0} K_i < \infty$. It also follows that $\sum_{M\le -1} |b_{M-1}| < \infty$ and $\sum_{N\ge 1} |c_{N+1}| < \infty$. Moreover, noting that $\lim_{N\to\infty} p_{N+1-i} = \lim_{M\to-\infty} p_{M-1+i} = 0$ for each $i \ge 1$, and that the sequences $\sum_{i=1}^{N} p_{N+1-i}$ and $\sum_{i=1}^{-M} p_{M-1+i}$ are both bounded (by 1), we have by Toeplitz' theorem (Knopp (1990)),

$$(2.5)\qquad \lim_{N\to\infty} \sum_{i=1}^{N} i K_i p_{N+1-i} = \lim_{M\to-\infty} \sum_{i=1}^{-M} i K_i p_{M-1+i} = 0.$$

Since $q_n \ge 0$, the left-hand sides of both equations (2.3) and (2.4) have limits. By (2.5) and the fact that $\sum_{M\le -1} |b_{M-1}|$ and $\sum_{N\ge 1} |c_{N+1}|$ both converge, these limits must be finite and $\lim_{N\to\infty} (N+1) c_{N+1} = \lim_{M\to-\infty} (1-M) b_{M-1} = 0$. Therefore,

$$\lim_{\substack{M\to-\infty \\ N\to+\infty}} (\alpha a^{-1} + p_1 h_0) \sum_{n=M}^{N} q_n = \alpha a^{-1} \sum_{n=M}^{N} p_n + p_1 h_0 = \alpha a^{-1} + p_1 h_0,$$

or $\sum_{-\infty}^{\infty} q_n = 1$. Conversely if $(q_n, n \in \boldsymbol{Z})$ of (2.2) is a lattice distribution, then (2.1) holds trivially. □

The following proposition offers a characterization of $[F;\alpha]$-unimodality in terms of generating functions (gf's).

PROPOSITION 2.3. *A $\boldsymbol{Z}$-valued rv X with distribution $(p_n, n \in \boldsymbol{Z})$ is $[F;\alpha]$-unimodal if and only if $P^+(z) = \sum_{n=0}^{\infty} p_n z^n$ and $P^-(z) = \sum_{n=0}^{\infty} p_{-n} z^n$ satisfy*

$$(2.6)\qquad \begin{aligned} P^+(z) &= \alpha[A_F(z)]^{-\alpha} \int_z^1 Q_1(v)A_F(v)^\alpha [U_F(v)]^{-1} dv \qquad \text{and} \\ P^-(z) &= \alpha[A_F(z)]^{-\alpha} \int_z^1 Q_2(v)A_F(v)^\alpha [U_F(v)]^{-1} dv, \end{aligned}$$

where $|z| \le 1$, and $Q_1(z)$ and $Q_2(z)$ are gf's. More specifically, $Q_1(z) = cQ_{11}(z)$, $Q_2(z) = (1 - c + p_0)Q_{22}(z)$, and $c = \sum_{n=0}^{\infty} p_n$, for some pgf's $Q_{11}(z)$ and $Q_{22}(z)$.

PROOF. Without loss of generality, we assume $c > 0$ and $c' = \sum_{n=0}^{\infty} p_{-n} = 1 - c + p_0 > 0$. By Proposition 2.1(i), $(p_n, n \in \boldsymbol{Z})$ is $[F;\alpha]$-unimodal if and only if the distributions (on $\boldsymbol{Z}_+$) $(p_n/c, n \ge 0)$ and $(p_{-n}/c', n \ge 0)$ are $[F;\alpha]$-monotone. Note that $P^+(z) = cP_1(z)$ and $P^-(z) = c'P_2(z)$ where $P_1(z)$ and $P_2(z)$ are the pgf's of $(p_n/c, n \ge 0)$ and $(p_{-n}/c', n \ge 0)$ respectively. By Proposition 2.2 in Aly and Bouzar (2002), a $\boldsymbol{Z}_+$-valued rv X' is $[F;\alpha]$-monotone if and only if its pgf $P(z)$ admits the representation

$$(2.7)\qquad P(z) = \alpha[A_F(z)]^{-\alpha} \int_z^1 Q(v)A_F(v)^\alpha [U_F(v)]^{-1} dv,$$

for some pgf $Q(z)$. The conclusion follows by applying (2.7) to $P_1(z)$ and $P_2(z)$. □

We next give a mixture representation theorem for $[F;\alpha]$-unimodal rv's.

PROPOSITION 2.4. *A $\boldsymbol{Z}$-valued rv X with distribution $(p_n, n \in \boldsymbol{Z})$ is $[F;\alpha]$-unimodal if and only if $\max(p_1 h_0, p_{-1} h_0) \le \alpha a^{-1} p_0$ and*

$$(2.8)\qquad X^+ \stackrel{d}{=} W_1^{1/\alpha} \odot_F Y_1 \qquad \text{and} \qquad X^- \stackrel{d}{=} W_2^{1/\alpha} \odot_F Y_2,$$

where for $i = 1, 2$, W_i is uniformly distributed over $(0,1)$, Y_i is a $\boldsymbol{Z}_+$-valued rv, Y_i and W_i are independent, and $\odot_F$ is as in (1.8).

PROOF. We recall from Aly and Bouzar (2002) that a $\boldsymbol{Z}_+$-valued rv X' is $[F;\alpha]$-monotone if and only if

$$(2.9)\qquad X' \stackrel{d}{=} W^{1/\alpha} \odot_F Y',$$

where W is uniformly distributed over $(0,1)$, Y is a $\boldsymbol{Z}_+$-valued rv, and Y and W are independent. By Proposition 2.1(ii), X^+ and X^- are $[F;\alpha]$-monotone. Hence, (2.8) follows by applying (2.9) to X^+ and X^-. □

A useful closure property is established next. Its proof is a straightforward consequence of Proposition 2.2 and is therefore omitted.

PROPOSITION 2.5. *Let $(X_k, k \ge 0)$ and X be $\boldsymbol{Z}$-valued (resp. $\boldsymbol{Z}_+$-valued) rv's. Assume that for each $k \ge 0$, X_k is $[F;\alpha]$-unimodal (resp. $[F;\alpha]$-monotone). If $(X_k, k \ge 0)$ converges in distribution to X, then X is $[F;\alpha]$-unimodal (resp. $[F;\alpha]$-monotone).*

The following two propositions explore the relationship between $[F;\alpha]$-unimodality and the concepts of discrete self-decomposabilty and stability due to van Harn *et al.* (1982). A $\boldsymbol{Z}_+$-valued rv X is said to be F-self-decomposable if for any $v \in (0,1)$, there exists a rv X_v such that $X \stackrel{d}{=} v \odot_F X + X_v$. The pgf of an F-self-decomposable rv X admits the following canonical representation

$$P(z) = \exp\left[-\lambda \int_z^1 \frac{1-Q(x)}{U_F(x)}dx\right], \tag{2.10}$$

for some unique (λ, Q), where $\lambda > 0$ and Q is a pgf with $Q(0) = 0$ (cf. van Harn *et al.* (1982)). X is said to be F-stable with exponent $\gamma > 0$ if there exits a sequence of iid rv's $(X_n, n \geq 0)$, $X_i \stackrel{d}{=} X$ for all i, such that for all $n > 0$, $X \stackrel{d}{=} n^{-1/\gamma} \odot_F \sum_{i=1}^n X_i$. F-self-decomposable distributions are infinitely divisible and F-stable distributions are necessarily F-self-decomposable and exist only when $0 < \gamma \leq \delta_F$ (cf. van Harn *et al.* (1982)). Moreover, the pgf of an F-stable rv X with exponent $0 < \gamma \leq \delta_F$ admits the canonical representation

$$P(z) = \exp[-\lambda A_F(z)^\gamma], \tag{2.11}$$

for some $\lambda > 0$.

PROPOSITION 2.6. (i) *Let X be an F-self-decomposable rv with pgf $P(z)$ described by* (2.10). *X is $[F;\alpha]$-unimodal for $\alpha > 0$ if and only if $\lambda \leq \alpha$.*

(ii) *Assume $\delta_F > 0$. Let X be an F-stable rv with exponent γ, $0 < \gamma \leq \delta_F$ and with pgf $P(z)$ described by* (2.11). *X is $[F;\alpha]$-unimodal for $\alpha > 0$ if and only if $\lambda\gamma \leq \alpha$.*

PROOF. By Proposition 2.10 in Aly and Bouzar (2002), an F-self-decomposable rv X is $[F;\alpha]$-unimodal if and only if $U_F(0)p_1/p_0 \leq \alpha$. Noting that $\frac{p_1}{p_0} = \frac{d}{dz}\log P(z)\mid_{z=0}$, it follows by (2.10) (resp. (2.11)) that $U_F(0)p_1/p_0 = \lambda$ for F-self-decomposable distributions (resp. $U_F(0)p_1/p_0 = \lambda\gamma$ for F-stable distributions). □

F-self-decomposable distributions with pgf's of the form

$$P(z) = (1 + cA_F(z)^\gamma)^{-r}, \tag{2.12}$$

for some $c > 0$, $r > 0$ and $0 < \gamma \leq \delta_F$, were shown by van Harn and Steutel (1993) to be solutions to important stability equations for $\boldsymbol{Z}_+$-valued processes with stationary independent increments (see also Pakes (1995)). By Proposition 2.6(i), a distribution with pgf given by (2.12) is $[F;\alpha]$-unimodal if and only if $\frac{cr\gamma}{1+c} \leq \alpha$. The special case where $r = \alpha/\gamma$ gives rise to an interesting characterization.

PROPOSITION 2.7. *Assume $\delta_F > 0$, $\alpha > 0$ and $0 < \gamma \leq \delta_F$. Let X be a $\boldsymbol{Z}_+$-valued, $[F;\alpha]$-unimodal rv with pgf $P(z)$. Then the pgf $Q(z)$ in the representation* (2.6) *of $P(z)$ satisfies $Q(z) = (P(z))^{1+\gamma/\alpha}$ if and only if $P(z) = (1 + cA_F(z)^\gamma)^{-\alpha/\gamma}$ for $z \in [-1,1]$.*

PROOF. First we note that in the $\boldsymbol{Z}_+$-valued case, (2.6) is equivalent to the following equation:

$$Q(z) = -\alpha^{-1}P'(z) + P(z). \tag{2.13}$$

A straightforward substitution argument in (2.13) implies the 'if' part. To prove the 'only if' part, we use the substitution $P(z) = (1 + f(z))^{-\alpha/\gamma}$ for some function $f(z)$. Equation (2.13) becomes

$$\frac{f'(z)}{f(z)} = -\frac{\gamma}{U_F(z)}$$

whose solution is $f(z) = cA_F(z)^\gamma$ for some $c > 0$. □

We conclude the section with some examples and remarks.

van Harn *et al.* (1982) give some rich examples of continuous composition semigroups of pgf's from which one can generate $[F;\alpha]$-unimodal distributions. We mention the parametrized family of semigroups $(F^{(\theta)}, \theta \in [0,1))$ described by

$$F_t^{(\theta)}(z) = 1 - \frac{\overline{\theta} e^{-\overline{\theta} t}(1-z)}{\overline{\theta} + \theta(1 - e^{-\overline{\theta} t})(1-z)}, \quad t \geq 0, \quad |z| \leq 1, \quad \overline{\theta} = 1 - \theta. \tag{2.14}$$

In this case we have $\delta_{F^{(\theta)}} = \overline{\theta}$, $U_{F^{(\theta)}}(z) = (1-z)(1-\theta z)$, $A_{F^{(\theta)}}(z) = (\frac{1-z}{1-\theta z})^{1/\overline{\theta}}$, $a = 1+\theta$, $h_0 = 1/(1+\theta)$, $h_1 = 0$, $h_2 = \theta/(1+\theta)$, $h_n = 0$, $n \geq 3$. A lattice distribution $(p_n, n \in \boldsymbol{Z})$ is $F^{(\theta)}$-unimodal if $\max(p_1, p_{-1}\theta) \leq \alpha p_0$ and

$$\begin{aligned} &(n-1)\theta p_{n-1} + (n+1)p_{n+1} \leq (\alpha + (1+\theta)n)p_n, \qquad n \geq 1 \\ &(1-n)p_{n-1} - (n+1)\theta p_{n+1} \leq (\alpha - (1+\theta)n)p_n, \qquad n \leq -1. \end{aligned} \tag{2.15}$$

We note that for $\theta = 0$, $F^{(\theta)}$ corresponds to the standard semigroup $F_t^{(0)}(z) = 1 - e^{-t} + e^{-t}z$ and the multiplication $\odot_{F^{(0)}}$ becomes the binomial thinning operator of Steutel and van Harn (1979). $[F^{(0)};\alpha]$-unimodality is the α-unimodality of Abouammoh (1987) and of Alamatsaz (1993) described by (1.2). The results obtained in this section are generalizations of these authors' results.

Another way of producing examples is to start out with a constant $a > 0$ and a pdf $(h_n, n \geq 0)$ with $h_1 = 0$. If the mean of $(h_n, n \geq 0)$ is less than or equal to 1 and its pgf $H(z)$ satisfies the non-explosion condition

$$\int_{1-\epsilon}^{1} |H(x) - x|^{-1} = \infty, \qquad (\epsilon > 0), \tag{2.16}$$

then there exists a unique continuous semigroup $F := (F_t, t \geq 0)$ admitting $U(z)$ given by (1.7) as its infinitesimal generator. For more on the construction of F we refer to van Harn *et al.* (1982) and references therein. We also note that the non-explosion condition (2.16) is satisfied under the assumptions given in the introduction (see also van Harn *et al.* (1982)).

Remarks. 1) Propositions 2.3 and 2.4 should be seen as the closest analogues to characteristic function and mixture representations in the non-lattice case (see Olshen and Savage (1970)).

2) In general, a lattice distribution $(p_n, n \in \boldsymbol{Z})$ will be said to be $[F;\alpha]$-unimodal

about an arbitrary mode $m_0 \in \boldsymbol{Z}$ if

$$
\begin{aligned}
&\sum_{i=m_0+1}^{n+1} (i-m_0)p_i h_{n-i+1} \le (\alpha a^{-1}+n-m_0)p_n, \qquad n \ge m_0 \\
&\sum_{i=1-m_0}^{-n+1} (i+m_0)p_{-i} h_{-n-i+1} \le (\alpha a^{-1}-n+m_0)p_n, \qquad n \le m_0.
\end{aligned}
\tag{2.17}
$$

Let $g_n = p_{n+m_0}$, $n \in \boldsymbol{Z}$. Then $(p_n, n \in \boldsymbol{Z})$ is $[F;\alpha]$-unimodal about m_0 if and only if $(g_n, n \in \boldsymbol{Z})$ is $[F;\alpha]$-unimodal about 0. Hence the results on $[F;\alpha]$-unimodality about 0 in this and subsequent sections carry over (through a translation argument as well as with the obvious modifications in the statements) to $[F;\alpha]$-unimodality about an arbitrary mode.

3) One can define the concepts of discrete monotonicity and discrete unimodality through (1.4) and (2.1), respectively, without referring to a continuous semigroup of pgf's. All that is really needed is a pdf $(h_n, n \ge 0)$ with $h_1 = 0$ and finite mean. One can even assume $a = 1$. It can be checked that in that case Propositions 2.1 and 2.2 (and their monotone analogues) also hold. However, the representations results in terms of gf's (Proposition 2.3) and mixtures (Proposition 2.4) do not carry over.

4) Proposition 2.7 is an extension of a result obtained by Sapatinas (1995) for the standard semigroup $F^{(0)}$ of (2.14).

3. Convolution properties

We start out with two useful properties of $[F;\alpha]$-monotonicity.

LEMMA 3.1. (Translation property) *Let $\alpha \ge a$. If $(p_n, n \ge 0)$ is an $[F;\alpha]$-monotone sequence of nonnegative numbers, then for any $i \ge 0$ $(p_{n+i}, n \ge 0)$ is $[F;\alpha]$-monotone.*

PROOF. We have by (1.4)

$$
\begin{aligned}
(n+\alpha a^{-1})p_{n+i} &= \frac{n+\alpha a^{-1}}{n+i+\alpha a^{-1}}(n+i+\alpha a^{-1})p_{n+i} \ge \frac{n+\alpha a^{-1}}{n+i+\alpha a^{-1}} \sum_{j=1}^{n+i+1} j p_j h_{n+i-j+1} \\
&\ge \frac{n+\alpha a^{-1}}{n+i+\alpha a^{-1}} \sum_{j=i+1}^{n+i+1} j p_j h_{n+i-j+1} \\
&= \frac{n+\alpha a^{-1}}{n+i+\alpha a^{-1}} \sum_{j=1}^{n+1} (j+i) p_{j+i} h_{n-j+1}.
\end{aligned}
$$

The conclusion follows by noting that $\alpha \ge a$ implies that for any $1 \le i \le n+1$, $\frac{(j+i)(n+\alpha a^{-1})}{n+i+\alpha a^{-1}} \ge j$. □

LEMMA 3.2. (Mixture property) *Let $((a_n^{(i)}, n \ge 0),\ i \ge 0)$ be $[F;\alpha]$-monotone sequences of nonnegative real numbers. Let $(w_i, i \ge 0)$ be a sequence of nonnegative* *numbers such that for every* $n \geq 0$,

$$b_n = \sum_{i=0}^{\infty} w_i a_n^{(i)} < \infty.$$

Then $(b_n, n \geq 0)$ *is* $[F;\alpha]$*-monotone.*

PROOF. Straightforward. □

We now state and prove the main result of this section.

PROPOSITION 3.1. *Let* $\alpha > 0$, $\beta > 0$ *be such that* $\alpha + \beta \geq a$. *The convolution of an* $[F;\alpha]$*-unimodal lattice distribution and an* $[F;\beta]$*-unimodal lattice distribution is* $[F;\alpha+\beta]$*-unimodal.*

PROOF. We use the same argument as in Wu and Dharmadhikari (1999). Assume $(p_n, n \in \boldsymbol{Z})$ (resp. $(q_n, n \in \boldsymbol{Z})$) is $[F;\alpha]$-unimodal (resp. $[F;\beta]$-unimodal). Let $(p * q)_n = \sum_{i=-\infty}^{\infty} p_i q_{n-i}$, $n \in \boldsymbol{Z}$, be the convolution of $\{p_n\}$ and $\{q_n\}$. We show that $((p*q)_n, n \geq 0)$ is $[F;\alpha]$-monotone. It's easy to see that

$$(p * q)_n = \sum_{i=1}^{\infty} p_{-i} q_{n+i} + \sum_{i=0}^{n} p_i q_{n-i} + \sum_{i=1}^{\infty} p_{n+i} q_{-i}. \tag{3.1}$$

By the translation property (Lemma 3.1) applied to $(q_n, n \geq 0)$ and the mixture property (Lemma 3.2) applied to $w_i = p_{-i}$ and $a_n^{(i)} = q_{n+i}$, $a_n = \sum_{i=1}^{\infty} p_{-i} q_{n+i}$ is $[F;\beta]$-monotone, and hence, by Corollary 2.4 (i) in Aly and Bouzar (2002), it is $[F;\alpha+\beta]$-monotone. Likewise, it can be shown that $c_n = \sum_{i=1}^{\infty} p_{n+i} q_{-i}$ is $[F;\alpha]$-monotone, and hence $[F;\alpha+\beta]$-monotone. The middle sum in (3.1), $b_n = \sum_{i=0}^{n} p_i q_{n-i}$, is the convolution of $(p_n, n \geq 0)$ and $(q_n, n \geq 0)$ and is therefore $[F;\alpha+\beta]$-monotone by Proposition 2.5 in Aly and Bouzar (2002) (obviously modified to apply to $[F;\alpha]$-monotone sequences of nonnegative numbers). Hence $(p*q)_n = a_n + b_n + c_n$ is $[F;\alpha+\beta]$-monotone. Similarly, one can show that $((p*q)_n, n \leq 0)$ is $[F;\alpha+\beta]$-monotone. □

The following proposition is an extension of a result due to Alamatsaz (1993) in the case of the standard semigroup $F^{(0)}$ of (2.14). We provide a simpler and more direct proof than Alamatsaz's.

PROPOSITION 3.2. *Let* $\alpha \geq a$, $\beta \geq a$, *and* $\delta = \max(\alpha, \beta)$. *Let* X *and* Y *be two independent* $\boldsymbol{Z}_+$*-valued rv's such that* X *and* Y *are* $[F;\alpha]$*-monotone and* $[F;\beta]$*-monotone, respectively. Then* $X - Y$ *is* $[F;\delta]$*-unimodal.*

PROOF. Let $g_n = P(X - Y = n)$, $n \in \boldsymbol{Z}$. Then one can easily derive that

$$g_n = \begin{cases} \displaystyle\sum_{i=0}^{\infty} p_{n+i} q_i, & \text{if } \ n \geq 0, \\ \displaystyle\sum_{i=0}^{\infty} p_i q_{-n+i}, & \text{if } \ n \leq 0. \end{cases} \tag{3.2}$$

By the translation property $(g_n, n \geq 0)$ (resp. $(g_{-n}, n \geq 0)$) is $[F;\alpha]$-monotone (resp. $[F;\beta]$-monotone). Hence, by Corollary 2.4 (i) in Aly and Bouzar (2002), $(g_n, n \geq 0)$ and $(g_{-n}, n \geq 0)$ are $[F;\delta]$-monotone. The conclusion follows from Proposition 2.1(i). □

The following corollary is a direct consequence of Proposition 3.2 above and Proposition 2.5 in Aly and Bouzar (2002). Its proof is omitted.

COROLLARY 3.1. *Let $\alpha \geq a$. The symmetrization of an $[F;\alpha]$-monotone distribution is $[F;\alpha]$-unimodal.*

Pakes (1995) extended the concept of discrete F-stability to lattice distributions as follows. A $\boldsymbol{Z}$-valued rv X is said to have an F-stable distribution with exponent γ, $0 < \gamma \leq \delta_F$, if $X \overset{d}{=} X_1 - X_2$ where X_1 and X_2 are independent, $\boldsymbol{Z}_+$-valued rv's with F-stable distributions with exponent γ. The characteristic function of X is given by

$$\phi(\tau) = \exp(-[\lambda_1 A(e^{i\tau})^\gamma + \lambda_2 A(e^{-i\tau})^\gamma]), \tag{3.3}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$.

COROLLARY 3.2. *Let $\alpha \geq a$ and let X be a $\boldsymbol{Z}$-valued rv with an F-stable distribution with exponent γ, $0 < \gamma \leq \delta_F$, and charactrisitic function* (3.3). *If* $\max(\lambda_1, \lambda_2) \leq \alpha\gamma^{-1}$, *then X has an $[F;\alpha]$-unimodal distribution.*

PROOF. Follows directly from Propositions 2.6 and 3.2. □

Pakes (1995) also introduced a two-sided version of the multiplication $\odot_F$ of (1.8). Let Y be a $\boldsymbol{Z}$-valued rv with the decomposition property $Y \overset{d}{=} Y_1 - Y_2$ where Y_1 and Y_2 are $\boldsymbol{Z}_+$-valued, independent rv's. Let W be a rv that is independent of Y_1 and Y_2 with support in $(0,1)$. Then $W \odot_F Y$ is defined by its characteristic function

$$\psi(\tau) = \int_0^1 P_1(F_{-\log w}(e^{i\tau}))P_2(F_{-\log w}(e^{-i\tau}))dF(w), \tag{3.4}$$

where P_i is the pgf of Y_i, $i = 1, 2$, and $F(w)$ is the distribution function of W. Note that if W is constant, $W = w$ with probability 1 ($w \in (0,1)$), then $w \odot_F Y \overset{d}{=} w \odot_F Y_1 - w \odot_F Y_2$.

PROPOSITION 3.3. *Let $\alpha \geq a$, $\beta \geq a$, and $\delta = \max(\alpha, \beta)$. Let Y, Y_1, Y_2 and W be as described above. If Y_1 and Y_2 are $[F;\alpha]$-monotone and $[F;\beta]$-monotone, respectively, then $W \odot_F Y$ is $[F;\delta]$-unimodal.*

PROOF. For $w \in (0,1)$, let $(q_n(w), n \in \boldsymbol{Z})$ be the pdf of $w \odot_F Y$ ($\overset{d}{=} w \odot_F Y_1 - w \odot_F Y_2$) and $(p_n, n \in \boldsymbol{Z})$ the pdf of $W \odot_F Y$. By the independence assumption, we have for every $n \in \boldsymbol{Z}$,

$$p_n = \int_0^1 q_n(w)dF(w), \tag{3.5}$$

where F is the distribution function of W. Since $w \odot_F Y_1$ and $w \odot_F Y_2$ are independent and, respectively, $[F;\alpha]$-monotone and $[F;\beta]$-monotone (cf. Aly and Bouzar (2002)), it follows by Proposition 3.2 that $w \odot_F Y$ is $[F;\delta]$-unimodal. This implies that $(q_n(w), n \in \boldsymbol{Z})$ satisfies (2.1). Consequently, by (3.5), $(p_n, n \in \boldsymbol{Z})$ satisfies (2.1) as well. □

Remarks. 1) The mixture property given in Lemma 3.2 holds too for $[F;\alpha]$-unimodality.

2) Corollary 3.1 is due to Alamatsaz (1993) for the standard semigroup $F^{(0)}$ of (2.14).

3) The results in this section remain valid under the semigroup-free definition of discrete unimodality given in Remark (3) in Section 2.

4. Variance bounds

Variance bounds for unimodal rv's have been discussed by several authors (see Abouammoh *et al.* (1994) and references therein). In this section we obtain lower bounds for the variances of $[F;\alpha]$-monotone and $[F;\alpha]$-unimodal rv's. In the case of $[F;\alpha]$-unimodality we derive a sharper lower bound than the one obtained by Abouammoh *et al.* (1994) by relaxing at the same time their assumption $\alpha \geq 1$.

We will assume throughout this section that $H''(1) = \sum_{i=2}^{\infty} i(i-1)h_i < \infty$.

PROPOSITION 4.1. *Let X be a $\boldsymbol{Z}_+$-valued, $[F;\alpha]$-monotone rv. Assume that the second moment of X exists and let μ and σ^2 denote the mean and variance of X, respectively. Then*

$$\sigma^2 \geq \frac{A\mu^2 + B\mu}{2\alpha\sqrt{A} + \alpha^2}, \tag{4.1}$$

where $A = a^2(1 - H'(1))^2$ and $B = a\alpha(1 - H'(1) + H''(1))$.

PROOF. First we recall from the assumptions on the semigroup F that $H'(1) \leq 1$. Let $P(z)$ be the pgf of X. Since X is $[F;\alpha]$-monotone, by Proposition 2.2 in Aly and Bouzar (2002) there exists a pgf $Q(z)$ such that

$$Q(z) = -\alpha^{-1}P'(z)U_F(z) + P(z). \tag{4.2}$$

The fact that X has a finite variance implies $P'(1) = \mu$ and $P''(1) = \sigma^2 + \mu^2 - \mu$. It follows by (4.2) that

$$\begin{aligned} Q'(1) &= (a\alpha^{-1}(1 - H'(1)) + 1)\mu \\ Q''(1) &= (2a\alpha^{-1}(1 - H'(1)) + 1)(\sigma^2 + \mu^2 - \mu) - a\alpha^{-1}H''(1)\mu. \end{aligned} \tag{4.3}$$

Therefore, the distribution with pgf $Q(z)$ has finite variance equal to $Q''(1) + Q'(1) - Q'(1)^2$, which implies the inequality $Q''(1) \geq Q'(1)^2 - Q'(1)$. Using (4.3) and solving for σ^2 in the latter inequality yields (4.1). □

PROPOSITION 4.2. *Let X be a $\boldsymbol{Z}$-valued, $[F;\alpha]$-unimodal rv. Assume that the second moment of X exists and let μ and σ^2 denote the mean and variance of X, respectively. Then*

$$\sigma^2 \geq \frac{\frac{1}{2}(\alpha + \sqrt{A})^2([E(|X|)]^2 - \mu^2) + BE(|X|) + A\mu^2}{2\alpha\sqrt{A} + \alpha^2}, \tag{4.4}$$

where A and B are as in Proposition 4.1.

PROOF. Let μ^+ and σ_+^2 (resp. μ^- and σ_-^2) be the mean and variance of X^+ (resp. X^-). Since $X^+X^- = 0$, we have

$$\sigma^2 = \sigma_+^2 + \sigma_-^2 + 2\mu^+\mu^-. \tag{4.5}$$

By Proposition 2.1(ii), X_+ and X_- are $[F;\alpha]$-monotone. Hence, by applying Proposition 4.1 to X^+ and X^- separately, it follows from (4.5) that

$$\sigma^2 \geq \frac{A(\mu^+)^2 + B\mu^+}{2\alpha\sqrt{A} + \alpha^2} + \frac{A(\mu^-)^2 + B\mu^-}{2\alpha\sqrt{A} + \alpha^2} + 2\mu^+\mu^-. \tag{4.6}$$

Since $\mu = \mu^+ - \mu^-$, $E(|X|) = \mu^+ + \mu^-$, and (consequently) $2\mu^+\mu^- = ([E(|X|)]^2 - \mu^2)/2$, (4.4) is deduced from (4.6) by straightforward algebraic manipulations. □

In the case of the semigroup $F^{(\theta)}$ of (2.14), $\theta \in [0,1)$, we have $H_\theta(z) = (1/1+\theta) + (\theta/1+\theta)z^2$. The quantities A, B, and C of Propositions 4.1 and 4.2 can be shown to be $A = \overline{\theta}^2$ and $B = \alpha(1+\theta)$, where $\overline{\theta} = 1-\theta$. If X is $\boldsymbol{Z}$-valued, $[F^{(\theta)};\alpha]$-unimodal, and with finite second moment, then its variance σ^2 satisfies

$$\sigma^2 \geq \frac{\frac{1}{2}(\alpha+\overline{\theta})^2([E(|X|)]^2 - \mu^2) + \alpha(1+\theta)E(|X|) + (\overline{\theta}\mu)^2}{2\overline{\theta}\alpha + \alpha^2}. \tag{4.7}$$

For the standard semigroup $F^{(0)}$, (4.7) at $\theta = 0$ becomes

$$\sigma^2 \geq \frac{\frac{1}{2}(\alpha+1)^2([E(|X|)]^2 - \mu^2) + \alpha E(|X|) + \mu^2}{2\alpha + \alpha^2}.$$

This is a sharper lower bound than the one obtained by Abouammoh *et al.* (1994) (see their Theorem 3.1): $\sigma^2 \geq (\mu^2 + \alpha|\mu|)/(2\alpha+\alpha^2)$. We also note that the assumption $\alpha \geq 1$ made by Abouammoh *et al.* (1994) to prove their result is not needed.

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