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THE THEORY OF THE NUMERICAL-ANALYTIC METHOD: ACHIEVEMENTS AND NEW TRENDS OF DEVELOPMENT. IV

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We analyze the application of the numerical-analytic method proposed by Samoilenko in 1965 to autonomous systems of differential equations and impulsive equations.

This paper is the fourth part of the series [1–3], and, therefore, we continue the enumeration of sections, theorems, lemmas, formulas, etc. Here, we continue the investigation of the numerical-analytic method which, for simplicity, is often called the "method."

3.4. Abstract Scheme of the Method

In Sec. 1 of [1], we suggested one possible "operator" formulation of the method. Moreover, the interpretation of the so-called Lyapunov–Schmidt equation given in Sec. 2.1 of [1] enabled us to discover a certain relationship between the numerical-analytic method and the Lyapunov–Schmidt equation. (To avoid misunderstanding, we deliberately speak of the Lyapunov–Schmidt equation and not of the "Lyapunov–Schmidt method" because, as far as we know, the latter notion is not used in the literature.)

Numerous developments and applications of the numerical-analytic method to various classes of boundary-value problems make us think that it would be useful to present the method in the following abstract form:

Let $C(D)$ be the space of continuous vector functions $y\colon D \to \mathbb{R}^n$, $(x_1, \dots, x_m) \to \operatorname{col}(y_1(x_1, \dots, x_m), \dots, y_n(x_1, \dots, x_m))$, $D \subset \mathbb{R}^m$. Denote by $K(D) \subset C(D)$ the space of functions from $C(D)$ that satisfy the additional condition

$$U(y) = \operatorname{col}(U_1(y), \dots, U_n(y)) = 0. \tag{103}$$

Taking into account that the main purpose of the method considered is the investigation of solutions of boundary-value problems, we call (103) an abstract boundary condition. It is obvious that, for example, in the case of periodic boundary conditions, $K(D)$ is the space of periodic functions.

Let $A\colon C(D) \to R_A$ be an operator with domain $D(A) = C(D)$ and range $R_A = A\,C(D) \subset C(D)$, which gives the equation

$$y = Ay. \tag{104}$$

The problem is to find a solution of Eq. (104) that satisfies the abstract boundary condition (103), i.e., a solution $y \in K(D)$.

If A were an operator of the type $A\colon K(D) \to K(D)$, then it would be possible to investigate the solvability of Eq. (104) in the space $K(D)$ by using the general theorems on the fixed points of operators. However, this is impossible in the general case where R_A contains points from $C(D) \setminus K(D)$. In such a situation, along with Eq. (104), one should properly introduce an auxiliary equation

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$$z = \tilde{A} z \tag{105}$$

so that

$$D_{\tilde{A}} = R_{\tilde{A}} = K(D).$$

It is natural that there should be a certain relationship between the operators A and $\tilde{A}$, namely, the following conditions should be satisfied:

(i) every solution of the abstract boundary-value problem (104), (103) is also a solution of Eq. (105);

(ii) every solution $z \in K(D)$ of Eq. (105) that satisfies a certain auxiliary condition

$$Bz = 0 \tag{106}$$

given on solutions of Eq. (105) and satisfied by all solutions of problem (104), (103) is, at the same time, a solution of the original abstract problem (104), (103).

Since $\tilde{A} : K(D) \to K(D)$, the solutions of Eq. (105) under appropriate conditions can be constructed with the use of various iteration processes, in particular, by the method successive approximations

$$z_{k+1} = \tilde{A} z_k, \qquad k = 0, 1, 2, \ldots,$$

in which there is a certain freedom in the choice of the zero approximation z_0. In the numerical-analytic method, this freedom is used to satisfy condition (106). In the method considered, the analytic apparatus of successive approximations of the Picard type is used for finding the limit function $z^* = \lim_{k\to\infty} \tilde{A} z_k$, and the auxiliary (algebraic) equation (106) is solved by numerical methods.

Note that, in a special case of problem (104), (103), it is not difficult to construct both an extension of the operator A, i.e., the operator $\tilde{A}$, and the operator B (see, e.g., [4, Chap. 1]).

Let us now present results concerning the application of the method to autonomous systems of differential equations and impulsive equations.

3.5. Autonomous Systems of Differential Equations

Consider the case where the right-hand side of the system

$$\frac{dx}{dt} = f(t, x), \qquad x, f \in \mathbb{R}^n,$$

is independent of t. It then follows from the main equations of the method (see relations (1) and (2) in [1])

$$x(t, z) = z + \int_0^t f(s, x(s, z))ds - \frac{t}{T}\int_0^T f(s, x(s, z))ds, \tag{107}$$

$$\Delta(z) = \frac{t}{T}\int_0^T f(s, x(s, z))ds = 0, \tag{108}$$

that, in the autonomous case, we have

$$x(t, z) = z, \quad \Delta(z) = f(z) = 0.$$

Therefore, the numerical-analytic method determined by the relation

$$x_m(t, z) = z + \int_0^t \left[f(s, x_{m-1}(s, z)) - \frac{t}{T} \int_0^T f(\tau, x_{m-1}(\tau, z)) d\tau \right] ds$$

(see formula (7) in [1]) "detects" only stationary solutions of the autonomous system

$$\begin{cases} \dfrac{dx}{dt} = f(x), \quad x \in D \subset \mathbb{R}^n, \quad f \in \mathbb{R}^n, \\ x(0) = x(T). \end{cases} \tag{109}$$

In contrast with nonautonomous systems, where the period of the required solution is known, in the autonomous case, in addition to the initial conditions for the periodic solution, one should also determine its period T. Furthermore, the T-periodic solutions $x = x(t)$ of system (109) are not isolated in the extended phase space. The latter makes the theorem on indexes completely inapplicable.

This suggests the idea of selecting a certain solution $x = x_0(t)$ among all hypothetical nontrivial T-periodic solutions $x = x(t + \varphi)$ according to a certain rule and constructing successive approximations on the basis of this solution. In this case, it is also natural to change the system of parameters determining the T-periodic solution and consider the vector $(c_1, c_2, \ldots, c_{n-1}, \mu)$ instead of the vector of initial values $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})$. Here, $(c_1, c_2, \ldots, c_{n-1})$ are the local coordinates on the hypersurface Γ of codimension 1 that is "transversal" to the required solution [an analog of the Poincaré section], and $\mu = T/(2\pi)$ determines the period T of a periodic solution.

Note that, by virtue of the transversality of Γ, the initial parameters $c_1, c_2, \ldots, c_{n-1}, \mu$ are, generally speaking, isolated in the new coordinates. However, since even the existence of a periodic solution is *a priori* unknown, the choice of Γ is neither unique nor evident.

To solve this problem, the following approach was suggested in [5].

It is obvious that every periodic solution intersects each of the sets

$$\Gamma_i = \{x \in \mathbb{R}^n : f_i(x) = 0, \ i = 1, 2, \ldots, n\},$$

where $f = \mathrm{col}(f_1, f_2, \ldots, f_n)$.

If at least one of the sets Γ_j is a hypersurface [as a rule, this is true in applications], we can set

$$\Gamma = \Gamma_j = \{x \in \mathbb{R}^n : x = x_0(c), \ c = (c_1, c_2, \ldots, c_{n-1}) \in D_c\}.$$

The next step is to change the variables

$$x = \Phi(\theta, y), \ \theta = \frac{t}{\mu} = t\frac{2\pi}{T}, \tag{110}$$

which enables one to study problem (109) near the hypothetical solution $x = x_0(t)$. The function $\Phi(\theta, y)$ is 2π-periodic in the first variable and is determined in every special case by the information about system (109) obtained from its first approximation. Moreover, the function

$$\Phi(\theta, y)\colon [0, 2\pi] \times D_1 \to D$$

satisfies all conditions usually imposed on the change of variables, namely, it is continuously differentiable with respect to the variables θ and y and satisfies the condition

$$\det\left[\frac{\partial\Phi(\theta, y)}{\partial y}\right] \neq 0.$$

After the coordinate transformation, at time $\theta = 0$, the hypersurface Γ turns into the hypersurface

$$\bar{\Gamma} = \{y \in \mathbb{R}^n\colon y = y_0(c)\},$$

where $x_0(c) = \Phi(0, y_0(c))$.

As a result of the change of variables (110), Eq. (109) takes the form

$$\frac{dy}{dt} = F(\theta, y, \mu), \tag{111}$$

where

$$F(\theta, y, \mu) = \left[\frac{\partial\Phi(\theta, y)}{\partial y}\right]^{-1} \left[\mu f(\Phi(\theta, y)) - \frac{\partial\Phi(\theta, y)}{\partial y}\right].$$

Assume that the scheme of the method enables one to find the 2π-periodic solution $y = y^*(\theta, c^*, \mu^*) = \lim_{m\to\infty} y_m(\theta, c^*, \mu^*)$ of Eq. (111), where

$$y_m(\theta, c, \mu) = y_0(c) + \int_0^\theta \Big[F(\theta, y_{m-1}(\theta, c, \mu), \mu)$$

$$- \frac{1}{2\pi}\int_0^{2\pi} F(s, y_{m-1}(\theta, c, \mu), \mu)\, ds\Big] d\theta, \qquad m = 1, 2, \ldots, \qquad y_0(\theta, c, \mu) = y_0(c),$$

and (c^*, μ^*) is the root of the determining equation

$$\Delta(c, \mu) = \frac{1}{2\pi}\int_0^{2\pi} F\big(\theta, y^*(\theta, c, \mu), \mu\big) d\theta = 0.$$

Then the function

$$x = x^*(t) = \Phi\left(\frac{t}{\mu^*}, y^*\left(\frac{t}{\mu^*}, c^*, \mu^*\right)\right)$$

is a nontrivial periodic solution of Eq. (109) with period $T^* = 2\pi\mu^*$. Furthermore, $x^*(0) = x_0(c^*)$ and $x^*(t)$ coincides with the hypothetical solution $x = x_0(t)$. It is obvious that $x = x^*(t + \varphi)$ determines a one-parameter family of solutions of Eq. (109).

With the use of the numerical-analytic method, autonomous systems were first studied by Le Lyong Tai [6, 7] and Samoilenko and Le Lyong Tai [8] with the use of the change of variables $x = \Phi(\theta, y) = e^{A\theta}y + f_0(\theta)$ under the assumption that the value of one of the coordinates of the required solution is *a priori* known.

Problem 11. Indicate classes of autonomous systems that, after a transformation of the form (110), admit investigation by the numerical-analytic method.

3.6. Autonomous Systems with Periodic External Influence

First, we establish one general statement concerning the solvability of a nonautonomous periodic boundary-value problem of the form

$$\frac{dx}{dt} = f(t, x), \quad x(0) = x(T), \quad x, f \in \mathbb{R}^n. \tag{112}$$

Definition 1 (Carathéodory conditions). *We say that the function $f(t, x)$ defined on the set $[0, T] \times \Omega$, $\Omega \subset \mathbb{R}^n$, satisfies the Carathéodory conditions if*

(HK1) *for every $x \in \Omega$, the function $f(\cdot, x)$ is measurable;*

(HK2) *for almost ali $t \in [0, T]$, the function $f(t, \cdot)$ is continuous;*

(HK3) *for all $x, y \in \Omega$ and almost all $t \in [0, T]$, the coordinatewise estimates*

$$|f(t, x)| \leq m(t), \quad |f(t, x) - f(t, y)| \leq L(t)|x - y|$$

are true and the components of the vector $m(t)$ and the matrix $L(t)$ are summable on the interval $[0, T]$.

We also assume that the domain Ω satisfies the following condition:

(HK4) $\Omega_\beta := \{z \in \Omega : B(z, \beta) \subset \Omega\} \neq \varnothing$, *where $B(z, \beta)$ denotes the convex set of $x \in \mathbb{R}^n$ such that, coordinatewise, we have*

$$|x - z| \leq \beta,$$

where

$$\beta = \max_{t \in [0, T]} (Km)(t),$$

K is the linear integral operator defined on the (classes of) summable vector functions by the formula

$$(Kx)(t) = \left(1 - \frac{t}{T}\right) \int_0^t x(s) ds + \frac{t}{T} \int_t^T x(s) ds, \tag{113}$$

and

$$|x| = |(x_1, x_2, \dots, x_n)| = \operatorname{col}(|x_1|, |x_2|, \dots, |x_n|),$$

$$\max_t |x(t)| = \operatorname{col}(\max_t |x_1(t)|, \dots, \max_t |x_n(t)|).$$

We introduce a linear transformation $\sim$ of the space of summable functions into itself as follows:

$$\tilde{x}(t) = \int_0^t \left[x(s) - \frac{t}{T}\int_0^T x(u)\,du \right] ds.$$

We also introduce preordering $\rhd$ of the cone of n-dimensional vectors $x = \operatorname{col}(x_1 \dots, x_n)$ with nonnegative components as follows:

$$x \rhd y \iff (\exists i \in \{1, 2, \dots, n\}:\ x_i > y_i).$$

Theorem 20. *Suppose that the function $f(t, x)$ in Eq. (112) satisfies the Carathéodory conditions (HK1)–(HK3), and the domain Ω satisfies condition (HK4). Assume that the spectral radius satisfies the inequality $r(S) < 1$, where S is the composition of the linear integral operator (113) and the multiplication by a variable matrix $L(t)$, i.e.,*

$$Sx = K(Lx). \tag{114}$$

Assume, in addition, that, for some m, the corresponding operator function (see (13) in [1])

$$\Delta_m(z) = \frac{1}{T}\int_0^T f(s, x_m(s, z))\,ds \tag{115}$$

satisfies the following relation on the boundary $\partial\Omega_1$ of a certain subdomain $\Omega_1 \subset \Omega$:

$$|\Delta_m(z)| \rhd \Theta_m(z), \qquad z \in \partial\Omega_1, \tag{116}$$

where

$$\Theta_m(z) = \frac{1}{T}\int_0^T L(t)\,(I - S)^{-1} S^m \left|\tilde{f}(t, z)\right| dt.$$

If, moreover, the Leray–Schauder degree $\deg(\Delta_m, \partial\Omega_1, 0) \neq 0$, then there exists at least one solution $x = x^*(t)$ of the periodic boundary-value problem (112) such that its initial value $x^*(0) \in \Omega_1$.

Proof. As shown in Lemma 17 and Corollary 16 in [9], it follows from the conditions of the theorem that the parametrized integral equation (107) corresponding to the boundary-value problem (112) (see (2) in [1]) has a unique solution for every $z \in \Omega_\beta$, and a single-valued continuous determining function $\Delta(z)$ of the form (108) is given on Ω_β.

In view of our assumptions, the linear deformation of a finite-dimensional vector field $\Delta_m(z)$ of the form (115) into $\Delta(z)$,

$$\Delta(\lambda, z) = \Delta_m(z) + \lambda[\Delta(z) - \Delta_m(z)], \quad \lambda \in [0, 1],$$

is nondegenerate on $\partial\Omega_1$. Indeed, otherwise, there exist $z_0 \in \partial\Omega_1$ and $\lambda_0 \in [0, 1]$ such that

$$\Delta_m(z_0) = -\lambda_0[\Delta(z_0) - \Delta_m(z_0)].$$

Since, according to Lemma 17 in [19],

$$|\Delta(z) - D_m(z)| \leq \Theta_m(z),$$

we have $|\Delta_m(z_0)| \leq |\Delta(z_0) - \Delta_m(z_0)| \leq \Theta_m(z_0)$, which contradicts inequality (116). This completes the proof of Theorem 20.

On the basis of Theorem 20, one can obtain the following result concerning the existence of periodic solutions of perturbed autonomous systems of the form

$$\frac{dx}{dt} = f(x) + g(t), \quad t \in [0, T], \quad x, f, g \in \mathbb{R}^n, \tag{117}$$

where the function $g(t)$ satisfies the condition $g(0) = g(T)$.

Theorem 21 ([9, Theorem 24]). *Suppose that a function $f\colon \Omega \to \mathbb{R}^n$ is continuous, satisfies the Lipschitz condition (HK3) with matrix L, and can be continuously extended from Ω to $\mathbb{R}^n$ with the same Lipschitz matrix. (We denote the corresponding extension by the symbol $\check{f}$.) Also assume that*

$$T\lambda_{\max}(L) < q_0 = 3.4161306\ldots \tag{118}$$

and the following relation holds for every $z \in \partial\Omega$ uniformly in $h \in [0, T]$:

$$|f(z)| \triangleright \Theta_0 := \frac{1}{T} L \int_0^T (I - KL)^{-1} |\tilde{g}_h(t)|\, dt, \tag{119}$$

where, by definition, we set

$$g_h(t) := g(t + h),$$

where the right-hand side contains the periodic extension of the function $g(\cdot)$ to the entire real axis denoted by the same symbol.

Also assume that, in the domain Ω, the function $f(\cdot)$ has a unique zero z_0 of nonzero topological index, i.e.,

$$\operatorname{ind}(f, z_0) \neq 0.$$

Then Eq. (117) has at least one T-periodic solution.

Proof. First, we note that, without loss of generality, one can assume that

$$\int_0^T g(t)dt \equiv 0.$$

Otherwise, the system should be rewritten in the form

$$\frac{dx}{dt} = f(x) + \frac{1}{T}\int_0^T g(s)ds + \left(g(t) - \frac{1}{T}\int_0^T g(s)ds \right).$$

We introduce the one-parameter family of differential equations

$$\frac{dx}{dt} = \check{f}(x) + g_h(x), \quad h \in [0, T]. \tag{120}$$

Let us now clarify the form of the zero determining functions $\check{\Delta}_0(z,h)$ constructed by the zero approximation $x_0(t, z) = z$. It is easy to see that, under our conditions, we have

$$\check{\Delta}_0(z,h) = \frac{1}{T}\int_0^T [\check{f}(z) + g_h(t)]dt = \check{f}(z) + \frac{1}{T}\int_0^T g(t)dt = \check{f}(z) \quad \forall z \in \mathbb{R}^n$$

and

$$\check{\Delta}(z,h) \equiv f(z) \quad \forall h \in [0, T]$$

for all $z \in \partial\Omega$.

Let us apply Theorem 20 to the T-periodic boundary-value problem for Eq. (120) in the case $m = 0$. Assumption (119) implies that relation (116) with $m = 0$ holds on $\partial\Omega$, where K and S are operators of the form (113), (114), because

$$\widetilde{f(z) + g} = \tilde{f}(z) + \tilde{g} = \tilde{g} \quad \forall z \in \partial\Omega.$$

The right-hand side of (116)

$$\frac{1}{T} L \int_0^T (I - S)^{-1} S^m |\tilde{g}_h(t)| dt,$$

is well defined by virtue of assumption (118), which, in this case, is equivalent to the inequality

$$r(S) = \frac{T}{q_0} \lambda_{\max}(L) < 1$$

for operator (114).

Since $\check{\Delta}_0(z,h) \rhd \Theta_0(z) \ \ \forall z \in \partial\Omega$ and $\deg(\check{\Delta}_0(z,h), \partial\Omega, 0) = \operatorname{ind}(f, z_0) \neq 0$, by virtue of Theorem 20 we can conclude that there exists a T-periodic solution $P(t, h)$ of Eq. (120) such that $P(0, h) \in \Omega$.

Let us show that $P(t, 0) : [0, T] \to \Omega$. Indeed, if this is not true, then there is $h^* \in [0, T]$ such that the inclusion $P(0, h^*) \in \partial\Omega$ holds. However, since $P(t, h^*)$ is a T-periodic solution of Eq. (120) for $h = h^*$, this implies that $\Delta(P(0, h^*), h^*) = 0$, which contradicts our assumption. Further, since $P(t, 0) : [0, T] \to \Omega$ and $\check{f}(z) \equiv f(z)\ \ \forall\, z \in \Omega$, we conclude that $P(t, 0)$ is a periodic solution of Eq. (117). The theorem is proved.

***Example** 7.* Consider the problem of 2π-periodic solutions of the equation

$$\frac{d^2x}{dt^2} + \alpha \sin x = e(t), \qquad t \in [0, 1], \quad x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, \tag{121}$$

where $\alpha > 0$ and $e(t) \in L^1[0, 1]$. Equation (121) is equivalent to the system

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dx}{dt} = -\alpha \sin x + e(t). \end{cases}$$

It is easy to verify that this system satisfies the Lipschitz condition (HK3) and the sublinear-growth condition $|f(t, x)| \le L(t)|x| + g(t)$ with the matrix

$$L(t) = L = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}.$$

Therefore, the spectral radius of operator (114) is given by

$$r(S) = \frac{1}{q_0} \lambda_{\max}(L) = \frac{\sqrt{\alpha}}{q_0}$$

and $r(S) < 1$ whenever

$$\alpha < q_0^2 \approx 11.669948\ldots. \tag{122}$$

Under these conditions, the determining function $\Delta(x, y)$ is uniquely defined for all $x, y \in \mathbb{R}^2$. By corresponding calculations, we obtain

$$\Delta_0(x, y) = \begin{pmatrix} y \\ -\alpha \sin x + \bar{e} \end{pmatrix},$$

where

$$\bar{e} := \frac{1}{T} \int_0^T e(t)\, dt.$$

Consequently,

$$\Delta_0(x_0, y_0) = 0 \iff y_0 = 0 \wedge \sin x_0 = \frac{\bar{e}}{\alpha}.$$

If $|\bar{e}| < \alpha$, then the approximate determining equation $\Delta_0(x, y)=0$ has two different roots in $S^1 \times \mathbb{R}$. Without loss of generality, we can assume that $\bar{e} \geq 0$. Then the two different roots are the following:

$$x_1 = \arcsin \frac{\bar{e}}{\alpha}, \qquad x_2 = \pi - \arcsin \frac{\bar{e}}{\alpha}.$$

Note that the validity of the inequality $|\bar{e}| < \alpha$ is also necessary for the existence of a root of Eq. (121). One can verify that this is true by integrating (121) over $[0, 1]$.

In what follows, we consider the neighborhood of the point $(x_1, 0)$. (The second point, $(x_2, 0)$, can be investigated by analogy.) First, we choose Θ_0. For this purpose, we estimate the integral

$$L \int_0^1 (I - KL)^{-1} \begin{pmatrix} 0 \\ |\tilde{e}| \end{pmatrix} dt,$$

where the operator K is defined according to (113), namely,

$$(Kx)(t) = (1-t) \int_0^t x(s)ds + t \int_t^1 x(s)ds.$$

If

$$(I - KL)^{-1} \begin{pmatrix} 0 \\ |\tilde{e}| \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

then

$$\begin{cases} \varphi_1 = K\varphi_2, \\ \varphi_2 = \alpha K \varphi_1 + |\tilde{e}|. \end{cases}$$

The direct calculation yields $\varphi_2 = \alpha K^2 \varphi_2 + |\tilde{e}|$. Hence, $\varphi_2 \leq \alpha |\varphi_2|_0 K^2 \cdot 1 + |\tilde{e}|$. Since

$$\max_{t \in [0,1]} \left[K^2 \cdot 1 \right](t) = \frac{1}{6},$$

we have

$$|\varphi_2|_0 \leq |\tilde{e}|_0 \frac{6}{6-\alpha}$$

whenever $\alpha < 6$.

Note that we have restricted ourselves to the simplest estimates. If necessary, the condition $\alpha < 6$ can be weakened to (122) by more accurate estimation.

Combining the last two formulas, we get

$$\varphi_2 \leq K^2 \cdot 1 |\tilde{e}|_0 \frac{6}{6-\alpha} + |\tilde{e}|.$$

Since

$$\int_0^1 K^2 \cdot 1\, dt = \frac{1}{10},$$

we get

$$\int_0^1 \varphi_2(s)\, ds \le |\tilde{e}|_0 \frac{0.6\alpha}{6-\alpha} + \int_0^1 |\tilde{e}(s)|\, ds.$$

Further, we have

$$\int_0^1 \varphi_1(s)\, ds = \int_0^1 K\varphi_2\, ds \le \int_0^1 \left(\left[K^3 \cdot 1\right] |\tilde{e}|_0 \frac{6}{6-\alpha} + K|\tilde{e}| \right) ds = \int_0^1 K|\tilde{e}|\, ds + |\tilde{e}|_0 \frac{0.2\alpha}{6-\alpha}$$

because

$$\int_0^1 K^3 \cdot 1 = \frac{1}{30}.$$

As a result, we obtain

$$\Theta_0 = \begin{pmatrix} \int_0^1 |\tilde{e}|\, ds + |\tilde{e}|_0 \dfrac{0.6\alpha}{6-\alpha} \\ \int_0^1 \alpha K|\tilde{e}|\, ds + |\tilde{e}|_0 \dfrac{0.2\alpha^2}{6-\alpha} \end{pmatrix}.$$

It is convenient to choose the domain Ω in the form of the rectangle bounded by the lines $x = \pm\pi/2$ and $y = \pm c$ $(c > 0)$. Obviously, we have $\Delta_0(x_0, y_0) \ge \Theta_0$, provided that

$$\alpha - \bar{e} > \alpha \int_0^1 K|\tilde{e}|\, ds + |\tilde{e}|_0 \frac{0.2\alpha^2}{6-\alpha},$$
$$c > \int_0^1 \alpha|\tilde{e}(s)|\, ds + |\tilde{e}|_0 \frac{0.6\alpha}{6-\alpha}. \tag{123}$$

Moreover,

$$\deg(\Delta_0(x,y), \partial\Omega, 0) = \operatorname{ind}(\Delta_0(x,y), (x_1, 0)) = \operatorname{sign}\det\begin{pmatrix} 0 & 1 \\ -\alpha\cos x_1 & 0 \end{pmatrix} \ne 0.$$

Consequently, all conditions of Theorem 21 are satisfied. Therefore, by virtue of this theorem, Eq. (121) has at least one periodic solution with the initial value $(t = 0)$ lying in the indicated rectangle.

Let us consider a particular case of Eq. (121). We set $e(t) = \pi\cos\pi t$. Then $\tilde{e}(t) = \sin\pi t$, $|\tilde{e}|_0 = 1$, and $\bar{e} = 0$. One can verify that inequality (123) holds whenever

$$\alpha < 4.81269\ldots, \qquad c = \frac{0.6\alpha}{6-\alpha} + \frac{2}{\pi}.$$

Therefore, with the periodic perturbation $e(t) = \pi\cos\pi t$, there exists at least one periodic solution $x(t, \alpha)$ of Eq. (121) for all $\alpha < 4.81269\ldots$. This solution is such that

$$\left|\frac{\partial}{\partial t}x(0,\alpha)\right| < \frac{0.6\alpha}{6-\alpha} + \frac{2}{\pi}, \qquad |x(0,\alpha)| < \frac{\pi}{2}.$$

Remark 8. Let us compare the estimates obtained above with those obtained by Granas, Guenther, and Lee [10, Theorem 10.3]. In [10], an equation of the form

$$\frac{d^2x}{dt^2} + g(x)\frac{dx}{dt} + \alpha\sin x = e(t)$$

was considered. Condition (B) in [10, Theorem 10.3] has the form

$$\alpha + \int_0^1 |e(t)|dt < \frac{\pi}{2}.$$

This condition is not satisfied for Eq. (121) with $e(t) = \pi\cos\pi t$. Therefore, Theorem 21 complements, to a certain extent, the results of [10].

3.7. Impulsive Equations

The numerical-analytic method was first applied to impulsive systems in 1967 by Samoilenko [11]. More precisely, a system of the following form was considered:

$$\begin{cases} \dfrac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \dfrac{dx}{dt}\right), \quad x \neq x_0, \\ \Delta \left.\dfrac{dx}{dt}\right|_{x=x_0} = \left.\dfrac{dx}{dt}\right|_{x=x_0+} - \left.\dfrac{dx}{dt}\right|_{x=x_0} = \begin{cases} \varepsilon I\left(\dfrac{dx}{dt}\right), & \left.\dfrac{dx}{dt}\right|_{x=x_0+} \geq 0; \\ 0, & \left.\dfrac{dx}{dt}\right|_{x=x_0-} < 0, \end{cases} \end{cases}$$

Within the framework of the modern classification of impulsive systems (see Samoilenko and Perestyuk [12, 13]), this system is referred to discontinuous dynamical systems.

In [11], with the use of the algorithm of the numerical-analytic method, the applicability of the averaging method to this system was justified and approximating solutions were constructed. Analyzing further applications of the method to impulsive systems, we outline the integrity and completeness of the investigation carried out in [11]. The subsequent applications of the method were more formal in spirit and established conditions of its applicability to systems with pulses at fixed moments of time [or, in fewer works and not always successfully, at nonfixed moments of time].

The paper [14] of Perestyuk and Shovkoplyas (see also Chap. 21 in [12] or Chap. 4.3 in [13]) should be regarded as the first and most important among these works. A somewhat improved version of [14] appeared as Sec. 16 of the monograph [15] by Bainov and Simeonov, where the application of the method to periodic impulsive systems of the form

$$\frac{dx}{dt} = f(t,x), \quad t \neq t_i, \quad 0 \leq t \leq T, \quad x \in \Omega \subset \mathbb{R}^n, \tag{124}$$

$$\Delta x\big|_{t_i} = H_i(x), \quad i = 1,2,\ldots,p, \quad 0 < t_1 < \ldots < t_p < T, \tag{125}$$

$$x(0) = x(T), \tag{126}$$

was considered.

In this case, as is customary, it is assumed that, in the compact domain Ω, the right-hand sides of (124) and (125) are continuous in the collection of variables and satisfy the Lipschitz condition with respect to x, namely,

$$\|f(t,x) - f(t,y)\| \leq K_0 \|x - y\|,$$

$$\|H_i(x) - H_i(y)\| \leq K_i \|x - y\|, \quad i = 1,2,\ldots,p, \quad x, y \in \Omega,$$

and, moreover, there exist constants M_i and a function $m(t) \in L_1[0,T]$ such that

$$\|H_i(x)\| \leq M_i \quad \forall x \in \Omega, \quad i = 1,2,\ldots,p, \quad \sup_{x \in \Omega} \|f(t,x)\| \leq m(t).$$

A solution of problem (124)–(126) was found as the limit of the uniformly convergent sequence of functions

$$x_m(t,z) = z + \int_0^t \left[f(s, x_m(s,z)) - \overline{fH^{(m)}} \right] ds + \sum_{0 < t_i < t} H_i(x_m(t_i, z)), \tag{127}$$

$$x_0(t,z) \equiv z, \quad m = 0,1,2,\ldots,$$

where

$$\overline{fH^{(m)}} = \frac{1}{T} \left[\int_0^T f(s, x_m(s,z))\,ds + \sum_{i=1}^p H_i(x_m(t_i, z)) \right].$$

The following theorem was proved:

Theorem 22. *Suppose that $m(t) \equiv M = M_i$ and the following conditions are satisfied:*

(I1) there exists a nonempty closed set Ω_β that belongs to Ω together with its $\beta = TM/2 + pM$-neighborhood;

(I2) the following inequality is true:

$$\frac{K_0 T}{3} + pK_1 + \frac{pTK_0K_1}{6} < 1. \tag{128}$$

If system (124)–(126) has a T-periodic solution $x = \varphi(t,z)$ passing through a point $z \in \Omega_\beta$ at time $t = 0$, then the solution is the limit of the uniformly convergent sequence of periodic functions

$$\varphi(t, z) = \lim_{m\to\infty} x_m(t,z) = x^*(t,z)$$

defined by relation (127).

Remark 9. In fact, in [12, 13], more restrictive conditions were considered, namely, the number $\beta' = TM/2 + 2pM$ was used instead of β and, instead of inequality (128), it was assumed that

$$\frac{TK_0}{3} + 2pK_1 + \frac{pTK_0K_1}{3} < 1.$$

In [15], more accurate computations were performed, which enabled one to weaken the requirements imposed on the impulsive system. More precisely, the following, even less restrictive, conditions were suggested in [15]:

$$\beta'' = \frac{TM}{2} + MQ,$$

$$-1 + \frac{TK_0}{3} + QK_1 < K_0K_1\left(\frac{TQ}{3} - S\right) < 1,$$

where

$$Q = \sup\left\{\left(1 - \frac{t}{T}\right)i[0,t] + \frac{t}{T}i[t,T]: t\in[0,t]\right\},$$

$$S = \sup\left\{\left(1 - \frac{t}{T}\right)\sum_{0<\tau_k<t}\alpha(\tau_k) + \frac{t}{T}\sum_{t<\tau_k<T}\alpha(\tau_k): t\in[0,t]\right\}.$$

Theorem 22 made it possible to reduce the problem of existence of T-periodic solutions to the problem of the roots of the determining function

$$\Delta(z) = \frac{1}{T}\left[\int_0^t f(s, x^*(s,z))ds + \sum_{i=1}^{p} H_i(x^*(t_i, z))\right].$$

The methods for the investigation of the determining equation $\Delta(z) = 0$ presented in [12] do not significantly differ from those in [11, 16] and, therefore, we do not dwell on them.

As applications of Theorem 22, the following scalar impulsive systems $(x \in \mathbb{R})$ in the standard, in the sense of Bogolyubov, form were also considered in [12]:

$$\frac{dx}{dt} = \varepsilon f(t,x), \quad t \neq t_i, \quad 0 \le t \le T, \tag{129}$$

$$\Delta x\big|_{t_i} = \varepsilon H_i(x), \quad i = 1,2,\ldots,p, \quad 0 < t_1 < \ldots < t_p < T. \tag{130}$$

Theorem 23. *Suppose that the right-hand sides of the T-periodic impulsive system (129), (130) satisfy the same requirements as in system (124), (125). If the averaged system*

$$\frac{dy}{dt} = \varepsilon f_0(y) \equiv \frac{\varepsilon}{T}\left[\int_0^T f(s,y)ds + \sum_{i=1}^{p} H_i(y)\right]$$

has an isolated equilibrium state $y = y_0$, $f_0(y_0) = 0$, *and the index of the mapping* $f_0(y)$ *at the point* y_0 *differs from zero, then, for sufficiently small values of the parameter* ε, *the system of equations (129), (130) has a T-periodic solution* $x = \varphi(t, \varepsilon)$ *such that* $\lim_{\varepsilon \to 0} \varphi(t, \varepsilon) = y_0$.

One can easily notice a certain "asymmetry" in the iterative relation (127) caused by a certain lack of regard for the "correction" of the impulsive part. Indeed, only the integrand is corrected in (127). One should expect that, suggesting a new modification of the method according to its general idea [see Sec. 3.4] such that the above-mentioned "asymmetry" disappears, we can improve the convergence of the method in certain cases. The results of Trofimchuk [17] show the validity of such assumptions.

For the determination of solutions of problem (124)–(126), the following recurrence sequence of functions $x_m(t,z)$ satisfying the boundary condition (126) was constructed in [17]:

$$x_{m+1}(t,z) = z + \int_0^T \left[f(s, x_m(s,z) - K_0 \, \overline{f H^{(m)}} \right] ds$$

$$+ \sum_{0<t_i<t} \left[H_i(x_m(t_i,z)) - K_i \overline{f H^{(m)}} \right], \qquad m = 0, 1, \ldots, \qquad x_0(t,z) \equiv z, \tag{131}$$

where the average $\overline{f H^{(m)}}$ is determined with regard for the weight coefficients as

$$\overline{f H^{(m)}} = \frac{1}{\tau} \left[\int_0^T f(s, x_m(s,z)) ds + \sum_{i=1}^{p} H_i(x_m(t_i, z)) \right]$$

for $\tau = K_0 T + K_1 + \ldots + K_p$. (We assume that the constants K_i are not simultaneously equal to zero because, otherwise, the problem becomes trivial.)

As is seen, the right-hand side of (131) already has the indicated symmetry property. Furthermore, the practical realization of the iteration process (131), in fact, is not more complicated than that of scheme (127) because the main auxiliary calculations (the computation of the averages $\overline{f H^{(m)}}$ and $\overline{f H^{m}}$) are the same in both cases.

We introduce a piecewise-continuous function $M(t)\colon I \to \mathbb{R}_+$ as follows:

$$M(t) = \left[\frac{1 - \left(K_0 t + \sum_{t_i<t} K_i\right)}{\tau} \right] \left[\int_0^T m(u)du + \sum_{t_i<t} M_i \right] + \left[\frac{K_0 t + \sum_{t_i<t} K_i}{\tau} \right] \left[\int_t^T m(u)du + \sum_{t_i<t} M_i \right].$$

Let $\beta^* = \sup_{t \in I} M(t)$. We have

$$\beta^* \le \int_0^T m(u)du + \sum_i M_i$$

because $(a - x)y + x(b - y) \le ab$ for arbitrary $x \in [0, a]$ and $y \in [0, b]$. Assume that the set Ω_{β^*} is nonempty. The functions $x_m(t,z)$ belong to Ω for $z \in \Omega_{\beta^*}$ because

$$\| x_m(t,z) - z \| \le M(t) \le \beta^*. \tag{132}$$

Prior to the formulation of the result on the convergence of scheme (131), we introduce the following notation:

$$\alpha_1^*(t) = \left[1 - \frac{\left(K_0 t + \sum_{t_i < t} K_i\right)}{\tau}\right]\left(K_0 t + \sum_{t_i < t} K_i\right) \le \frac{\tau}{2},$$

$$\gamma = \frac{1}{\tau}\sum_{i=1}^{p} \max\left\{0, \left[2\left(K_0 t_i + \sum_{j=1}^{i-1} K_j\right) - \tau + \frac{2}{3}K_i\right]K_i^2\right\} \le \sum_{j=1}^{p} K_j^2.$$

Theorem 24. *Suppose that the set* Ω_{β^*} *is nonempty and the inequality* $q = \tau/3 + \gamma < 1$ *is satisfied. Then the integral equation corresponding to formula (131) has a unique solution* $x^*(t, z)$ *in the domain considered, and it can be found by the method of successive approximations (131). Furthermore, the following estimates are true:*

$$\left\| x^*(t, z) - x_m(t, z) \right\| \le \beta^*\left[q\alpha_1^*(t) + \gamma\right] q^{m-2}(1-q)^{-1}, \qquad m \ge 2,$$

$$\left\| x^*(t, z) - z \right\| \le M(t) \le \beta^*,$$

$$\left\| x^*(t, z) - x_1(t, z) \right\| \le \min\left\{2\beta^*, \beta^*(q\alpha_1^*(t) + \gamma)(1-q)^{-1}\right\}.$$

The function $x^*(t, z^*)$ *is a solution of the boundary-value problem (124)–(126) whenever the continuous function*

$$\Delta^*(z) = \frac{K_0}{\tau}\left[\int_0^T f(s, x^*(s, z))\,ds + \sum_{i=1}^{p} H_i(x^*(t_i, z))\right]$$

vanishes at the point $z = z^*$.

It is interesting to note that, later, considerations similar to those presented above were independently developed by Samoilenko and Teplinskii [18]. In [18], the periodic problem of control for impulsive systems was studied, which required the examination of an integral equation of the form

$$x(t, z) = z + \int_0^t [\,f(s, x(s,z) - u_1\,]\,ds + \sum_{0 < t_i < t} [\,H_i(x(s,z) - u_2\,]. \tag{133}$$

In this connection, versions of "pulse control" with

$$u_1 = 0, \qquad u_2 = \frac{1}{p}\left[\int_0^T f(s, x(s, z))\,ds + \sum_{i=1}^{p} H_i(x(t_i, z))\right],$$

"differential control" with

$$u_1 = \frac{1}{T}\left[\int_0^T f(s, x(s, z))\,ds + \sum_{i=1}^{p} H_i(x(t_i, z))\right], \qquad u_2 = 0,$$

and "mixed control" with

$$u_1 = \frac{1}{T}\left[\int_0^T f(s, x(s,z))ds + \sum_{i=1}^{p} H_i(x(t_i,z))\right],$$

$$u_2 = \frac{1}{p}\left[\int_0^T f(s, x(s,z))ds + \sum_{i=1}^{p} H_i(x(t_i,z))\right]$$

were considered. It is clear that the first two versions can be used for the determination of a T-periodic solution of an impulsive system. This was carried out in [18], again under the condition of control over the differential part. It should be noted that less accurate estimates of convergence were obtained there. For instance, instead of (128), it was assumed that

$$\frac{K_0 T}{2} + 2pK_1 < 1, \quad K_0 = K_1.$$

Note that the symmetry properties of impulsive systems were used only in [18].

The following statement is true:

Theorem 25. *Let the T-periodic system of equations*

$$\begin{cases} \dfrac{dx}{dt} = \varepsilon f(t,x), & t \neq t_i, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\ \Delta x|_{t_i} = \varepsilon H(t)|_{t_i}, & i \in Z, \end{cases}$$

be such that $H(-t) = -H(t)$, $f(t,x) = -f(-t,x)$, and, on the interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$, the moments of pulse influence are located symmetrically with respect to zero.

Then every point $z \in \Omega_\beta$ is the initial value of a T-periodic solution of this system.

Note that the application of the method to the investigation of impulsive systems with nonfixed moments of pulse influence is justified in [12]. This monograph indicates the possibility of such an application; however, the important technical Lemma 21.3 in [12], as well as the corresponding statements in [19], contains certain inaccuracies. We do not discuss these results here.

Problem 12. Correct the inaccuracy in Lemma 21.3 of [12] for systems with pulse influence at fixed moments of time of the form

$$\frac{dx}{dt} = f(t,x), \quad t \neq \tau_i(x),$$

$$\Delta x|_{t=\tau_i(x)} = I_i(x).$$

Among the other works on this subject, one should also mention the papers of Akhmetov [20] and Akhmetov and Perestyuk [21]. In [21], the method was applied to the investigation of a periodic impulsive system of the form

$$\frac{dx}{dt} = Ax + f(t,x,y), \quad t \neq t_i,$$

$$\frac{dy}{dt} = g(t,x,y), \quad t \neq t_i,$$

$$\Delta x\Big|_{t=t_i} = Bx + I_i^{(1)}(x,y), \qquad \Delta y\Big|_{t=t_i} = I_i^{(2)}(x,y),$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A and B are real $n \times n$ matrices, and $\det(E+B) \neq 0$. The results obtained in [21] complement and generalize the investigations of [14].

We also mention the works of Hristova and Bainov [19, 22, 23]. In [19], problem (124), (125) is studied with the use of the numerical-analytic method with the two-point boundary conditions

$$Ax(0) + Cx(T) = d, \quad A, C \in \mathbb{R}^{n\times n}, \quad \det(C) \neq 0.$$

In these works, systems with fixed and nonfixed moments of pulse influence are considered. We have already noted the inaccuracy of the lemma from [19] in the case of nonfixed moments of pulse influence. In [23], an impulsive delay system of the form

$$\frac{dx}{dt} = f(t, x(t), x(t-h)), \qquad t \neq t_i,$$

$$\Delta x\Big|_{t=t_i} = I_i(x(t_i)),$$

where $x \in \mathbb{R}^n$ and the sequence of points $\{t_i \in \mathbb{R},\ t_{i+1} > t_i\}$, $i \in \mathbb{Z}$, is fixed, was considered. In [23], certain conditions were indicated under which a method for finding periodic solutions based on the ideas of the numerical-analytic method and the Galerkin method can be applied to this problem. Simultaneously with [19, 22, 23], the Sarafova and Bainov [24] applied the method to the investigation of periodic solutions of the impulsive system of integro-differential equations

$$\frac{dx}{dt} = f(t, x, \int_0^t \varphi(t,s,x(s))\,ds), \qquad t \neq t_i(x),$$

$$\Delta x\Big|_{t=t_i(x)} = I_i(x), \qquad i = 0, \pm 1, \pm 2, \ldots .$$

Existence theorems were proved for both cases $t = t_i$ and $t = t_i(x)$.

We also mention the papers of Gul'ka [25] and Butris [26], where the application of the ideas of the method to integro-differential equations with pulse influence is considered. Thus, the first part of [26] is devoted to the determination of periodic solutions of nonlinear [globally Lipschitzian in the domain of definition] systems of differential operator equations with pulse influence of the form

$$\frac{dx}{dt} = f(t, x, Ax), \qquad t \neq t_i,$$

$$\Delta x\Big|_{t=t_i} = I_i(x, Ax).$$

The indicated work mostly follows the ideas of [14]. It should be noted that rather restrictive conditions are imposed on the operator A, e.g.,

$$(Ax)(t) = (Ax)(t+\tau), \qquad \|(Ax)(t) - (Ay)(t)\| \leq q\|x(t) - y(t)\|.$$

This disadvantage was partially eliminated in the second part of that work [26] where, in a similar way, the system

$$\frac{dx}{dt} = f(t, x, \int_{a(t)}^{b(t)} g(s, x(s))ds), \qquad t \neq t_i,$$

$$\Delta x\big|_{t=t_i} = I_i(x, \int_{a(t_i)}^{b(t_i)} g(s, x(s))ds)$$

with periodic functions $a(t)$ and $b(t)$ was investigated.

In [25], a combination of the ideas of the method with the results obtained by Tsidylo and Gul'ka in [27] enabled one to prove, under certain conditions, the existence of a unique T-periodic solution of the system

$$\frac{dx}{dt} = f\left(t, x, \int_0^t \varphi(t, s, x(s))ds + \sum_{0<s_j<t} I_j(x(s_j - 0)) \right), \qquad t \neq t_i,$$

$$\Delta x\big|_{t=t_i} = I_i(x), \qquad i, j = 0, 1, \ldots .$$

Finally, we note that the results of [11] were generalized by Samoilenko and Perestyuk in [28].

Problem 13. Justify the application of the method to problem (124)–(126) in the case where the constants K_0 and K_i in the corresponding Lipschitz conditions are replaced by matrices.

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