Ukrainian Mathematical Journal, Vol. 50, No. 12, 1998

STRONG SUMMABILITY OF MULTIPLE FOURIER SERIES AND SIDON-TYPE INEQUALITIES

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UDC 517.518.476

We study different versions of strong summation of N-dimensional Fourier series over polyhedrons and related estimates for integral norms of linear means of the Dirichlet kernels (Sidon-type inequalities).

Let

$$f \in L(T^N), \quad \|f\|_L = \int_{T^N} |f(u)|\,du < \infty, \quad T^N = [-\pi, \pi)^N.$$

We define the Fourier series of this function

$$f(x) \sim \sum_{k \in Z^N} \hat{f}(k) e^{ikx}, \quad kx = k_1 x_1 + \ldots + k_N x_N, \tag{1}$$

where

$$\hat{f}(k) = (2\pi)^{-N} \int_{T^N} f(u) e^{-iku} du$$

and Z^N is the integer lattice in R^N. In contrast to the one-dimensional case, there is no canonical method for determining partial sums of series (1) in a multiple case. Let V be a closed bounded domain in R^N containing the origin O inside it. For $n > 0$, we set

$$S_{nV} f(x) = \sum_{k \in nV \cap Z^N} \hat{f}(k) e^{ikx}$$

($nV = \{x \in R^N : x/n \in V\}$ is a homothet of V). It is natural to define the sum of series (1) as the limit (if it exists) of partial sums $S_{nV} f$ as $n \to \infty$, where

$$S_{nV} f = (2\pi)^{-N} f * D_{nV}$$

is a convolution of the function f and the Dirichlet kernel

$$D_{nV}(x) = \sum_{k \in nV \cap Z^N} e^{ikx}$$

corresponding to the set V.

Institute of Applied Mathematics and Mechanics, Ukrainian Academy of Sciences, Donetsk. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 50, No. 12, pp. 1630–1635, December, 1998. Original article submitted December 16, 1996.

 0041-5995/98/5012-1860 $20.00

In what follows, V is a closed bounded polyhedron in R^N star-shaped with respect to O, $O \in \operatorname{int} V$, and is such that the extension of any face of it does not pass through O, and W is the set of polyhedrons with the indicated properties. In [1], Podkorytov studied sufficient conditions of regularity of linear methods of summation in $C(T^N)$ over polyhedrons of the set W and showed that all conditions in the definition of W are necessary.

Denote by W_b the subset of W defined as follows: A collection of real numbers $(\alpha_1, \ldots, \alpha_N)$ is "poorly" approximated (by rational numbers) if the inequality

$$\| \alpha_1 k_1 + \ldots + \alpha_N k_N \| < a^{-N-1}, \qquad a = \max_{1 \le i \le N} |k_i|, \tag{2}$$

has at most finitely many solutions in integer numbers $k = (k_1, \ldots, k_N)$ ($\|x\|$ is the distance from x to the nearest integer number). A polyhedron V belongs to the set W_b if the coefficients in the equations of hyperplanes $\sum \alpha_i x_i - 1 = 0$ that determine it form collections of "poorly" approximated numbers. Note that the set of collections α for which inequality (2) has infinitely many solutions in integer numbers is the set of Lebesgue measure zero in R^N [2, p. 36], i.e., almost all collections α are "poorly" approximated.

Denote by W_a the subset of W_b that consists of polyhedrons for which α are collections of algebraic numbers.

The following theorems on the strong summation of series (1) are true:

Theorem 1. *Let a polyhedron $V \in W_b$. Then, for arbitrary $f \in C(T^N)$, $p \ge 1$, and $n \in \mathbb{N}$, we have*

$$(n+1)^{-1} \sum_{l=0}^{n} \left| S_{lV} f(0) \right|^p \le c^p p^{Np} \| f \|_\infty^p, \tag{3}$$

where $c = c(N, V) > 0$ is a constant.

Let Q be the set of functions φ defined on the semiaxis $[0, \infty)$, bounded and continuous at zero, and such that $\varphi(0) = 0$. For $V \in W_b$ and $n \in \mathbb{N}$, we set

$$h_n(f, \varphi, V, x) = (n+1)^{-1} \sum_{l=0}^{n} \varphi\left(\left| f(x) - S_{lV} f(x) \right| \right).$$

Theorem 2. *Let $V \in W_b$ be a polyhedron.*

(i) If $\varphi \in Q$ is such that

$$\limsup_{u \to \infty} \varphi(u) u^{-1/N} < \infty,$$

then the equality

$$\lim_{n \to \infty} h_n(f, \exp(\varphi) - 1, V, x) = 0$$

holds for any $f \in C(T^N)$ uniformly in x.

(ii) If

$$\limsup_{u\to\infty} \varphi(u)u^{-1/N} = \infty,$$

then there exists a function $F \in C(T^N)$ *for which*

$$\limsup_{n\to\infty} h_n(f, \exp(\varphi)-1, V, 0) = \infty.$$

Theorem 3. *Let* $V \in W_b$ *be a polyhedron,* $1 \le p < \infty$, *and let* $\{\nu_j\}$ *be an increasing sequence of natural numbers. In order that the inequality*

$$(n+1)^{-1}\sum_{j=0}^{n}\left|S_{\nu_j V}f(0)\right|^p \le c\|f\|_\infty^p, \qquad n \in \mathbb{N}, \tag{4}$$

hold for any $f \in C(T^N)$, *it is necessary and, if* $\{\nu_j\}$ *is a convex sequence* $\{\nu_{j+1}-\nu_j \uparrow\}$, *sufficient that*

$$\log \nu_j \le c_1 j^{\min\left(\frac{1}{2N},\frac{1}{pN}\right)}. \tag{5}$$

For $V \in W_a$, Theorems 1–3 were proved in [3, 4] (see also the history of the problem and detailed bibliography therein). As proved in these works, assertion (ii) of Theorem 2 and the necessity of condition (5) in Theorem 3 hold for any polyhedron V, $0 \in \operatorname{int} V$, and follow from the estimate of the norm of the operator of taking a partial sum in $C(T^N)$ [5]

$$\sup_{|f|\le 1}\|S_{nV}f\|_\infty = (2\pi)^{-N}\int_{T^N}|D_{nV}(x)|dx \asymp \log^N n \tag{6}$$

and, moreover, $\{\nu_j\}$ is not necessarily a convex sequence. The proof of sufficiency in Theorems 1–3 is analogous to the case $V \in W_a$ because polyhedrons from W_b (as well as polyhedrons from W_a) possess the following property:

Lemma 1. *For an arbitrary polyhedron* $V \in W_b$, *there exists a constant* $d = d(V) > 0$ *such that the sets* $(n + dn^{-N-1})V \setminus nV$, $n \in \mathbb{N}$, *do not contain the points of the lattice* Z^N.

Proof. If $\Gamma = \{x \in R^N\colon \sum \alpha_i x_i - 1 = 0\}$ is one of the hyperplanes that determine the polyhedron V, then, since inequality (2) has at most finitely many solutions in the integer numbers $k \in Z^N$, we have either $\sum \alpha_i k_i - n = 0$ for certain $k \in Z^N$ and $n \in \mathbb{N}$ (α is a collection of rational numbers), or there exists a constant $c_\Gamma > 0$ such that

$$\left|\sum \alpha_i k_i - n\right| > c_\Gamma a^{-N-1}, \qquad a = \max|k_i|,$$

for any $n \in \mathbb{N}$ and $k \in Z^N$. For $k \in (n+1)V \setminus nV$, we have $a \le cn$ for certain $c > 0$. Since the distance from the point k to the face $n\Gamma$ is equal to $\left|\sum \alpha_i k_i - n\right|\left(\sum \alpha_i^2\right)^{-1/2}$, we set

$$d = c^{-N-1} \min_{\Gamma} \left(c_{\Gamma} \left(\sum \alpha_i^2 \right)^{-1/2} \right).$$

Theorem 2 implies that, for arbitrary $A > 0$ and $f \in C(T^N)$, we have

$$\frac{1}{n} \sum_{l=0}^{n} \exp\left(A \left| f(x) - S_{lV} f(x) \right|^{1/N} \right) - 1 \to 0 \quad \text{as} \quad n \to \infty$$

uniformly on T^N.

Lemma 2. *Let $V \in W_b$. There exist constants $A > 0$ and $c > 0$ such that*

$$\sup_{|f| \le 1} \frac{1}{n} \sum_{l=1}^{n} \exp\left(A \left| S_{lV} f(0) \right|^{1/N} \right) \le c, \qquad n \in \mathbb{N}. \tag{7}$$

Proof. We expand the function e^z in a series and use estimate (3) and the Stirling formula. Then

$$\sup_{|f| \le 1} \frac{1}{n} \sum_{l=1}^{n} \exp\left(A \left| S_{lV} f(0) \right|^{1/N} \right) = \sup_{|f| \le 1} \frac{1}{n} \sum_{p=0}^{\infty} \frac{A^p}{p!} \sum_{l=1}^{n} \left| S_{lV} f(0) \right|^{p/N} \le c_1 \sum_{p=0}^{\infty} (Aec^{1/N})^p.$$

It remains to choose $A < \dfrac{1}{ec^{1/N}}$.

For $N = 1$, the assertions presented below are called Sidon-type inequalities [6].

Theorem 4. *Let $V \in W_b$ and let $\{a_j\}$ be a sequence of real numbers. Then*

$$\int_{T^N} \left| \sum_{j=1}^{n} a_j D_{jV}(x) \right| dx \le c \sum_{j=1}^{n} |a_j| \left[1 + \left(\log^+ \frac{|a_j|}{n^{-1} \sum_{j=1}^{n} |a_j|} \right)^N \right]. \tag{8}$$

Proof. For $N = 1$ and $V = [-1, 1]$, this theorem was proved by Fridli [7]. We prove this theorem for $N > 1$. Let L_Φ be the Orlicz space [8] of functions defined on the segment $[0, 1]$ that is determined by the $\mathcal{N}$-function

$$\Phi(u) = \int_0^{|u|} \varphi(t)\, dt,$$

where

$$\varphi(t) = \begin{cases} \left(\dfrac{Ae}{N-1} \right)^{N-1} \dfrac{t}{t_N} & \text{for} \quad 0 \le t < t_N, \\ e^{At^{1/N}} t^{(1-N)/N} & \text{for} \quad t \ge t_N, \end{cases}$$

A is the constant from Lemma 2, $t_N = \left(\dfrac{N-1}{A} \right)^N$, φ is a strictly increasing function, and $\varphi(0) = 0$. Let ψ be the function inverse to φ. Then

$$\Psi(u) = \int_0^{|u|} \psi(t)\,dt$$

is the $\mathcal{N}$-function complementary to Φ. Since $\psi(t) = O(\ln^N t)$ for sufficiently large t and $\psi(t) = ct$ for t close to zero, there exists a constant $c_1 = c_1(N, A)$ such that

$$\Psi(u) \le c_1 |u| \left[1 + \left(\log^+ |u|\right)^N\right]. \tag{9}$$

For all u, we have $\Phi(u) = O\left(e^{A|u|^{1/N}}\right)$. Following [6], for a collection of real numbers $(a_1, \ldots, a_n)$ we set

$$\Gamma(a_1, \ldots, a_n) = \sum_{j=1}^{n} a_j \chi_{[(j-1)n^{-1},\, jn^{-1}]},$$

where χ_A is the characteristic function of the set $A \subset R$. Let

$${}_n f(0) = \sum_{j=1}^{n} S_{jV} f(0) \chi_{[(j-1)n^{-1},\, jn^{-1}]}$$

for $f \in C(T^N)$. Then

$$\frac{1}{n} \int_{T^N} \left| \sum_{j=1}^{n} a_j D_{jV}(x) \right| dx = \sup_{|f| \le 1} \frac{1}{n} \left| \sum_{j=1}^{n} a_j S_{jV} f(0) \right| = \sup_{|f| \le 1} \int_0^1 |\Gamma(a_1, \ldots, a_n) I_n f(0)|\,dt.$$

Applying the Hölder inequality [8, p. 91], we obtain

$$\frac{1}{n} \int_{T^N} \left| \sum_{j=1}^{n} a_j D_{jV}(x) \right| dx \le \|\Gamma(a_1, \ldots, a_n)\|_{\Psi} \sup_{|f| \le 1} \| I_n f(0) \|_{\Phi}. \tag{10}$$

The definition of norm in the space L_Φ implies that

$$\| g \|_{\Phi} \le \int_0^1 \Phi(g)\,dt + 1.$$

Therefore,

$$\sup_{|f| \le 1} \| I_n f(0) \|_{\Phi} \le \sup_{|f| \le 1} \int_0^1 \Phi(I_n f(0))\,dt + 1 \le c \sup_{|f| \le 1} \int_0^1 \exp\left(A |I_n f(0)|^{1/N}\right) dt + 1$$

$$= c \sup_{|f| \le 1} \frac{1}{n} \sum_{j=1}^{n} \exp\left(A \left| S_{jV} f(0) \right|^{1/N}\right) + 1.$$

According to Lemma 2, the last sum is finite for any n. Consequently, it follows from (10) that

$$\frac{1}{n}\int_{T^N}\left|\sum_{j=1}^{n} a_j D_{jV}(x)\right|dx \le c\|\Gamma(a_1,\dots,a_n)\|_{\psi} \le c\int_0^1 \Psi(\Gamma(a_1,\dots,a_n))dt + c.$$

In view of (9), the last inequality yields

$$\frac{1}{n}\int_{T^N}\left|\sum_{j=1}^{n} a_j D_{jV}(x)\right|dx \le c_1\int_0^1 |\Gamma|\left[1+\left(\log^+|\Gamma|\right)^N\right]dt + c_1 = \frac{c_1}{n}\sum_{j=1}^{n}|a_j|\left[1+\left(\log^+|a_j|\right)^N\right] + c_1. \tag{11}$$

Denote

$$\|a_n\|_1 = n^{-1}\sum_{j=1}^{n}|a_j|.$$

Then, replacing a_j by $a_j/\|a_n\|_1$ in (11), we obtain estimate (8).

Theorem 5. *Let* $V \in W_b$, $1 < q \le 2$, *let* $\{a_j\}$ *be a sequence of real numbers, and let* $\{\nu_j\}$ *be an increasing sequence of natural numbers. In order that the inequality*

$$\frac{1}{n}\int_{T^N}\left|\sum_{j=1}^{n} a_j D_{\nu_j V}(x)\right|dx \le c\left(\frac{1}{n}\sum_{j=1}^{n}|a_j|^q\right)^{1/q} \tag{12}$$

hold for any $n \in \mathbb{N}$, *it is necessary and, if* $\{\nu_j\}$ *is convex, sufficient that*

$$\log \nu_j \le c_1 j^{(q-1)/qN}.$$

Proof. To prove the sufficiency, we apply the Hölder inequality to the sum

$$\sup_{|f|\le 1}\frac{1}{n}\left|\sum_{j=1}^{n} a_j S_{\nu_j V} f(0)\right|$$

and then use Theorem 3 [estimate (4)] with $p = \dfrac{q}{q-1}$.

To prove the necessity, we set $a_1 = \ldots = a_{n-1} = 0$ and $a_n = 1$ in (11). Then (11) yields

$$\int_{T^N}\left|D_{\nu_j V}(x)\right|dx \le cn^{1-\frac{1}{q}}$$

and the necessity of the condition of Theorem 5 follows from (6).

For $N = 1$ and $\nu_j = j$, inequality (11) (in fact, obtained in [9]) was proved in [10].

Finally, note that the duality between the strong summability and Sidon-type inequalities for $N = 1$ was established in [6]. Without changing the proof, one can obtain an analogous result in a multiple case for any bounded set V, $0 \in \operatorname{int} V$.

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