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INVESTIGATIONS OF DNEPROPETROVSK MATHEMATICIANS RELATED TO INEQUALITIES FOR DERIVATIVES OF PERIODIC FUNCTIONS AND THEIR APPLICATIONS

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We present a survey of investigations of Dnepropetrovsk mathematicians related to Kolmogorov-type exact inequalities for norms of intermediate derivatives of periodic functions and their applications in approximation theory.

1. Introduction

Let G be a Lebesgue measurable subset of R such that $\mu G > 0$. Consider the spaces $L_p(G)$, $0 < p \le \infty$, of measurable functions $x: G \to R$ such that

$$\|x\|_p = \|x\|_{L_p(G)} := \left\{ \int_G |x(t)|^p \, dt \right\}^{1/p} < \infty \quad \text{if} \quad 0 < p < \infty$$

or

$$\|x\|_\infty = \|x\|_{L_\infty(G)} := \sup_{t \in G} \operatorname{vrai} |x(t)| < \infty \quad \text{if} \quad p = \infty.$$

In what follows, G denotes either the number axis R, or the semiaxis R_+, or a finite interval I, or a unit circle T realized as the segment $[0, 2\pi]$ with identified endpoints.

Denote by $L_s^r(G)$, $r \in \mathbf{N}$, $1 \le s \le \infty$, the space of functions x that have the locally absolutely continuous derivative $x^{(r-1)}$ and are such that $x^{(r)} \in L_s(G)$. For $1 \le p \le \infty$, we set $L_{p,s}^r(G) = L_p(G) \cap L_s^r(G)$. Note that if $G = I$ or $G = T$, then $L_s^r(G) \subset L_p(G)$ for any p. In some cases, instead of $L_p^r(\mathbf{T})$, we write L_p^r.

An important role in numerous problems of analysis and its applications is played by inequalities for norms of intermediate derivatives of functions $x \in L_{p,s}^r(G)$ of the form

$$\left\| x^{(k)} \right\|_q \le \Phi\left(\|x\|_p, \left\| x^{(r)} \right\|_s \right), \tag{1}$$

where $\Phi: R^2 \to R_+$ is a certain fixed function. For the first time, inequalities of this type were considered by Hardy and Littlewood [1] in the case $p = q = s = \infty$. It is customary to write these inequalities in the additive form, namely,

$$\left\| x^{(k)} \right\|_q \le A \|x\|_p + B \left\| x^{(r)} \right\|_s, \tag{2}$$

or in the multiplicative form:

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$$\left\| x^{(k)} \right\|_q \leq K \| x \|_p^{\alpha} B \left\| x^{(r)} \right\|_s^{\beta}. \tag{3}$$

The problem of derivation of inequalities of the type (2) or (3) with unimprovable constants (exact inequalities) was studied by many mathematicians.

The first exact results were obtained by Landau [2] (for $x \in L^2_{\infty,\infty}(R_+)$ or $x \in L^2_\infty(I)$, $k = 1$) and Hadamard [3] (for $x \in L^2_{\infty,\infty}(R)$, $k = 1$).

One of the first complete and outstanding results in this direction was obtained by Kolmogorov (see, e.g., [4]). Later, inequalities of this type were named "Kolmogorov-type inequalities." Kolmogorov proved the following statement: If $x \in L^2_{\infty,\infty}(R)$, then, for any $k \in \mathbf{N}$, $k < r$, we have

$$\left\| x^{(k)} \right\|_\infty \leq \frac{\| \varphi_{r-k} \|_\infty}{\| \varphi_r \|_\infty^{1-k/r}} \| x \|_\infty^{1-k/r} \left\| x^{(r)} \right\|_\infty^{k/r}, \tag{4}$$

where φ_r is the rth periodic integral of the function $\varphi_0(t) = \operatorname{sgn} \sin x$ with mean value zero on the period (note that, for $2 < r < 5$ and $r = 5$, $k = 2$, this fact was proved by Shilov [5]). Inequality (4) turns into the equality for any function of the form $\varphi_r(\lambda t)$, where $\lambda \in R$, $\lambda > 0$.

To establish inequality (4), Kolmogorov proved a theorem on the comparison of derivatives. Later, this theorem proved to be very useful for the the exact solution of numerous extremal problems in approximation theory and analysis in general. In particular, numerous inequalities for the norms of intermediate derivatives and various inequalities of Markov–Bernstein type for polynomials and splines were obtained with the use of this theorem and different versions and generalizations of it (for more details, see, e.g., [6, 7]). Furthermore, this theorem is one of the main premises of the Korneichuk method of comparison of permutations and Σ-permutations, which is a powerful method for the exact solution of numerous extremal problems in approximation theory (see, e.g., [8, 9]).

Later, proofs of the Kolmogorov inequality on the basis of different ideas were proposed. Among these proofs, we note the proof of Bang proposed in 1941 (see, e.g., [10]) and the proof of Ligun [11, 12].

As already noted, numerous mathematicians studied the problem of finding the exact solutions of Kolmogorov-type inequalities for functions given on the entire number axis, a semiaxis, unit circle, or finite segment. However, at present, only several cases are known in which, for certain values of p, q, and s, exact constants in inequalities of the type (3) were obtained for all pairs $k, r \in \mathbf{N}$, $k < r$. Besides the Kolmogorov inequality mentioned above, these cases are as follows:

for $G = R$:

(i) $p = q = s = 2$ (Hardy, Littlewood, and Polya [13]);

(ii) $p = q = s = 1$ (Stein [14]);

(iii) $q = \infty$ and $p = s = 2$ (Taikov [15]);

for $G = R_+$:

(i) $p = q = s = \infty$ (Landau [2], Matorin [16], and Shoenberg and Cavaretta [17]);

(ii) $p = q = s = 2$ (Lyubich [18] and Kuptsov [19]);

(iii) $q = \infty$ and $p = r = 2$ (Gabushin [20]).

Exact results for $G = R$ or R_+ and different values of r, k, p, q, and s were obtained by Arestov, Berdyshev, Buslaev, Gabushin, Magaril-Il'yaev, Sz.-Nagy, Solyar, and many other mathematicians (see [21–24]).

For exact inequalities for intermediate derivatives of functions given on a finite segment, see [25–28] and the bibliography therein. In the case of functions given on a unit circle (periodic functions), many exact inequalities were obtained by Dnepropetrovsk mathematicians. The description of known results in this direction and applications of known inequalities to the investigation of numerous extremal problems of approximation theory and extremal properties of polynomials and splines is the main subject of the present paper.

The following question is especially interesting: How should the parameters r, k, p, q, and s relate to one another in order that inequalities of the type (1)–(3) be possible in principle?

It is known that, for $G = I$ and any given $1 \le p, q, s \le \infty$, $k, r \in \mathbf{Z}$, $0 \le k < r$, there exist constants A and B such that inequality (2) holds for any function $x \in L^r_{p,s}(G)$. If G is either R or R_+, then inequality (2) holds for all functions $x \in L^r_{p,s}(G)$ if and only if [29]

$$\frac{r-k}{p} + \frac{k}{s} \ge \frac{r}{q}; \tag{5}$$

in this case, inequality (2) is equivalent to inequality (3) with

$$\alpha = \frac{r-k-s^{-1}+q^{-1}}{r-s^{-1}+p^{-1}} \quad \text{and} \quad \beta = \frac{k-q^{-1}+p^{-1}}{r-s^{-1}+p^{-1}}.$$

For $p, q, s \ge 1$, general conditions for the existence of inequalities of the type (3) in the periodic case were established in [30], where it was proved that inequality (3) holds for all functions $x \in L^r_s(\mathbf{T})$ and $k \in \mathbf{N}$, $k < r$, if and only if

$$\alpha \le \alpha_{kr} := \min\left\{1-\frac{k}{r}, \frac{r-k-s^{-1}+q^{-1}}{r-s^{-1}+p^{-1}}\right\}. \tag{6}$$

Note that inequalities of the form (3) with $\alpha = \alpha_{kr}$ are most interesting.

Inequalities of the type (1)–(3) were generalized in different directions. Thus, Hormander [31] proved the following inequality: Let $E_0(x)_\infty$ be the best uniform approximation of a function x by a subspace of constants. Also assume that $\varphi_r(\cdot;\alpha,\beta)$ is the rth periodic integral with mean value zero on the period of the 2π-periodic function $\varphi_0(t;\alpha,\beta)$ that is equal to α for $t \in [0, 2\pi\beta/(\alpha+\beta))$ and $-\beta$ for $t \in [2\pi\beta/(\alpha+\beta), 2\pi)$. It was proved in [31] that, for any function $x \in L^r_{\infty,\infty}(R)$, we have

$$\|x^{(k)}_\pm\|_\infty \le \frac{\left\|\varphi_{r-k}\left(\cdot;\|x^{(r)}_+\|_\infty, \|x^{(r)}_-\|_\infty\right)_\pm\right\|_\infty}{E_0\left(\varphi_r\left(\cdot;\|x^{(r)}_+\|_\infty, \|x^{(r)}_-\|_\infty\right)\right)_\infty^{1-k/r}} E_0(x)_\infty^{1-k/r}, \tag{7}$$

where $x_\pm(t) = \max\{\pm x(t), 0\}$. It should be noted that inequality (7) and its numerous generalizations proved to be useful for the investigation of different problems of one-sided and nonsymmetric approximation.

Another way of generalization of Kolmogorov-type inequalities consists of the replacement of the operators d^k/dx^k and d^r/dx^r in this inequality by more general differential operators or operators of different kind (see [32–36] and the detailed bibliography in [36]). In what follows, we consider several generalizations of this sort.

Since the norm in the space L_p is a support function of a unit ball in the dual space, inequalities for the norms of intermediate derivatives can be interpreted as inequalities for support functions of convex sets. From this point of view, taking into account the duality of the best approximations by convex sets (see [8, Chap. 1]), the result obtained by Korneichuk in 1961 [37] for the best approximation of the class H^{ω} of continuous periodic functions with a given majorant $\omega(t)$ of the modulus of continuity by the class NW_{∞}^{1} (in what follows, NW_p^r is the class of functions $x \in L_p^r$ such that $\|x^{(r)}\| \le N$) can be regarded as an inequality of the type (2): For $x \in L_1^1$ and any $N > 0$, we have

$$S_{H^{\omega}}(x') := \sup_{f \in H^{\omega}} \int_0^{2\pi} f(t)x'(t)dt \le N\|x\|_1 + \frac{\|x'\|_1}{2} \max_{0 \le t \le \pi} (\omega(t) - Nt).$$

If fact, this was the first exact Kolmogorov-type inequality obtained in Dnepropetrovsk University. The results of Korneichuk [38, 39] concerning the estimation of upper bounds of functionals on classes of periodic functions can also be regarded as Kolmogorov-type inequalities. These results and other results of this type are discussed in detail in [40, 41].

In Sec. 2, we present different approaches to the complete solution of the problem of exact constants in Kolmogorov-type inequalities for the norms of intermediate derivatives of periodic functions. The list of these approaches is significantly broader than in the nonperiodic case. In Sec. 2, we also present exact inequalities in the case where $r \in \mathbf{N}$ is arbitrary, $k = 0$, and q and p are quite arbitrary. In Sec. 3, we consider the case of low smoothness. In this case, the range of the parameters q, p, and s for which it is possible to obtain exact solutions is also broader than in the nonperiodic case. Section 4 is devoted to inequalities for derivatives in which specific additional properties of the functions under consideration (the number of changes of the sign of certain derivatives) are taken into account. In Sec. 5, we discuss the relationship between Kolmogorov-type inequalities and other problems. In the same section, we present several versions of the theorem on equivalence between the problem of finding exact inequalities of the type (1) and several other problems. In Sec. 6, we consider applications of Kolmogorov-type inequalities to the exact solution of extremal problems in approximation theory. In Sec. 7, we consider applications of these inequalities to the investigation of extremal properties of polynomials and splines. Finally, in Sec. 8, we briefly discuss other investigations of Dnepropetrovsk mathematicians related to inequalities for intermediate derivatives.

2. Cases of Complete Solution of the Problem on Exact Kolmogorov-Type Inequalities for Periodic Functions

First, we note the following cases, which are either particular cases of inequalities for the entire axis or can be obtained by analogy:

(i) $p = q = r = \infty$ (Hadamard, Shilov, and Kolmogorov);

(ii) $p = q = r = 2$ (Hardy, Littlewood, and Polya);

(iii) $p = q = r = 1$ (Stein);

(iv) $q = \infty$ and $p = r = 2$ (Shadrin [42]).

The Hadamard–Shilov–Kolmogorov inequality (4) is presented above. The Hardy–Littlewood–Polya inequality for functions $x \in L_{2,2}^r(\mathbf{T})$ has the form

$$\left\|x^{(k)}\right\|_2 \le \|x\|_2^{1-k/r} \left\|x^{(r)}\right\|_2^{k/r}. \tag{8}$$

Using the Stein method [14], one can easily derive from (4) the following unimprovable inequality for functions $x \in L_1^r$ and all $k \in \mathbf{N}$, $k < r$:

$$\left\| x^{(k)} \right\|_1 \le \frac{\| g_{r-k} \|_1}{\| g_r \|_1^{1-k/r}} \| x \|_1^{1-k/r} \left\| x^{(r)} \right\|_1^{k/r} \tag{9}$$

(here and below, $g_r(t) := 1/4\varphi_{r-1}(t)$). Note that the Stein method [14] can also be used in some other situations (for different versions and generalizations of this method, see [43–45]). The Shadrin inequality, which is a periodic analog of the Taikov and Gabushin inequalities, is not presented here because its formulation is rather cumbersome.

The other results presented in this section were obtained by Dnepropetrovsk mathematicians. We begin the presentation of these results with one of the most interesting inequalities proved by Ligun in [46] and formulated as follows: For any function $x \in L_\infty^r(\mathbf{T})$, any $k \in \mathbf{N}$, $k < r$, and any $p \in [1, \infty]$, the following unimprovable inequality is true:

$$\left\| x^{(k)} \right\|_p \le \frac{\| \varphi_{r-k} \|_p}{\| \varphi_r \|_\infty^{1-k/r}} \| x \|_\infty^{1-k/r} \left\| x^{(r)} \right\|_\infty^{k/r}. \tag{10}$$

Inequality (10) turns into the equality for functions of the form $\varphi_{n,r}(t) := n^{-r}\varphi_r(nt)$. In the proof of this inequality, the method of comparison of permutations was essentially used.

Since inequality (10) is important for the investigation of numerous extremal problems in approximation theory, it was generalized in numerous directions. Thus, various nonsymmetric versions of this inequality (Hormander-type inequalities) were obtained for the best one-sided and nonsymmetric approximations. Various "one-sided" inequalities of the type (10) and their numerous applications are described in detail in [9]. For more general "nonsymmetric" inequalities and generalizations of inequality (10) to the case of linear differential operators with constant coefficients, see [47, 34–36], [6, Chap. 1], and [7, Chap. 1].

The following inequality was proved in [48]: For any $k \in \mathbf{N}$, $k < r$, and $x \in L_\infty(\mathbf{T})$, we have

$$\left\| \tilde{x}^{(k)} \right\|_1 \le \frac{\left\| \tilde{\varphi}_{r-k} \right\|_1}{\| \varphi_r \|_\infty^{1-k/r}} \| x \|_\infty^{1-k/r} \left\| x^{(r)} \right\|_\infty^{k/r} \tag{11}$$

($\tilde{x}$ is the function trigonometrically conjugate to x).

Let

$$a_0(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t)\,dt$$

for $x \in L_1$. If $r \in R$, $r > 0$, then we set

$$B_r(t) = \frac{1}{\pi} \sum_{\nu=1}^{\infty} \nu^{-r} \cos(\nu t - \pi r/2).$$

A function $g \in L_1$ such that $a_0(g) = 0$ is called the rth Weyl derivative of $x \in L_1$ ($x^{(r)} = g$) if

$$x(t) = a_0(x) + (B_r * g)(t) = a_0(x) + \int_0^{2\pi} B_r(t-u)g(u)\,du.$$

Denote by L_p^r the set of functions $x \in L_1$ such that $x^{(r)} \in L_p$. For $r \in R$, $r > 0$, we set

$$\varphi_r(t) = \int_0^{2\pi} B_r(t-u)\varphi_0(u)\,du.$$

In [49, 50], the following inequality was obtained for derivatives of half-integral order: Let $k, r \in \mathbf{N}$ and $r/2 \le k < r$. Then, for any function $x \in L_\infty^r$, the following unimprovable inequality holds:

$$\left\| x^{(k+1/2)} \right\|_2 \le \frac{\left\| \varphi_{r-k-1/2} \right\|}{\left\| \varphi_r \right\|_\infty^{1-(k+1/2)/r}} \| x \|_\infty^{1-(k+1/2)/r} \left\| x^{(r)} \right\|_\infty^{(k+1/2)/r}. \tag{12}$$

In the proof of inequality (10), the theorem on comparison of derivatives, based on the Rolle theorem for the operator d/dt, was essentially used. However, the operators of trigonometric conjugation and fractional differentiation do not possess such properties. In the proof of inequality (11), this problem was solved by using the Stein–Weiss lemma (see, e.g., [6, Sec. 1.6]). Inequality (12) is proved by using (10) and (11).

In [51], one can find certain generalizations of the results of [48].

Inequality (10) relates to the case where [see (6)]

$$\alpha_{kr} := \min\left\{ 1 - \frac{k}{r},\ \frac{r-k-s^{-1}+q^{-1}}{r-s^{-1}+p^{-1}} \right\} = 1 - \frac{k}{r}.$$

In this case, inequalities of the type (3) are impossible in principle for functions defined on the entire axis or semi-axis. The following group of results obtained by Babenko, Kofanov, and Pichugov [52–55] is related to the case where

$$\alpha_{kr} := \min\left\{ 1 - \frac{k}{r},\ \frac{r-k-s^{-1}+q^{-1}}{r-s^{-1}+p^{-1}} \right\} = \frac{r-k-s^{-1}+q^{-1}}{r-s^{-1}+p^{-1}}.$$

Let $k, r \in \mathbf{N}$, $k < r$, and $p \in [1, \infty)$. Then, for any function $x \in L_\infty^r$, the following inequality holds:

$$\left\| x^{(k)} \right\|_\infty \le \frac{\left\| \varphi_{r-k} \right\|_\infty}{\left\| \varphi_r \right\|_p^{(r-k)/(r+p^{-1})}} \| x \|_p^{(r-k)/(r+p^{-1})} \left\| x^{(r)} \right\|_\infty^{(k+p^{-1})/(r+p^{-1})}. \tag{13}$$

Inequality (13) turns into the equality for functions of the form $x(t) = a\varphi_r(t)$, $a \in R$. .

For any function $x \in L_1^r$, we have

$$\left\| x^{(k)} \right\|_\infty \le \frac{\left\| g_{r-k} \right\|_\infty}{\left\| g_r \right\|_p^{(r-k-1)/(r-1+p^{-1})}} \| x \|_p^{(r-k-1)/(r-1+p^{-1})} \left\| x^{(r)} \right\|_1^{(k+p^{-1})/(r-1+p^{-1})}. \tag{14}$$

For $p = \infty$, this inequality was obtained by Ligun (see, e.g., [9, Sec. 6.4].

Finally, for any function $x \in L_1^r$, we have

$$\left\| x^{(k)} \right\|_2 \le \frac{\left\| g_{r-k} \right\|_2}{\left\| g_r \right\|_p^{(r-k-1/2)/(r-1+p^{-1})}} \| x \|_p^{(r-k-1/2)/(r-1+p^{-1})} \left\| x^{(r)} \right\|_1^{(k-1/2+p^{-1})/(r-1+p^{-1})}. \tag{15}$$

Inequality (13) is the main inequality. Inequalities (14) and (15) are proved with the use of (13). The proof of inequality (13) is based on the Kolmogorov comparison theorem and method of comparison of permutations.

In 1941, Sz.-Nagy [56] obtained exact inequalities of the form

$$\|x\|_{L_q(G)} \le K\|x\|^{\alpha}_{L_q(G)}\left\|x^{(r)}\right\|^{1-\alpha}_{L_s(G)} \tag{16}$$

for functions defined on the axis in the case where $r = 1$, $q \ge p > 0$, and $1 \le s \le \infty$. Inequalities of the form (16) are called Sz.-Nagy-type inequalities. Below, we present the inequalities of the type (16) for periodic functions x with zeros and essentially bounded derivatives of order $r \ge 1$ obtained by Babenko, Kofanov, and Pichugov in [57–59].

Let $r \in \mathbf{N}$ and $p, q \in (0, \infty]$, $q > p$. Then, for functions $x \in L^r_\infty(\mathbf{T})$ with zeros, the following unimprovable inequality is true:

$$\|x\|_q \le \sup_{0 \le c \le \|\varphi_r\|_\infty} \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_p^{(r+1/q)/(r+1/p)}} \|x\|_p^{(r+1/q)/(r+1/p)}\left\|x^{(r)}\right\|_\infty^{(1/p-1/q)/(r+1/p)}. \tag{17}$$

Let $E_0(x)_p$ be the ordinary best approximation and let $E_0^{\pm}(x)_p$ denote the best approximations of a function $x \in L_p(\mathbf{T})$ from below (+) and above (–) by constants in the space $L_p(\mathbf{T})$. For $1 \le p \le \infty$, we denote by $c_p(x)$ the constant of the best approximation of a function $x \in L_p(\mathbf{T})$ in $L_p(\mathbf{T})$. The following exact inequalities are true: Let $r \in \mathbf{N}$, $p, q \in (0, \infty]$, $q > p$, and $m = p + 1$ or $m = \infty$. Then, for any function $x \in L^r_\infty$, we have

$$\|x - c_m(x)\|_q \le \frac{\|\varphi_r\|_q}{\|\varphi_r\|_p^{(r+1/q)/(r+1/p)}} \|x - c_m(x)\|_p^{(r+1/q)/(r+1/p)}\left\|x^{(r)}\right\|_\infty^{(1/p-1/q)/(r+1/p)}, \tag{18}$$

$$E_0^{\pm}(x)_q \le \frac{\big\|\varphi_r - \|\varphi_r\|_\infty\big\|_q}{\big\|\varphi_r - \|\varphi_r\|_\infty\big\|_p^{(r+1/q)/(r+1/p)}} E_0^{\pm}(x)_p^{(r+1/q)/(r+1/p)}\left\|x^{(r)}\right\|_\infty^{(1/p-1/q)/(r+1/p)}, \tag{19}$$

$$E_0(x)_\infty \le \frac{\|\varphi_r\|_\infty}{\|\varphi_r\|_p^{r/(r+1/p)}}\|x\|_p^{r/(r+1/p)} E_0(x^{(r)})_\infty^{(1/p)/(r+1/p)}. \tag{20}$$

As a consequence of (20), for any function $x \in L^r_\infty$, $x \perp$ const, the following exact inequality holds for $r \in \mathbf{N}$ and $q \in (1, \infty]$:

$$\|x\|_q \le \frac{\|\varphi_r\|_q}{\|\varphi_r\|_1^{(r+1/q)/(r+1)}}\|x\|_1^{(r+1/q)/(r+1)}\left\|x^{(r)}\right\|_\infty^{(1-1/q)/(r+1)}. \tag{21}$$

The proof of inequalities (18)–(21) is also based on the method of comparison of derivatives and permutations.

Also note that Babenko and Vakarchuk [60] obtained exact Kolmogorov-type inequalities for functions bounded on a discrete lattice.

Summarizing the results presented above, we conclude that the list of known (in the periodic case) sufficiently complete results (i)–(iv) of determination of exact constants for Kolmogorov-type inequalities can be extended as follows:

(v) $q \in [1, \infty)$ and $p = s = \infty$ (Ligun);

(vi) $q = \infty$, $p \in [1, \infty)$, and $s = \infty$ (Babenko, Kofanov, and Pichugov);

(vii) $q = \infty$, $p \in [1, \infty)$, and $s = 1$ (Babenko, Kofanov, and Pichugov);

(viii) $q = 2$, $p \in [1, \infty]$, $s = 1$, and $k > r/2$ (Babenko, Kofanov, and Pichugov);

(ix) $q, p \in (0, \infty]$, $q > p$, $r \in \mathbf{N}$, and $k = 0$ (Babenko, Kofanov, and Pichugov).

3. Kolmogorov-Type Inequalities for Periodic Functions in the Case of Low Smoothness

We begin with the case of functions with essentially bounded higher derivative. In particular, Gabushin [61] proved that, for $r = 2$ and $r = 3$, $1 \le k \le r-1$, any $p \ge 1$, and $q = rp/(r-k)$, the following unimprovable inequality holds for functions from $L^r_{p,\infty}(R)$:

$$\left\| x^{(k)} \right\|_q \le 2^{\alpha/p - 1/q} \frac{\left\| \varphi_{r-k} \right\|_q}{\left\| \varphi_{r-k} \right\|_p^{\alpha}} \left\| x \right\|_p^{\alpha} \left\| x^{(r)} \right\|_{\infty}^{1-\alpha},$$

where $\alpha = \dfrac{r - k + 1/q}{r + 1/p}$.

The following statement for periodic functions was proved in [62]: For functions $f \in L^r_{p,\infty}(\mathbf{T}^1)$, the inequality

$$\left\| x^{(k)} \right\|_q \le \frac{\left\| \varphi_{r-k} \right\|_q}{\left\| \varphi_{r-k} \right\|_p^{\alpha}} \left\| x \right\|_p^{\alpha} \left\| x^{(r)} \right\|_{\infty}^{1-\alpha}, \tag{22}$$

where

$$\alpha = \alpha_{kr} = \min\left\{ 1 - \frac{k}{r}, \frac{r - k + q^{-1}}{r + p^{-1}} \right\},$$

holds for any $p, q \in [1, \infty]$ in the case where $r = 2$ and $k = 1$ or $r = 3$ and $k = 1, 2$, and for $p = 1$ and $q \in [1, \infty]$ in the case where $r = 4$ and $k = 3$ or $r = 6$ and $k = 4, 5$. Inequality (22) is exact. For $q > rp/(r-k)$, it turns into the equality for functions of the form $x(t) = a\varphi_r(t + c)$, $a, c \in R$. For $1 \le q \le rp/(r-k)$, it turns into the equality for functions of the form $x(t) = a\varphi_r(nt + c)$, $a, c \in R$, $n \in \mathbf{N}$.

The next group of inequalities (see also [62]) was obtained for functions with summable higher derivative. For any $p, q \in [1, \infty]$, the following exact inequality holds for functions $f \in L^2_{p,1}(\mathbf{T}^1)$:

$$\left\| x' \right\|_q \le \sup_{\gamma + \delta = 1/2} \frac{\left\| \varphi_0(\cdot; \gamma, \delta) \right\|_q}{\left\| \varphi_1(\cdot; \gamma, \delta) \right\|_p^{\alpha}} \left\| x \right\|_p^{\alpha} \left\| x'' \right\|_1^{1-\alpha}, \tag{23}$$

where

$$\alpha = \min\left\{\frac{1}{2}, \frac{p}{q(p+1)}\right\}.$$

If $q \le 2$, then

$$\sup_{\gamma+\delta=1/2} \frac{\|\varphi_0(\cdot;\gamma,\delta)\|_{L_q(\mathbf{T}^1)}}{\|\varphi_1(\cdot;\gamma,\delta)\|^{\alpha}_{L_p(\mathbf{T}^1)}} = \frac{\|\varphi_0(\cdot;1/4,1/4)\|_{L_q(\mathbf{T}^1)}}{\|\varphi_1(\cdot;1/4,1/4)\|^{\alpha}_{L_p(\mathbf{T}^1)}} = \frac{\|g_1\|_{L_q(\mathbf{T}^1)}}{\|g_2\|^{\alpha}_{L_p(\mathbf{T}^1)}}$$

for all $p \in [1,\infty]$. Moreover, for any $p \in [1,\infty]$, if $q > \frac{3}{2} + \frac{1}{2}\sqrt{\frac{p+9}{p+1}}$, then

$$\sup_{\gamma+\delta=1/2} \frac{\|\varphi_0(\cdot;\gamma,\delta)\|_{L_q(\mathbf{T}^1)}}{\|\varphi_1(\cdot;\gamma,\delta)\|^{\alpha}_{L_p(\mathbf{T}^1)}} > \frac{\|g_1\|_{L_q(\mathbf{T}^1)}}{\|g_2\|^{\alpha}_{L_p(\mathbf{T}^1)}}.$$

For functions $f \in L^2_{p,1}(R)$, an exact inequality of the type (23) was proved in [63] for $q = 2p/(p+1)$.

In addition to (23), the following statement is true: Let $p \in [1,\infty)$ and let $r = 3$ or $r = 4$. Also assume that if $r = 3$, then $k = 1$ and $q \ge 2p$, and if $r = 4$, then $k = 1$ and $q \ge 3p/2$ or $k = 2$ and $q \ge 3p$. Then, for functions $x \in L^r_{p,1}(\mathbf{T}^1)$, the following unimprovable inequality holds:

$$\|x^{(k)}\|_q \le \frac{\|g_{r-k}\|_q}{\|g_r\|^{\alpha}_p} \|x\|^{\alpha}_p \|x^{(r)}\|^{1-\alpha}_1,$$

where

$$\alpha = \alpha_{kr} = \frac{r-k-1+q^{-1}}{r-1+p^{-1}}.$$

The next two assertions [62] relate to the case of arbitrary smoothness r and certain specific values of k.

Assume that $r \in \mathbf{N}$ and $k = r-1$, $k = r-2$, or $k = r-3$ and, moreover, $1 \le q \le 2$ if $k = r-1$, $q \ge 2$ if $k = r-2$, and $q \ge 3/2$ if $k = r-3$. Then the following exact inequality holds for functions $x \in L^r_{1,1}(\mathbf{T}^1)$:

$$\|x^{(k)}\|_q \le \frac{\|g_{r-k}\|_q}{\|g_r\|^{\alpha}_1} \|x\|^{\alpha}_1 \|f^{(r)}\|^{1-\alpha}_1,$$

where

$$\alpha = \alpha_{kr} = \frac{r-k-1+q^{-1}}{r}.$$

Finally, assume that $q, p \in (1,\infty]$, $r \in \mathbf{N}$, and $k = r-1$, $k = r-2$, or $k = r-3$ and let $4/3 \le q \le 2$ and $r \ge 5$ if $k = r-1$, $q \ge 4$ and $r \ge 7$ if $k = r-2$, and $q \ge 3$ and $r \ge 9$ if $k = r-3$. Then functions $x \in L^r_{p,1}(\mathbf{T}^1)$ satisfy the exact inequality

$$\left\| x^{(k)} \right\|_q \le \frac{\| g_{r-k} \|_q^{\alpha}}{\| g_r \|_p} \| x \|_p^{\alpha} \left\| x^{(r)} \right\|_1^{1-\alpha},$$

where $\alpha = \alpha_{kr} = \dfrac{r-k-1+q^{-1}}{r-1+p^{-1}}$.

We now present more inequalities for functions with the higher derivative from L_s, $1 < s < \infty$. In [64], for functions $x \in L^r_{\infty, s}(R)$, Arestov established an exact inequality of the type (3) for the following values of parameters: $r = 2$, $k = 1$, $p = \infty$, $s \in [1, \infty]$, $q \ge 2s$ and $r = 3$, $k = 1, 2$, $q = p = \infty$, $s \in [1, \infty]$. For periodic functions, the following two statements were proved in [62, 65]:

Let $s \in (1, \infty)$. Then functions $x \in L^2_{\infty, s}(\mathbf{T})$ satisfy the inequality (as usual, $p' = p/(p-1)$)

$$\| x' \|_\infty \le 2^{(1-s')/(1+s')} \left(\frac{s'+1}{s'} \right)^{s'/(1+s')} E_0(x)_\infty^{\alpha} E_0(x'')_s^{1-\alpha} \tag{24}$$

with the exponent $\alpha = \alpha_{kr} = 1/(s'+1)$. For functions $x \in L^3_{\infty, s}(\mathbf{T})$, $s \in (1, \infty)$, the following inequality holds:

$$E_0(x')_\infty \le 2^{(2-3s')/(2s'+1)} \frac{(2s'+1)^{2s'/(2s'+1)}}{(s')^{s'/(2s'+1)} (s'+1)^{(s'+1)/(2s'+1)}} E_0(x)^{\alpha_1}_{L_\infty(\mathbf{T}^1)} E_0(x''')^{1-\alpha_1}_{L_s(\mathbf{T}^1)}, \tag{25}$$

where

$$\alpha_1 = \frac{2 - 1/s}{3 - 1/s} = \frac{1 + 1/s'}{2 + 1/s'}$$

is the critical exponent. The constants in inequalities (24) are unimprovable.

4. Inequalities That Take into Account the Number of Changes of the Sign of Derivatives

For a summable 2π-periodic function x, we denote by $\nu(x)$ the number of essential changes of the sign of x on the period (see, e.g., [8, p. 80]). By virtue of [30], inequalities of the form (3) with $\alpha > \alpha_{kr}$ are impossible. Nevertheless, as proved by Ligun [66], if we transform an inequality of the form (3) so that it takes into account certain additional properties of the function x such as, e.g., the number of changes of the sign of its derivatives, then Kolmogorov-type inequalities with exponent $\alpha > \alpha_{kr}$ are possible. Let us formulate the indicated result of Ligun. For $r, k \in \mathbf{N}$, $k < r$, $p \in [1, \infty]$, and $x \in L_1^r$, the following unimprovable inequality is true:

$$\left\| x^{(k)} \right\|_1 \le \left(\frac{\nu(x')}{2} \right)^{(1-1/p)\alpha} \frac{\| g_{r-k} \|_1}{\| g_r \|_p^{\alpha}} \| x \|_p^{\alpha} \left\| x^{(r)} \right\|_1^{1-\alpha}, \tag{26}$$

where

$$\alpha = \frac{r-k}{r-1+1/p}.$$

Note that, for $p = 1$, inequality (26) turns into the Stein inequality (6). In [66], Ligun also presented several instances of application of inequality (26) to approximation theory. In [67], inequality (26) was generalized to the case of differential operators with constant coefficients.

Consider inequalities of the form [68]

$$\left\| x^{(k)} \right\|_q \le M \prod_{i=1}^{m} \nu\left(x^{(i)}\right)^{\alpha_i} \| x \|_p^{\alpha} \left\| x^{(r)} \right\|_s^{1-\alpha} \tag{27}$$

(here, $\alpha, \alpha_1, \ldots, \alpha_m$ are nonnegative numbers, $\alpha \in (0, 1)$) for functions $x \in L_s^r$ (in this case, $m = r$) and $x \in L_1^{r+1}$ (in this case, $m = r + 1$). We consider inequalities satisfying the following conditions:

(i) $\alpha > \alpha_{kr} := \min\left\{ 1 - \frac{k}{r}, \frac{r-k-s^{-1}+q^{-1}}{r-s^{-1}+p^{-1}} \right\};$

we established that, in several cases, as in (26), it is possible to take

$$\alpha = \max\left\{ 1 - \frac{k}{r}, \frac{r-k-s^{-1}+q^{-1}}{r-s^{-1}+p^{-1}} \right\};$$

(ii) $\sum_{i=1}^{m} \alpha_i = k - r(1-\alpha).$

Note that $k - r(1-\alpha)$ is the minimum possible value of $\sum_{i=1}^{m} \alpha_i$ for which inequality (27) holds with a constant M independent of f. Indeed, if $\sum_{i=1}^{m} \alpha_i < k - r(1-\alpha)$, then, for sufficiently large $n \in \mathbf{N}$, inequality (27) is not satisfied for $x(nt)$, $x \ne \text{const}$. The fact that inequality (26) possesses properties (i) and (ii) plays an important role in its applications.

We begin the formulation of results with inequalities of the type (27), which can easily be derived from inequalities (4) and (9).

Let $k, r \in \mathbf{N}$ and $k < r$. Then, for any function $x \in L_\infty^{r+1}$, the following inequality is true:

$$\left\| x^{(k)} \right\|_1 \le \left(\frac{\nu(x^{(r+1)})}{2} \right)^{k/(r+1)} \frac{\| \varphi_{r-k} \|_1}{\| \varphi_r \|_1^{(r-k+1)/(r+1)}} \| x \|_1^{(r-k+1)/(r+1)} \left\| x^{(r)} \right\|_\infty^{k/(r+1)}. \tag{28}$$

If $k \ge 2$, then

$$\left\| x^{(k)} \right\|_1 \le \left(\frac{\nu(x')}{2} \right)^{(r-k)/(r-1)} \frac{\| g_{r-k} \|_1}{\| g_r \|_1^{(r-k)/(r-1)}} \| x \|_\infty^{(r-k)/(r-1)} \left\| x^{(r)} \right\|_1^{(k-1)/(r-1)} \tag{29}$$

or

$$\left\| x^{(k)} \right\|_1 \le \left(\frac{\nu(x')}{2} \right)^{(r-k+1)/r} \left(\frac{\nu(x^{(r+1)})}{2} \right)^{(k-1)/r} \frac{\| \varphi_{r-k} \|_1}{\| \varphi_r \|_\infty^{(r-k+1)/r}} \| x \|_\infty^{(r-k+1)/r} \left\| x^{(r)} \right\|_\infty^{(k-1)/r}. \tag{30}$$

Moreover, for any function $x \in L_\infty^r$ and $k \ge 2$, we have

$$\left\| x^{(k)} \right\|_1 \le \frac{\nu(x^{(k)})}{2} \frac{\| \varphi_{r-k} \|_1}{\| \varphi_r \|_\infty^{(r-k+1)/r}} \| x \|_\infty^{(r-k+1)/r} \left\| x^{(r)} \right\|_\infty^{(k-1)/r}. \tag{31}$$

Inequalities (28)–(31) are exact.

Note that inequality (29) is a particular case of inequality (26). Here, it is presented for completeness of exposition and for the reason that, unlike inequality (26), it easily follows from (9).

In inequalities (30) and (31), the norm $\|x^{(k)}\|_1$ is estimated in terms of $\|x\|_\infty$ and $\|x^{(r)}\|_\infty$ in equal powers. However, the number of changes of the sign of derivatives of the function x is taken into account differently in these inequalities, and one can easily give examples of functions for which (30) gives a better estimate of $\|x^{(k)}\|_1$ than (31), and vice versa.

Using inequality (26), we can establish the following inequality, which is more general than (28) and (30): Let $k, r \in \mathbf{N}$, $k < r$, and $p \in [1, \infty]$. Then, for any function $x \in L_1^{r+1}$, we have

$$\|x^{(k)}\|_1 \le \left(\frac{\nu(x')}{2}\right)^{(1-1/p)(r-k+1)/(r+1/p)} \left(\frac{\nu(x^{(r+1)})}{2}\right)^{(k-1+1/p)/(r+1/p)}$$

$$\times \frac{\|\varphi_{r-k}\|_1}{\|\varphi_r\|_p^{(r-k+1)/(r+1/p)}} \|x\|_p^{(r-k+1)/(r+1/p)} \|x^{(r)}\|_\infty^{(k-1+1/p)/(r+1/p)}. \tag{32}$$

Inequality (32) is exact.

Note two more inequalities. Let $k, r \in \mathbf{N}$, $r/2 < k < r$, and $p \in [1, \infty]$. Then, for any function $x \in L_1^{r+1}$, the following inequality is true:

$$\|x^{(k)}\|_2 \le \left(\frac{\nu(x')}{2}\right)^{(1-1/p)(r-k+1/2)/(r+1/p)} \left(\frac{\nu(x^{(r+1)})}{2}\right)^{1/2-(r-k+1/2)/(r+1/p)}$$

$$\times \frac{\|\varphi_{r-k}\|_2}{\|\varphi_r\|_p^{(r-k+1/2)/(r+1/p)}} \|x\|_p^{(r-k+1/2)/(r+1/p)} \|x^{(r)}\|_\infty^{1-(r-k+1/2)/(r+1/p)}. \tag{33}$$

Moreover,

$$\|x^{(k)}\|_2 \le \left(\frac{\nu(x^{(2k-r)})}{2}\right)^{1/2} \frac{\|\varphi_{r-k}\|_2}{\|\varphi_r\|_\infty^{(r-k+1/2)/r}} \|x\|_\infty^{(r-k+1/2)/r} \|x^{(r)}\|_\infty^{(k-1/2)/r} \tag{34}$$

for $x \in L_\infty^r$. Inequalities (33) and (34) are exact.

Comparing inequalities (33) (for $p = \infty$) and (34), we can draw the same conclusions as in the case of inequalities (30) and (31). For more inequalities of the type (27), see [68].

5. Relationship between Kolmogorov-Type Inequalities and Approximation Problems

In the course of solution of numerous extremal problems in approximation theory, it was established that they are closely related to exact inequalities of the form (1)–(3). In this connection, one should mention the important works devoted to the method of intermediate approximation [37, 39] and to the approximation of unbounded operators by bounded operators [69, 70]. The investigation of this relationship was carried out in [71, 72] and [30, 46] (for functions defined on R or R_+ and periodic functions, respectively).

We restrict ourselves to the consideration of the relationship between Kolmogorov-type inequalities and problems of approximation of functional classes. We present two theorems that form the basis of such applications. Let $k, r \in \mathbf{N}$, $p, q, s \in [1, \infty]$, and $N \in R_+$. We set

$$W_p^k := \left\{x \in L_p^k(\mathbf{T}): \left\|x^{(k)}\right\|_p \le 1\right\}, \qquad W_p^0 := \left\{x \in L_p(\mathbf{T}): \|x\|_p \le 1\right\}$$

and

$$E\left(W_q^k, NW_p^r\right)_s := \sup_{x \in W_q^k} \inf_{u \in NW_p^r} \|x-u\|_s.$$

Assume that, for any seminorm ψ on $L_s(\mathbf{T})$ and any subset $A \subset L_s(\mathbf{T})$, we have

$$\psi(A) := \sup\{\psi(x): x \in A\}.$$

Theorem 1. *Let* $r \in \mathbf{N}$, $k = 1, \ldots, r-1$, $\alpha \in (0, 1)$, $N > 0$, *and* $p, q, s \in [1, \infty]$. *Then the following assertions are equivalent:*

(i) for any function $x \in L_{s'}^r(\mathbf{T})$, *we have*

$$E_0(x^{(r-k)})_{q'} \le K E_0(x)_{p'}^{\alpha} \left\|x^{(r)}\right\|_{s'}^{1-\alpha};$$

(ii) for any $N > 0$, *we have*

$$E\left(W_q^k, NW_p^r\right)_s \le \frac{1-\alpha}{\alpha}\left(\frac{N^{\alpha}}{K\alpha}\right)^{-1/(1-\alpha)};$$

(iii) for any seminorm ψ *on* $L_s(\mathbf{T})$, *we have*

$$\psi\left(W_q^k\right)_s \le K\psi^{\alpha}\left(W_p^r\right)\psi^{1-\alpha}\left(W_s^0\right);$$

(iv) for any function $x \in W_q^k$ *and any* $t > 0$, *we have*

$$\inf_{x_1 \in L_p^r(\mathbf{T})} \left\{\|x - x_1\|_s + t\left\|x_1^{(r)}\right\|_p\right\} \le Kt^{\alpha}.$$

The equivalence of assertions (i)–(iii) of this theorem was proved by Ligun (see Theorem 6.1.1 in [9]). Assertion (iv) was proved by Babenko, Kofanov, and Pichugov [53].

The next theorem [73, 74] describes the relationship between Kolmogorov-type inequalities for support functions of convex sets and other problems.

Assume that X is a real linear space, θ_X is zero in X, $p(x)$ is a certain (generally speaking, nonsymmetric) norm on X, $H_{X,p} := \{x \in X: p(x) \le 1\}$, $X'(p)$ is the space of linear bounded (with respect to p) functionals on X, $\langle x, y\rangle$ is the value of a functional $y \in X'(p)$ on the element $x \in X$, and $p^*(y) := \sup\{\langle x, y\rangle: x \in H_{X,p}\}$ is the (nonsymmetric) norm in $X'(p)$. Note that if X is a normed space and $p(x) = \|x\|_X$, then $X'(p) = X^*$, where X^* is the space of all linear bounded functionals on X.

Assume that, for $M, M_1 \subset X$, $x \in X$, $y \in X'(p)$, and an arbitrary sublinear functional ψ on X,

$$S_M(y) \;:=\; \sup\{\langle x, y\rangle:\; x \in M\}$$

is the support function of the set M and

$$M^0 \;:=\; \{y \in X'(p):\; S_M(y) \le 1\}, \qquad E(x, M_1)_{X,p} \;:=\; \inf\{p(x-u):\; u \in M_1\},$$

$$E(M, M_1)_{X,p} \;:=\; \sup\{\, E(x, M_1)_{X,p} : x \in M\}, \qquad \psi(M) \;:=\; \sup\{\psi(x):\; x \in M\}.$$

If $H_1, \ldots, H_m \subset X$, then, for $x \in X$ and any $t \in (t_1, \ldots, t_m) \in R_+^m$, we set

$$K_p(X; H_1,\ldots, H_m; x; t) \;:=\; \inf_{\substack{x_j \in \operatorname{cone} H_j \\ j = 1,\ldots, m}} \left\{ p\left(x - \sum_{j=1}^{m} x_j \right) + \sum_{j=1}^{m} t_j S_{H_j^0}(x_j) \right\}.$$

Denote by $\mathcal{F}_m$ the set of lower-semicontinuous convex functions $\Phi:\; R_+^m \in R_+$. For $\Phi \in \mathcal{F}_m$, we set $\overline{\Phi}(z) = -\Phi(z)$ if $z \in R_+^m$, and $\overline{\Phi}(z) = +\infty$ if $z \notin R_+^m$. Also assume that $\overline{\Phi}^*$ is the Legendre transformation of the function $\overline{\Phi}$, i.e., $\overline{\Phi}^*(y) \;:=\; \sup\left\{ \langle x, y\rangle - \overline{\Phi}(x):\; x \in R^m \right\}$, $y \in R^m$, and $\sum_{j=1}^m N_j H_j$ is the algebraic sum of the sets $N_j H_j$.

Theorem 2. *Let $H_1, \ldots, H_m$ be arbitrary convex sets in X that contain θ_X and let $\Phi \in \mathcal{F}_m$. Then the following assertions are equivalent:*

(i) for any $x \in X'(p)$ such that $p^*(x) \ne 0$, we have

$$S_H(x) \;\le\; p^*(x)\Phi\left(\frac{S_{H_1}(x)}{p^*(x)}, \ldots, \frac{S_{H_m}(x)}{p^*(x)} \right);$$

(ii) for any $x \in X'(p)$ and $N = (N_1, \ldots, N_m) \in R_+^m$, we have

$$S_H(x) \;\le\; \overline{\Phi}^*(-N)p^*(x) + \sum_{j=1}^{m} N_j S_{H_j}(x);$$

(iii) for any $N = (N_1, \ldots, N_m) \in R_+^m$, we have

$$E\left(H; \sum_{j=1}^{m} N_j H_j \right)_{X,p} \;\le\; \overline{\Phi}^*(-N);$$

(iv) for any $N \in R_+^m$ and any sublinear functional ψ on X for which the values $\psi(H_{X,p})$ and $\psi(H_j)$, $j = 1, \ldots, m$, are finite, we have

$$\psi(H) \le \overline{\Phi}^*(-N)\psi(H_{X,p}) + \sum_{j=1}^{m} N_j \psi(H_j);$$

(v) for any functional ψ, $\psi \neq 0$, *from assertion (iv), we have*

$$\psi(H) \le \psi(H_{X,p})\Phi\left(\frac{\psi(H_1)}{\psi(H_{X,p})}, \ldots, \frac{\psi(H_m)}{\psi(H_{X,p})}\right);$$

(vi) for any $z \in H$ *and* $t \in R_+^m$, *we have*

$$K(X; H_1, \ldots, H_m; z; t) \le \Phi(t).$$

Assertions (i) and (ii) of this theorem are abstract versions of a Kolmogorov-type inequality in the multiplicative and additive form, respectively. Assertion (iii) is an estimate of approximation of a class by a class. Assertions (iv) and (v) are abstract versions of inequalities for upper bounds of seminorms. Finally, assertion (vi) is an estimate (on the class H) of a K-functional-type characteristic of a collection of m spaces.

6. Applications to Approximation Theory

Applications of Kolmogorov-type inequalities that are based on Theorems 1 and 2 are described in detail, e.g., in [9, 36, 41, 53–55]. In the present paper, we restrict ourselves to the formulation of several corollaries of the inequalities presented in Sec. 2. The estimates for approximation of a class by a class presented below are established by using inequalities (13)–(15) and Theorem 1.

Let $k, r \in \mathbf{N}$, $k < r$, and $p \in [1, \infty]$. Then, for any $N > 0$, we have

$$E\left(W_1^k, NW_p^r\right)_1 \le \frac{r-k+1/p'}{k} \frac{\left(\|\varphi_k\|_\infty \dfrac{k}{r+1/p'}\right)^{(r+1/p')/(r-k+1/p')}}{\|\varphi_r\|_{p'}^{k/(r-k+1/p')}} N^{-k/(r-k+1/p')} \tag{35}$$

(as usual, $p' = p/(p-1)$). The constant coefficient of $N^{-k/(r-k+1/p')}$ is unimprovable.

Let $r, k \in \mathbf{N}$, $k < r$, and $p \in [1, \infty]$. Then, for any $N > 0$, we have

$$E\left(W_1^k, NW_p^r\right)_\infty \le \frac{r-k+1/p'}{k} \frac{\left(\|g_k\|_\infty \dfrac{k-1}{r-1+1/p'}\right)^{(r-1+1/p')/(r-k+1/p')}}{\|g_r\|_{p'}^{(k-1)/(r-k+1/p')}} N^{-(k-1)/(r-k+1/p')}.$$

The constant coefficient of $N^{-(k-1)/(r-k+1/p')}$ is unimprovable.

Let $r, k \in \mathbf{N}$, $k < r/2$, and $p \in [1, \infty]$. Then, for any $N > 0$, we have

$$E\left(W_2^k, NW_p^r\right)_\infty \le \frac{r-k-1/2+1/p'}{k-1/2} \frac{\left(\|g_k\|_2 \dfrac{k-1/2}{r-1+1/p'}\right)^{(r-1+1/p')/(r-k+1/p'-1/2)}}{\|g_r\|_{p'}^{(k-1/2)/(r-k-1/2+1/p')}} N^{-(k-1/2)/(r-k+1/p'-1/2)}.$$

The constant coefficient of $N^{-(k-1/2)/(r-k+1/p'-1/2)}$ is unimprovable.

Using inequality (12), we can get the following estimate:

Let $r, k \in \mathbf{N}$ and $k \le r/2$. Then, for any $N > 0$, we have

$$E\left(W_2^{k-1/2}, NW_1^r\right)_\infty \le \frac{r-k+1/2}{r} \left\| \varphi_{k-1/2} \right\|_{L_2(\mathbf{T})}^{r/(r-k-1/2)} \left(\frac{k-1/2}{rN\| \varphi_r \|_{L_\infty(\mathbf{T})}} \right)^{(k-1/2)/(r-k-1/2)} \tag{36}$$

It is well known (see, e.g., [8, Chaps. 4 and 5]) that, for $r, n \in \mathbf{N}$, we have

$$E\left(W_p^r, H\right)_1 \le \left\| \varphi_{n,r} \right\|_{L_{p'}(\mathbf{T})} = \frac{\| \varphi_r \|_{L_{p'}(\mathbf{T})}}{n^r}, \tag{37}$$

where H is either $\mathcal{T}_{2n-1}$ (the set of trigonometric polynomials whose degree does not exceed $n-1$) or $S_{2n,\mu}$, $\mu \in \mathbf{N}$, $\mu \ge r-1$ (the set of polynomial splines of degree μ, defect one, and nodes at the points $\nu\pi/n$, $\nu \in \mathbf{Z}$), and $p' = p/(p-1)$. For fractional r, inequality (37) is known only for $H = \mathcal{T}_{2n-1}$ and $p = 1$ (see [8, pp. 171, 172]).

Using relations (36) and (37) and the method of intermediate approximation, we obtain the following statement:

Theorem 3. *Let $k, n \in \mathbf{N}$ and let H be either $\mathcal{T}_{2n-1}$ or $S_{2n,\mu}$, $\mu \ge 2k-2$. Then*

$$E\left(W_2^{k-1/2}, H\right)_1 \le \left\| \varphi_{n,k-1/2} \right\|_{L_2(\mathbf{T})} = \frac{\left\| \varphi_{k-1/2} \right\|_{L_2(\mathbf{T})}}{n^{k-1/2}}.$$

7. Investigation of Extremal Properties of Polynomials and Splines

As above, we denote by $S_{n,r}$ $(n, r \in \mathbf{N})$ the set of 2π-periodic polynomial splines of degree r, defect one, and nodes at the points $\nu\pi/n$, $\nu \in \mathbf{Z}$, and by $\mathcal{T}_n$ the set of trigonometric polynomials whose degree does not exceed n. We present inequalities of the Bernstein–Nikol'skii type for splines and polynomials, which can be established by using the inequalities presented in Sec. 2 and which are unimprovable in a certain sense. We restrict ourselves to the formulation of applications of inequalities (13)–(15). For known exact inequalities of the Bernstein–Nikol'skii type for polynomials and splines, see [6, Chaps. 3 and 6], [7, Chaps. 3 and 6], and [75].

Let $n, k \in \mathbf{N}$ and $p \in [1, \infty)$. Then

$$\sup_{n \in \mathbf{N}} \sup_{\substack{T_n \in \mathcal{T}_n \\ T_n \ne 0}} \frac{\left\| T_n^{(k)} \right\|_\infty}{n^{k+1/p} \| T_n \|_p} = \frac{1}{\| \cos(\cdot) \|_p}.$$

An analogous result is true for splines. Let $n, r, k \in \mathbf{N}$, $k < r$, and $p \in [1, \infty)$. Then

$$\sup_{n \in \mathbf{N}} \sup_{\substack{s \in S_{n,r} \\ s \ne 0}} \frac{\left\| s^{(k)} \right\|_\infty}{n^{k+1/p} \| s \|_p} = \frac{\| \varphi_{r-k} \|_\infty}{\| \varphi_r \|_p}.$$

The statements below contain exact inequalities of the Nikol'skii type in various metrics for trigonometric polynomials and 2π-periodic polynomial splines; these inequalities are proved by using inequalities (18)–(21). Certain known results in this direction can be found in [6, Chap. 3], [7, Chap. 3], and [75].

Let $p, q \in (0, \infty]$ and $q > p$. Then

$$\sup_{n \in \mathbf{N}} \sup_{\substack{T_n \in \mathcal{T}_n \\ T_n \neq 0}} \frac{\|T_n\|_q}{n^{1/p-1/q}\|T_n\|_p} = \sup_{c \in [0,1]} \frac{\|\cos(\cdot)+c\|_q}{\|\cos(\cdot)+c\|_p}.$$

If $m = p+1$ or $m = \infty$, then

$$\sup_{n \in \mathbf{N}} \sup_{\substack{T_n \in \mathcal{T}_n \\ T_n \neq 0}} \frac{\|T_n - c_m(T_n)\|_q}{n^{1/p-1/q}\|T_n - c_m(T_n)\|_p} = \frac{\|\cos(\cdot)\|_q}{\|\cos(\cdot)\|_p}.$$

Furthermore,

$$\sup_{n \in \mathbf{N}} \sup_{\substack{T_n \in \mathcal{T}_n \\ T_n \neq 0}} \frac{E_0^{\pm}(T_n)_q}{n^{1/p-1/q}E_0^{\pm}(T_n)_p} = \frac{\|\cos(\cdot)+1\|_q}{\|\cos(\cdot)+1\|_p}.$$

Analogous results are also valid for splines. Let $r \in \mathbf{N}$, $p, q \in (0, \infty]$, and $q > p$. Then

$$\sup_{n \in \mathbf{N}} \sup_{\substack{s \in S_{n,r_n} \\ s \neq 0}} \frac{\|s\|_q}{n^{1/p-1/q}\|s\|_p} = \sup_{c \in [0, \|\varphi_r\|_\infty]} \frac{\|\varphi_r(\cdot)+c\|_q}{\|\varphi_r(\cdot)+c\|_p}.$$

If $m = p+1$ or $m = \infty$, then

$$\sup_{n \in \mathbf{N}} \sup_{\substack{s \in S_{n,r} \\ s \neq 0}} \frac{\|s - c_m(s)\|_q}{n^{1/p-1/q}\|s - c_m(s)\|_p} = \frac{\|\varphi_r\|_q}{\|\varphi_r\|_p}$$

and

$$\sup_{n \in \mathbf{N}} \sup_{\substack{s \in S_{n,r} \\ s \neq 0}} \frac{E_0^{\pm}(s)_q}{n^{1/p-1/q}E_0^{\pm}(s)_q} = \frac{\|\varphi_r - \|\varphi_r\|_\infty\|_q}{\|\varphi_r - \|\varphi_r\|_\infty\|_p}.$$

Applications (of somewhat different kind) of Kolmogorov-type inequalities to the investigation of extremal properties of splines can be found in [76].

8. Other Results Related to Kolmogorov-Type Inequalities

Here, we only mention other directions of investigations of Dnepropetrovsk mathematicians related to the determination of Kolmogorov-type exact inequalities for functions of one and many variables and give the corresponding references. First of all, we note that these directions of investigations are more or less completely described in [54, 55, 77].

In [78, 28, 79], the problem of exact constants in additive inequalities for intermediate derivatives of functions defined on a finite interval is studied. In [80, 81], analogous problems are considered for differentiable mappings of Banach spaces and, in particular, for functions. In [82], a Hormander-type exact inequality is obtained for functions defined on a semiaxis. The relationship between exact constants in Kolmogorov-type inequalities for periodic and nonperiodic functions is studied in [83–85]. Finally, the problem of determination of exact constants in Kolmogorov-type inequalities for functions of many variables is investigated in [86–89, 73].

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