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# Strong previsions of random elements

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**Abstract.** Let $C$ be a class of arbitrary real random elements and $P$ an extended real valued function on $C$. Two definitions of coherence for $P$ are compared. Both definitions reduce to the classical de Finetti's one when $C$ includes bounded random elements only. One of the two definitions (called strong coherence) is investigated, and some criteria for checking it are provided. Moreover, conditions are given for the integral representation of a coherent $P$, possibly with respect to a $\sigma$-additive probability. Finally, the two definitions and the integral representation theorems are extended to the case where $C$ is a class of random elements taking values in a given Banach space.

**Key words:** Banach space, coherence, finite additivity, integral representation, strong coherence

**AMS classification:** Primary 60A05, Secondary 28C05

## 1 Introduction

Nowadays, there is a well established theory of coherence for the prevision of real bounded random elements (r.e.'s). It was started by de Finetti in the decade 1930-1940, and developed in 1985 by Holzer [12] and Regazzini [14] in the conditional case. Instead, the case of real, not necessarily bounded, r.e.'s is not so much settled. In fact, in that case, the term "prevision" has different meanings according to different authors. The first definition, due to Berti, Regazzini and Rigo in 1994 [1], retains more features of the bounded case than the second, due to Crisma, Gigante and Millossovich in 1997 [5]. In its turn, the second definition is closer to de Finetti's ideas. Throughout, to avoid misunderstandings, a function $P$ on a class $C$ of real r.e.'s is said to be *coherent* (or a *prevision*) if it satisfies the definition in

[5], and *strongly coherent* (or a *strong prevision*) if it meets the definition in [1]. We note by now that a strong prevision is also a prevision, and that both definitions reduce to the classical de Finetti's one if each element of $\mathcal{C}$ is bounded.

In this paper, the theory of strong previsions is resumed and some new results are given. Denoting by $P$ a function on a class $\mathcal{C}$ of r.e.'s, two distinct situations are dealt with. Firstly, $\mathcal{C}$ is a class of real, not necessarily bounded, r.e.'s and $P$ is allowed to be extended real valued. Subsequently, $\mathcal{C}$ is a class of r.e.'s taking values in a Banach space $\mathcal{Y}$ and $P : \mathcal{C} \to \mathcal{Y}$. Moreover, two problems are considered with particular attention. The first is the checking of strong coherence of a given $P$, and the second is the possible integral representation for $P$. Checking strong coherence is, in general, a difficult task; hence, some criteria are provided. For example, checking strong coherence notably simplifies when $\mathcal{C}$ is a vector lattice including the constants. Representing a prevision $P$ as an integral, with respect to (w.r.t.) some finitely additive probability, is always possible when $\mathcal{C}$ contains only real bounded r.e.'s. Instead, when $\mathcal{C}$ is a class of arbitrary real r.e.'s, integral representation of $P$ can fail even if $P$ is a prevision or a strong prevision. Thus, various conditions for such a representation are given. In particular, we also give conditions for a $\sigma$-additive integral representation. Precisely it is shown that, under strong assumptions on $\mathcal{C}$, the only real valued, coherent function on $\mathcal{C}$ is the integral w.r.t. a $\sigma$-additive probability.

### *1.1 Motivations and preliminaries*

Let $\Omega$ be a non-empty set and $\mathcal{P}(\Omega)$ its power set. Any real function $X$ on $\Omega$ is called a *random quantity* (r.q.). The class of all r.q.'s is denoted by $\mathcal{V}$, and that of bounded r.q.'s by $\mathcal{V}_b$. Apart from the final Section 3, $\mathcal{C}$ always denotes a class of r.q.'s and $P$ a function on $\mathcal{C}$, possibly extended real valued. Moreover, throughout all the paper, a *probability* is meant as a finitely additive probability measure defined on some field of subsets of $\Omega$.

Let us consider first the classical case, where $\mathcal{C} \subset \mathcal{V}_b$, $P$ is real valued on $\mathcal{C}$, and one has to decide whether $P$ is coherent. Then, de Finetti's definition can be restated as follows: $P$ is a *prevision* on $\mathcal{C} \subset \mathcal{V}_b$ if

$$\inf \sum_{i=1}^{n} c_i X_i \leq \sum_{i=1}^{n} c_i P(X_i) \leq \sup \sum_{i=1}^{n} c_i X_i \tag{1}$$

whenever $c_1, \dots, c_n \in R$, $X_1, \dots, X_n \in \mathcal{C}$ and $n \geq 1$. Some consequences of (1) are remarkable:

(i) *A prevision $P$ on $\mathcal{C} \subset \mathcal{V}_b$ can be extended, preserving coherence, to $\mathcal{V}_b$;*

(ii) *$P$ on $\mathcal{C} \subset \mathcal{V}_b$ is a prevision if and only if it can be extended as a linear functional $P'$ on $\mathcal{V}_b$ such that* $\inf X \leq P'(X) \leq \sup X$ *for each $X$ in $\mathcal{V}_b$;*

(iii) *$P$ on $\mathcal{C} \subset \mathcal{V}_b$ is a prevision if and only if there is a probability $\pi$ on $\mathcal{P}(\Omega)$ such that $P(X) = \int X d\pi$ for every $X \in \mathcal{C}$.*

As it is well known, however, the strength of the "principle of coherence" (1) is not only due to its consequences, but also to the fact that it is quite in line with the heuristic meaning one attaches to the term "prevision". To explain this point, given $c_1, \dots, c_n \in R$ and $X_1, \dots, X_n \in \mathcal{C}$, define $G$ to be

$$G(\omega) = G(c_1, \dots, c_n, X_1, \dots, X_n)(\omega) = \sum_{i=1}^{n} c_i(X_i(\omega) - P(X_i)) \quad \forall \quad \omega \in \Omega.$$

If $P(X)$ is regarded as the price of a bet on $X$, then $G$ can be thought of as the gain of a gambler who bets on $X_1, \dots, X_n$ with stakes $c_1, \dots, c_n$. Suppose now that $P$ is assigned by a bookie, and that the bookie is obliged to accept any finite combination of bets proposed by the gambler. Since $P$ is a prevision on $\mathcal{C}$ if and only if

$$\inf G \leq 0 \leq \sup G \quad \forall \ G,$$

no betting system with uniformly positive gains for the gambler can be found when $P$ is a prevision, and this property looks very attractive for the bookie.

Now, let us consider the case when $\mathcal{C}$ is a class of real, not necessarily bounded, r.q.'s and let us try to define what a prevision on $\mathcal{C}$ should be. Whichever the definition is, the starting point is that (i) must hold with $\mathcal{V}$ in the place of $\mathcal{V}_b$. In fact, the possibility of extending our prevision to a larger class is one of the major merits of de Finetti's theory of probability. Another property that seems unavoidable is monotonicity: if $X \geq Y$, then we want $P(X) \geq P(Y)$. A simple example shows that these two requirements suffice to force $P$ not to be real valued. In fact, let $\Omega = (0, +\infty)$, $\mathcal{C}$ the linear space generated by all the indicators, $Y$ the identity function, and $\pi$ a probability on $\mathcal{P}(\Omega)$ such that $\pi((x, +\infty)) = 1$ for every $x > 0$. Then, $P_0(X) := \int X d\pi$ for all $X \in \mathcal{C}$ is a prevision on $\mathcal{C}$ (by (iii)), $Y \geq \alpha I_{(\alpha,+\infty)}$ and $P_0(\alpha I_{(\alpha,+\infty)}) = \alpha$ for all $\alpha > 0$. Thus, $P_0$ cannot be extended to $\mathcal{C} \cup \{Y\}$, in such a way that the extension is real valued and monotone.

Of course, when $P(X)$ is not real, it can not be interpreted as the price of a bet on $X$, and the nice interpretation of coherence in terms of betting schemes falls down.

It is clear at this point that, in order to define a prevision on a class $\mathcal{C}$ of arbitrary r.q.'s , more than one solution can be sensible.

A first definition of coherence for $P$ (*strong coherence*, in the sequel) is in [1]. Taking such definition as a starting point, Crisma, Gigante and Millossovich [5] give another definition (*coherence*, in the sequel) which follows closely de Finetti's idea that coherence is a *minimal* requisite for $P$ to be called a prevision. This latter definition coincides with de Finetti's one in the bounded case, is interpretable in the usual way when $P$ is real valued, allows the extension theorem, leads to linearity when it makes sense, and to the property $\inf X \leq P(X) \leq \sup X$ for every $X \in \mathcal{C}$. However, it also allows previsions $P$ which do not take care of extreme evaluations like $P(A) = 0$ for some event $A$. For example, for a coherent $P$, it is possible that $P(X > 5) = 0$ and $P(X) = 7$ for some *unbounded* r.q. X. It is well possible to accept and justify such previsions; at the same time, we feel that a form of coherence, like strong coherence, that excludes them a priori deserves some attention. Loosely speaking, strong coherence may be thought of as coherence plus the additional request that, if $P(A) = 0$ for some $A \subset \Omega$, then $P(XA) = 0$ for all r.q.'s $X$. We note that this request is satisfied in the bounded case, if $P$ is coherent.

Considerations of this type lead us to study strong coherence, as in [1].

## 2 Strong coherence for previsions of arbitrary random quantities

From now on, we adopt the useful convention of denoting a subset of $\Omega$ and its indicator by the same symbol. Thus, if $A$ is a subset of $\Omega$, then $A$ also denotes the indicator function of $A$. If $\mathcal{A}$ and $\mathcal{B}$ are subclasses of $\mathcal{P}(\Omega)$ and $\mathcal{V}$, respectively, we write $\mathcal{A} \subset \mathcal{B}$ to mean that the indicators of the elements of $\mathcal{A}$ belong to $\mathcal{B}$. Moreover, all integrals are intended in the sense of [11]; see also [3]. In particular, the class of all r.q.'s which are integrable w.r.t. $\pi$, where $\pi$ is any probability on a field, is denoted by $\mathcal{L}^1(\pi)$.

### *2.1 Definition and basic results*

Given a probability $\pi$ on $\mathcal{P}(\Omega)$ and a r.q. $X$, let

$$\text{ess sup}\, X = \inf\{a \in R : \pi(X > a) = 0\},$$

$$\text{ess inf}\, X = \sup\{a \in R : \pi(X < a) = 0\},$$

with the usual conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

**Definition 1 (strong coherence)** Let $\mathcal{C} \subset \mathcal{V}$, $P : \mathcal{C} \to \bar{R}$ and let $\pi$ be a probability on $\mathcal{P}(\Omega)$. Then, $P$ is said to be *strongly coherent with* $\pi$ if

(*) $\text{ess inf}\,(\sum_{i=1}^m a_i A_i + \sum_{i=1}^n b_i X_i)$
$\leq \sum_{i=1}^m a_i \pi(A_i) + \sum_{i=1}^n b_i P(X_i) \leq \text{ess sup}\,(\sum_{i=1}^m a_i A_i + \sum_{i=1}^n b_i X_i)$

for every $n, m \geq 1, a_1, \ldots, a_m, b_1, \ldots, b_n$ in $R$, $A_1, \ldots, A_m$ in $\mathcal{P}(\Omega)$, and $X_1, \ldots, X_n$ in $\mathcal{C}$ such that the quantity $\sum_{i=1}^n b_i P(X_i)$ is well defined (i.e., it is not of the form $+\infty - \infty$). Moreover, $P$ is called *strongly coherent* (or a *strong prevision*) whenever, for some probability $\pi$ on $\mathcal{P}(\Omega)$, $P$ is strongly coherent with $\pi$.

We remark that, if $P$ is strongly coherent with $\pi$, then $P(A) = \pi(A)$ for all $A \in \mathcal{C} \cap \mathcal{P}(\Omega)$. (Given $A \in \mathcal{C} \cap \mathcal{P}(\Omega)$, just apply condition (*) with $m = n = 1$, $a_1 = 1, b_1 = -1$, and $A_1 = X_1 = A$). Moreover, in the sequel, even if $\pi$ is defined only on a field (and not on all $\mathcal{P}(\Omega)$), we will say that $P$ is strongly coherent with $\pi$ whenever $P$ is strongly coherent with some probability on $\mathcal{P}(\Omega)$ extending $\pi$.

Plainly, when $\mathcal{C} \subset \mathcal{V}_b$, $P$ is a strong prevision if and only if it satisfies the usual de Finetti's condition of coherence. Moreover, a strong prevision is also a *prevision*, i.e., it is coherent according to the following definition (cf. [5]):

**Definition 2 (coherence)** Let $\mathcal{C} \subset \mathcal{V}$ and $P : \mathcal{C} \to \bar{R}$. Then $P$ is *coherent* if

$$\inf \sum_{i=1}^n b_i X_i \leq \sum_{i=1}^n b_i P(X_i) \leq \sup \sum_{i=1}^n b_i X_i$$

for every $n \geq 1$, $b_1, \ldots, b_n$ in $R$, and $X_1, \ldots, X_n$ in $\mathcal{C}$ such that the quantity $\sum_{i=1}^{n} b_i P(X_i)$ is well defined.

Loosely speaking, when passing from strong coherence to coherence, the essential extremes are replaced by the actual extremes. Of course, this gives rise to some advantages for Definition 2: its meaning is more transparent, and closer to de Finetti's ideas. However, as noted in Section 1, it may be that $P$ is coherent, $P(X > 5) = 0$, while $P(X) = 7$ for some unbounded r.q. $X$. It is to avoid situations of this type that, in 1994, we discarded the present Definition 2 in favour of Definition 1. In fact, in a sense, Definition 1 amounts to Definition 2 plus the additional request that, if $P(A) = 0$ for some $A \subset \Omega$, then $P(XA) = 0$ for all r.q.'s $X$. The price to be paid is a more involved definition, since one needs a probability $\pi$ to calculate the essential extremes, and generally $\mathcal{C}$ does not include sufficiently many indicators. However, when $\mathcal{C}$ contains indicators enough, Definition 1 strongly simplifies. For example, if $\mathcal{C} \supset \mathcal{P}(\Omega)$, then $P$ is a strong prevision when the restriction $\pi$ of $P$ to $\mathcal{P}(\Omega)$ is a probability, and

$$\operatorname{ess\,inf} \sum_{i=1}^{n} c_i X_i \leq \sum_{i=1}^{n} c_i P(X_i) \leq \operatorname{ess\,sup} \sum_{i=1}^{n} c_i X_i$$

for all $n \geq 1$, $c_1, \ldots, c_n$ in $R$ and $X_1, \ldots, X_n$ in $\mathcal{C}$ such that $\sum_{i=1}^{n} c_i P(X_i)$ is well defined.

Let us turn now to review the main properties of strong previsions.

**Theorem 1** *Let $\mathcal{C} \subset \mathcal{V}$, $P : \mathcal{C} \to \bar{R}$, and $\pi$ a probability on $\mathcal{P}(\Omega)$. If $P$ is strongly coherent with $\pi$, then:*

*(i)* $P(cX) = cP(X)$ *for every $c$ in $R$ and $X$ in $\mathcal{C}$ such that $cX \in \mathcal{C}$;*
*(ii)* $P(X+Y) = P(X) + P(Y)$ *for every $X$, $Y$ in $\mathcal{C}$ such that $P(X) + P(Y)$ is well defined and $X + Y \in \mathcal{C}$;*
*(iii)* $\operatorname{ess\,inf} X \leq P(X) \leq \operatorname{ess\,sup} X$ *for each $X$ in $\mathcal{C}$.*

*Moreover, if $\mathcal{C}$ is a linear space including $\mathcal{P}(\Omega)$ and $P = \pi$ on $\mathcal{P}(\Omega)$, then (i), (ii) and (iii) imply that $P$ is strongly coherent with $\pi$.*

*Proof.* Condition (iii) is a direct consequence of Definition 1, and (i) trivially holds when $P(cX) - cP(X)$ is not well defined. On the other hand, if $P(cX) - cP(X)$ is well defined, (i) follows from Definition 1 with : $m = 1$, $n = 2$, $a_1 = 0$, $b_1 = 1$, $b_2 = -c$, $X_1 = cX$, $X_2 = X$. Analogously, (ii) is immediate when $P(X) + P(Y) - P(X+Y)$ is not well defined. Otherwise, (ii) follows from Definition 1 with: $m = 1$, $n = 3$, $a_1 = 0$, $b_1 = b_2 = 1$, $b_3 = -1$, $X_1 = X$, $X_2 = Y$, $X_3 = X + Y$. Finally, the proof of the second part of the theorem is quite direct. □

Even if $P$ is extended real valued, we will say that $P$ is *linear* in case $P$ meets conditions (i)–(ii) of Theorem 1.

**Theorem 2 (extension theorem)** *Let $\mathcal{C} \subset \mathcal{V}$, $P : \mathcal{C} \to \bar{R}$ and $\pi$ a probability on $\mathcal{P}(\Omega)$. If $P$ is strongly coherent with $\pi$, then $P$ admits an extension $P'$ to $\mathcal{V}$ which is strongly coherent with $\pi$.*

The proof is only sketched, since it consists of a straightforward modification of the proof of the analogous theorem in the classical case $\mathcal{C} \subset \mathcal{V}_b$.

*Sketch of the proof.* Without loss of generality, it can be assumed $\mathcal{P}(\Omega) \subset \mathcal{C}$. The first step is showing that $P$ admits a strongly coherent (with $\pi$) extension to $\mathcal{C} \cup \{X\}$, for every $X$ in $\mathcal{V} \setminus \mathcal{C}$. Fix $X \in \mathcal{V} \setminus \mathcal{C}$. Given $\alpha$ in $\bar{R}$, say that $\alpha$ is too small or too large according to whether

$$\alpha + \sum_{i=1}^{n} c_i P(X_i) < \operatorname{ess\,inf} \left( X + \sum_{i=1}^{n} c_i X_i \right)$$

or

$$\alpha + \sum_{i=1}^{n} c_i P(X_i) > \operatorname{ess\,sup} \left( X + \sum_{i=1}^{n} c_i X_i \right)$$

for some $c_1, \ldots, c_n$ in $R$ and $X_1, \ldots, X_n$ in $\mathcal{C}$ such that $\alpha + \sum_{i=1}^{n} c_i P(X_i)$ is well defined. It is easily seen that there is $\alpha$ which is neither too small nor too large, so that $P'$ defined on $\mathcal{C} \cup \{X\}$ by $P' = P$ on $\mathcal{C}$ and $P'(X) = \alpha$, is strongly coherent with $\pi$. At this stage, a standard argument based on Zorn's lemma concludes the proof. □

The following result is a direct consequence of Theorems 1 and 2.

**Theorem 3** *Let $\mathcal{C} \subset \mathcal{V}$ and $P : \mathcal{C} \to \bar{R}$. Then, $P$ is a strong prevision if and only if there is an extension $P'$ of $P$ to $\mathcal{V}$ which satisfies the following conditions:*

*(i)* *The restriction $\pi$ of $P'$ to $\mathcal{P}(\Omega)$ is a probability;*
*(ii)* $P'(cX) = cP'(X)$ *for every $X$ in $\mathcal{V}$ and $c$ in $R$;*
*(iii)* $P'(X+Y) = P'(X) + P'(Y)$ *for every $X, Y$ in $\mathcal{V}$ such that $P'(X) + P'(Y)$ is well defined;*
*(iv)* $\operatorname{ess\,inf} X \leq P'(X) \leq \operatorname{ess\,sup} X$ *for every $X$ in $\mathcal{V}$.*

### 2.2 Criteria to check strong coherence

In principle, to check strong coherence one has to single out a probability $\pi$ to calculate the essential extremes. This fact makes Definition 1 ill–suited from an operative point of view. However, as already noted, checking strong coherence becomes easier when $\mathcal{C}$ includes sufficiently many indicators. In fact, in that case, it is enough to involve the actual extremes (and not the essential extremes) made on complements of those sets $A \in \mathcal{C}$ such that $P(A) = 0$. Basing on this idea, we now give two criteria for checking strong coherence. Moreover, we provide a third criterion obtained with a slightly different argument.

**Theorem 4** *Let $\mathcal{C} \subset \mathcal{V}$, $P : \mathcal{C} \to \bar{R}$, and let $\mathcal{G}$ be the class of the indicators in $\mathcal{C}$. Assume that:*

*(i)* *For each $X$ in the linear space generated by $\mathcal{C}$, $\{X < a\} \in \mathcal{G}$ for every $a$ in some dense set $D(X) \subset R$;*

*(ii)* $\emptyset \in \mathcal{C}$ *and* $P(\emptyset) = 0$;
*(iii)* $\inf_{A^c} \sum_{i=1}^n c_i X_i \leq \sum_{i=1}^n c_i P(X_i) \leq \sup_{A^c} \sum_{i=1}^n c_i X_i$ *for all* $n \geq 1$, $c_1, \dots, c_n$ *in* $R$, $X_1, \dots, X_n$ *in* $\mathcal{C}$ *such that* $\sum_{i=1}^n c_i P(X_i)$ *is well defined, and for all* $A$ *in* $\mathcal{G}$ *with* $P(A) = 0$.

*Then,* $P$ *is a strong prevision. In fact,* $P$ *is strongly coherent with* $\pi$ *for every probability* $\pi$ *on* $\mathcal{P}(\Omega)$ *which agrees with* $P$ *on* $\mathcal{G}$.

*Proof.* Since $P(\emptyset) = 0$, condition (iii) holds at least for $A^c = \Omega$, and this implies that $P \mid_{\mathcal{G}}$ is coherent. Let $\pi$ be any probability on $\mathcal{P}(\Omega)$ extending $P \mid_{\mathcal{G}}$; we show that $P$ is strongly coherent with $\pi$. In view of (i), fixed $c_1, \dots, c_n$ in $R$ and $X_1, \dots, X_n$ in $\mathcal{C}$ such that $\sum_{i=1}^n c_i P(X_i)$ is well defined, condition (iii) gives

$$\operatorname{ess\,inf} \sum_{i=1}^n c_i X_i \leq \sum_{i=1}^n c_i P(X_i) \leq \operatorname{ess\,sup} \sum_{i=1}^n c_i X_i \tag{2}$$

where the essential extremes are made w.r.t. $\pi$. Now, define $Q$ on $\mathcal{C} \cup \mathcal{P}(\Omega)$ by $Q(X) = P(X)$ for $X \in \mathcal{C}$, and $Q(A) = \pi(A)$ for $A \in \mathcal{P}(\Omega)$, and choose $c_1, \dots, c_n$ in $R$ and $X_1, \dots, X_n$ in $\mathcal{C} \cup \mathcal{P}(\Omega)$ in such a way that $\sum_{i=1}^n c_i Q(X_i)$ is well defined. By (2), condition (*) in Definition 1 holds if $X_i$ belongs to $\mathcal{C}$ for each $i$, and clearly (*) holds if $X_i$ belongs to $\mathcal{P}(\Omega) \setminus \mathcal{C}$ for each $i$. Hence, let us assume that $X_i$ is in $\mathcal{C}$ whenever $i \leq k$ and $X_i$ is in $\mathcal{P}(\Omega) \setminus \mathcal{C}$ if $i > k$, for some $k$ in $\{1, \dots, n-1\}$. Setting $X = \sum_{i=1}^k c_i X_i$ and $Z = \sum_{i=k+1}^n c_i X_i$, we prove that $\sum_{i=1}^n c_i Q(X_i) \leq \operatorname{ess\,sup}(X + Z)$. Let $s := \operatorname{ess\,sup} X$. If $s = +\infty$ then, since $Z$ is bounded, $\operatorname{ess\,sup}(X+Z) = +\infty$. If $s = -\infty$, then $\sum_{i=1}^k c_i P(X_i) \leq s = -\infty$, which in turn implies $\sum_{i=1}^n c_i Q(X_i) = -\infty$. Hence, let $s \in R$. Let us choose $b < s$ and $\epsilon > 0$ in such a way that $b$ and $s + \epsilon$ belong to $D(X)$, and let us introduce the subdivision $b = a_0 < a_1 < \dots < a_m = s + \epsilon$ of $[b, s + \epsilon]$ with: $a_i \in D(X)$ and $a_i - a_{i-1} < \epsilon$ for every $i$. Moreover, put $A_i = \{X < a_i\}$ for $i = 0, \dots, m$ and

$$V = a_0 A_0 + \sum_{i=1}^m a_i (A_i - A_{i-1}).$$

By applying (2) to the r.q. $V - X$,

$$0 \leq \operatorname{ess\,inf}(V - X) \leq a_0 P(A_0) + \sum_{i=1}^m a_i \left(P(A_i) - P(A_{i-1})\right) - \sum_{i=1}^k c_i P(X_i)$$

$$= \int V d\pi - \sum_{i=1}^k c_i P(X_i).$$

Hence, setting $X_b := bA_0 + (1 - A_0)X$, one has

$$\sum_{i=1}^k c_i P(X_i) \leq \int V d\pi \leq \epsilon + \int X_b d\pi. \tag{3}$$

Since $Z$ is bounded, $b$ can be determined in such a way that

$$\operatorname{ess\,sup}(X_b + Z) = \operatorname{ess\,sup}(X + Z),$$

and from (3):

$$\sum_{i=1}^{n} c_i Q(X_i) = \sum_{i=1}^{k} c_i P(X_i) + \int Z d\pi$$

$$\leq \epsilon + \int (X_b + Z) d\pi \leq \epsilon + \operatorname{ess\,sup}(X_b + Z).$$

Thus, $\sum_{i=1}^{n} c_i Q(X_i) \leq \operatorname{ess\,sup}(X + Z)$. In a similar way, one proves that $\sum_{i=1}^{n} c_i Q(X_i) \geq \operatorname{ess\,inf}(X + Z)$. □

**Corollary 1** *Let $\mathcal{C}$, $P$ and $\mathcal{G}$ be as in Theorem 4. Assume that:*

*(i)* *$\mathcal{C}$ is a linear space including the constants;*
*(ii)* *For each $X$ in $\mathcal{C}$, $\{X < a\} \in \mathcal{G}$ for every $a$ in some dense set $D(X) \subset R$;*
*(iii)* $P(\Omega) = 1$;
*(iv)* *$P$ is linear, i.e., assumptions (i)-(ii) of Theorem 1 hold;*
*(v)* *$P(X) \geq 0$ for each $X$ in $\mathcal{C}$ such that $P(X < -\epsilon) = 0$ for every $\epsilon > 0$ with $-\epsilon$ in $D(X)$.*

*Then, $P$ is a strong prevision. In fact, $P$ is strongly coherent with $\pi$ for every probability $\pi$ on $\mathcal{P}(\Omega)$ which agrees with $P$ on $\mathcal{G}$.*

*Proof.* It suffices to prove (iii) of Theorem 4. Fix $c_1, \ldots, c_n$ in $R$ and $X_1, \ldots, X_n$ in $\mathcal{C}$ such that $\sum_{i=1}^{n} c_i P(X_i)$ is well defined, and let $A \in \mathcal{G}$ with $P(A) = 0$. Setting

$$\alpha = \inf_{A^c} \sum_{i=1}^{n} c_i X_i,$$

we now show that $\alpha \leq \sum_{i=1}^{n} c_i P(X_i)$. This is trivial if $\alpha = -\infty$, so that assume $\alpha \in R$. By (i), $\sum_{i=1}^{n} c_i X_i - \alpha$ belongs to $\mathcal{C}$ and

$$P\left(\sum_{i=1}^{n} c_i X_i - \alpha < -\epsilon\right) = 0 \text{ for every } \epsilon > 0 \text{ with } -\epsilon \in D\left(\sum_{i=1}^{n} c_i X_i - \alpha\right).$$

Hence, (iii), (iv) and (v) yield

$$\sum_{i=1}^{n} c_i P(X_i) - \alpha = P\left(\sum_{i=1}^{n} c_i X_i - \alpha\right) \geq 0.$$

Likewise, one proves that $\sum_{i=1}^{n} c_i P(X_i) \leq \sup_{A^c} \sum_{i=1}^{n} c_i X_i$. □

Finally, we give two more theorems. The first one is needed to prove the second, and furthermore it will be useful in next Subsection 2.3. The second one provides a further criterion for checking strong coherence. In both theorems, $\mathcal{C}$ is a vector lattice containing the constants, i.e., $\mathcal{C}$ is a linear space, $\Omega \in \mathcal{C}$ , and $X \vee Y$ and $X \wedge Y$ are in $\mathcal{C}$ whenever $X, Y \in \mathcal{C}$. Further, we let

$$\mathcal{F}_0 := \{A \subset \Omega : \forall \epsilon > 0, \exists X, Y \in \mathcal{C} \text{ with } X \leq A \leq Y \text{ and } P(Y - X) < \epsilon\}.$$

We also recall that, when the domain of $P$ is a linear space, $P$ is a prevision if and only if it is linear (in the sense that (i)-(ii) of Theorem 1 hold) and $\inf X \leq P(X) \leq \sup X$ for each $X$ in the domain; cf. [5], Theorem 3.3.

**Theorem 5** *Let $\mathcal{C}$ be a vector lattice including the constants, and $P : \mathcal{C} \to \bar{R}$ a prevision. Then, $\mathcal{F}_0$ is a field, and there is a unique probability $\pi_0$ on $\mathcal{F}_0$ such that $\mathcal{C} \cap \mathcal{V}_b \subset \mathcal{L}^1(\pi_0)$ and $P(X) = \int X d\pi_0$ for every $X \in \mathcal{C} \cap \mathcal{V}_b$. Moreover,*

$$X \in \mathcal{L}^1(\pi_0) \text{ and } P(X) \geq \int X d\pi_0$$

*whenever $X \in \mathcal{C}$, $P(X) < +\infty$, and $X \geq a$ for some real $a$.*

*Proof.* Let $Q$ be the restriction of $P$ to $\mathcal{C} \cap \mathcal{V}_b$. Then, $Q$ is real valued, and thus it is a linear positive functional on the vector lattice $\mathcal{C} \cap \mathcal{V}_b$ which includes the constants. Let

$$\mathcal{F}_1 = \{A \subset \Omega : \forall \epsilon > 0, \exists X, Y \in \mathcal{C} \cap \mathcal{V}_b \text{ with } X \leq A \leq Y \text{ and } Q(Y - X) < \epsilon\}.$$

By a result in [13] (Lemma 1, p. 171), $\mathcal{F}_1$ is a field and, for each $X \in \mathcal{C} \cap \mathcal{V}_b$ with $X \geq 0$, one has $\{X > \alpha\} \in \mathcal{F}_1$ for all but countably many $\alpha > 0$. Since $\mathcal{C}$ is a vector lattice including the constants, $\mathcal{F}_0 = \mathcal{F}_1$. Thus, $\mathcal{F}_0$ is a field and $\mathcal{C} \cap \mathcal{V}_b \subset \mathcal{L}^1(\nu)$ for every probability $\nu$ on $\mathcal{F}_0$. Next, fix any probability $\pi$ on $\mathcal{P}(\Omega)$ such that $P(X) = \int X d\pi$ for each $X \in \mathcal{C} \cap \mathcal{V}_b$, and call $\pi_0$ the restriction of $\pi$ to $\mathcal{F}_0$. If $X \in \mathcal{C} \cap \mathcal{V}_b$, since $X \in \mathcal{L}^1(\pi_0)$ then $P(X) = \int X d\pi = \int X d\pi_0$. Further, let $\nu$ be a probability on $\mathcal{F}_0$ such that $P(X) = \int X d\nu$ for $X \in \mathcal{C} \cap \mathcal{V}_b$, and let $A \in \mathcal{F}_0$. Given $\epsilon > 0$, take $X, Y \in \mathcal{C} \cap \mathcal{V}_b$ with $X \leq A \leq Y$ and $P(Y - X) < \epsilon$. Then,

$$\nu(A) \leq \int Y d\nu = P(Y) < P(X) + \epsilon = \int X d\pi_0 + \epsilon \leq \pi_0(A) + \epsilon.$$

Hence, $\nu(A) \leq \pi_0(A)$ for each $A \in \mathcal{F}_0$, and taking complements yields $\nu = \pi_0$. Finally, fix $X \in \mathcal{C}$ with $X \geq a$ for some real $a$ and $P(X) < +\infty$. By replacing $X$ with $X - a$, it can be assumed $a = 0$. Since $X \wedge n \in \mathcal{C} \cap \mathcal{V}_b$, one has

$$\int (X \wedge n) d\pi_0 = P(X \wedge n) \leq P(X) < +\infty \text{ for all } n,$$

and thus $\lim_{n,m} \int \mid X \wedge n - X \wedge m \mid d\pi_0 = 0$. Further, the set $\{X > \alpha\}$ belongs to the family $\mathcal{A}_\alpha = \{\{X \wedge k > \alpha\} : k \in N\}$, and $\mathcal{A}_\alpha \subset \mathcal{F}_0$ for all but countably many $\alpha > 0$. Hence, for all but countably many $\epsilon > 0$, one has

$$\{\mid X - X \wedge n \mid > \epsilon\} = \{X > n + \epsilon\} \in \mathcal{F}_0 \text{ for all } n.$$

Given $\epsilon > 0$ as above, one obtains

$$\pi_0(| X - X \wedge n |> \epsilon) = \pi_0(X > n + \epsilon) \leq \frac{1}{n+\epsilon} \int (X \wedge (n+\epsilon)) d\pi_0$$
$$\leq \frac{1}{n} P(X) \to 0.$$

By Theorem 4.4.20 in [3], p. 114, it follows that $X \in \mathcal{L}^1(\pi_0)$ and $\int X d\pi_0 = \sup_n \int (X \wedge n) d\pi_0 = \sup_n P(X \wedge n) \leq P(X)$. □

Next Theorem 6, in addition to give a criterion for checking strong coherence, makes clear the differences between previsions and strong previsions. Define

$$\mathcal{C}^+ = \{X \in \mathcal{C} : X \geq 0\},$$

$$\mathcal{C}^{++} = \{X \in \mathcal{C} : \forall \epsilon > 0, \exists Y \in \mathcal{C}^+ \text{ with } P(Y) < \epsilon \text{ and } Y \geq 1 \text{ on the set } \{X < -\epsilon\}\},$$

and note that, if $\emptyset \in \mathcal{C}$ and $P(\emptyset) = 0$, then $\mathcal{C}^+ \subset \mathcal{C}^{++}$. Then, a strong prevision takes non negative values on all $\mathcal{C}^{++}$, while a prevision $P$ only meets the ordinary positivity property $P(X) \geq 0$ for $X \in \mathcal{C}^+$. At least in the particular case where $\mathcal{C}$ is a vector lattice including the constants, this is the only difference between a prevision and a strong prevision.

**Theorem 6** *Let $\mathcal{C}$ be a vector lattice containing the constants and $P : \mathcal{C} \to \bar{R}$. If $P$ is linear (i.e., conditions (i)-(ii) of Theorem 1 hold), $P(\Omega) = 1$, and*

$$P(X) \geq 0 \text{ whenever } X \in \mathcal{C}^{++}, \tag{4}$$

*then $P$ is strongly coherent.*

*Proof.* First note that $P$ is a prevision, and thus Theorem 5 applies. Let $\pi$ be a probability on $\mathcal{P}(\Omega)$ extending $\pi_0$. We prove that $P$ is strongly coherent with $\pi$. Fix $a_1, \ldots, a_m, b_1, \ldots, b_n$ in $R$, $A_1, \ldots, A_m$ in $\mathcal{P}(\Omega)$, and $X_1, \ldots, X_n$ in $\mathcal{C}$ such that $\sum_{i=1}^n b_i P(X_i)$ is well defined. Let $X = \sum_{i=1}^n b_i X_i$ and $Z = \sum_{i=1}^m a_i A_i$. Since $X \in \mathcal{C}$ and $\sum_{i=1}^n b_i P(X_i)$ is well defined, $P(X) = \sum_{i=1}^n b_i P(X_i)$. Hence, we have to show that $P(X) + \int Z d\pi \geq \operatorname{ess\,inf}(X + Z)$. To this end, it can be assumed that $P(X) < +\infty$ and $\operatorname{ess\,inf} X > -\infty$. (In fact, $\operatorname{ess\,inf}(X + Z) = -\infty$ whenever $\operatorname{ess\,inf} X = -\infty$). Let $a$ be a real number such that $a < \operatorname{ess\,inf} X$, and let $\epsilon > 0$. Since $(X \vee a - X) \in \mathcal{C}^+$, by arguing as in the proof of Theorem 5, there is $\delta \in (0, \epsilon]$ such that $A := \{X - X \vee a < -\delta\} \in \mathcal{F}_0$. Since $A \subset \{X < a\}$, $\pi_0(A) = 0$. By definition of $\mathcal{F}_0$, there are $V, Y \in \mathcal{C} \cap \mathcal{V}_b$ with $V \leq A \leq Y$ and $P(Y - V) < \epsilon$. By Theorem 5, $P(V) = \int V d\pi_0 \leq \pi_0(A) = 0$, and thus $P(Y) \leq P(Y) - P(V) < \epsilon$. Also, $Y \geq A \geq 0$ and, since $\delta \leq \epsilon$, $Y \geq 1$ on the set $\{X - X \vee a < -\epsilon\}$. It follows that $(X - X \vee a) \in \mathcal{C}^{++}$, and thus (4) implies $P(X - X \vee a) \geq 0$. Since $X \vee a \geq X$, one has $P(X) = P(X \vee a)$. In particular, Theorem 5 implies $X \vee a \in \mathcal{L}^1(\pi_0)$ and $P(X \vee a) \geq \int (X \vee a) d\pi_0 = \int (X \vee a) d\pi$. Hence,

$$P(X) + \int Z d\pi = P(X \vee a) + \int Z d\pi \geq \int (X \vee a) d\pi + \int Z d\pi$$

$$= \int X d\pi + \int Z d\pi \geq \operatorname{ess\,inf}(X + Z).$$

Likewise, one proves that $P(X) + \int Z d\pi \leq \operatorname{ess\,sup}(X + Z)$. □

### *2.3 Integral representation of previsions*

The prevision of a r.q. is the counterpart, within the theory of coherence, of the usual expected value of a random variable. It is therefore natural to investigate when a prevision is, like an expected value, an integral w.r.t. some probability. Of course, when a prevision $P$ admits an integral representation, it is also a strong prevision. However, not all strong previsions admit an integral representation.

Let $\pi$ be a probability on some field $\mathcal{F}$ of subsets of $\Omega$.

If $\mathcal{C} \subset \mathcal{L}^1(\pi)$ and $P(X) = \int X d\pi$ for every $X \in \mathcal{C}$, then $P$ is strongly coherent with $\pi$. Indeed, let $\pi'$ be a probability which extends $\pi$ to $\mathcal{P}(\Omega)$, and let $P'(X) = \int X d\pi'$ for every $X$ in $\mathcal{L}^1(\pi')$. Then, Corollary 1 implies that $P'$ is strongly coherent with $\pi'$, and in particular $P$ is strongly coherent with $\pi$. Conversely, assume that $\mathcal{C} \subset \mathcal{L}^1(\pi)$ and $P$ is strongly coherent with $\pi$. Then, $P$ coincides with the integral w.r.t. $\pi$ on $\mathcal{C} \cap \mathcal{V}_b$, but not necessarily on $\mathcal{C} \setminus \mathcal{V}_b$. For instance, let $\mathcal{C} = \mathcal{F} \cup \{X\}$, where $X \in \mathcal{L}^1(\pi)$, $X \geq 0$ and $\operatorname{ess\,sup} X = +\infty$, and let

$$P(A) = \pi(A) \text{ for all } A \text{ in } \mathcal{F} \text{ and } P(X) = c.$$

Then, $P$ is strongly coherent with $\pi$ for each $c \geq \int X d\pi$.

The integral has, however, a special status among the various previsions. For instance, if $P$ is strongly coherent with $\pi$, then $P(X) \geq \int X d\pi$ for each $X$ in $\mathcal{C} \cap \mathcal{L}^1(\pi)$ such that $X \geq a$ for some real $a$; cf. Theorem 5.

To our knowledge, the first integral representation theorem for previsions (and, in particular, for strong previsions) is in [1] (cf. Theorem (2.13)) and covers the case $\mathcal{C} = \mathcal{L}^1(\pi)$. It states that, if $P : \mathcal{L}^1(\pi) \to R$ is a linear positive functional which agrees with $\pi$ on $\mathcal{F}$, then $P$ is the integral w.r.t. $\pi$. Since a prevision is linear and positive, it follows that the only real valued prevision on $\mathcal{L}^1(\pi)$, which agrees with $\pi$ on $\mathcal{F}$, is the integral w.r.t. $\pi$.

One more related reference is [10].

Basing on results in [2], in the present section we improve the earlier integral representation theorem. Roughly speaking, we will obtain that, if $P$ is coherent and real valued on sufficiently many r.q.'s then $P$ is the integral. One consequence is that when $\mathcal{C}$ is sufficiently large, any real valued prevision is necessarily a *strong* prevision. In a sense, this supports the notion of strong prevision. In fact, even if a prevision $P$ is allowed to be extended real valued, from the point of view of interpretation it is desirable that $P$ is finite as often as possible (cf. [5], Remark 5.3). Hence, each sufficiently good (i.e., real valued for sufficiently many r.q.'s) prevision must be a strong prevision.

The following three results give conditions under which the integral, possibly w.r.t. a $\sigma$-additive probability, is the *unique* real valued prevision. Since a real valued prevision on a linear space is a linear positive functional, they are just corollaries of Theorems 2, 3 and 8 in [2], and thus proofs are omitted.

Let $\pi_0$ be the probability introduced in Theorem 5. As usual, $\mathcal{L}^1(\pi_0)$ is a normed space under the norm $\|X\|_1 = \int \mid X \mid d\pi_0$ (and by quotienting according to $X \sim Y$ if and only if $\pi_0^*(\mid X - Y \mid > \epsilon) = 0$ for all $\epsilon > 0$, where $\pi_0^*$ denotes the $\pi_0$-outer measure). By $\bar{\mathcal{C}}$, it is denoted the closure of $\mathcal{C}$ in the $\mathcal{L}^1(\pi_0)$-norm.

**Theorem 7** *Let $\mathcal{C}$ be a vector lattice including the constants, $P$ a real valued prevision on $\mathcal{C}$, and $\pi_0$ the unique probability on $\mathcal{F}_0$ such that $P(X) = \int X d\pi_0$ for all $X \in \mathcal{C} \cap \mathcal{V}_b$. Then, $\mathcal{C} \subset \mathcal{L}^1(\pi_0)$. Moreover, $P(X) = \int X d\pi_0$ for all $X \in \mathcal{C}$ if and only if*

$$\sup\{P(Z) : 0 \leq Z \leq Y, Z \in \mathcal{C}\} < +\infty \textit{ for every } Y \in \bar{\mathcal{C}}, Y \geq 0.$$

**Theorem 8** *Let $\mathcal{C}$, $P$ and $\pi_0$ be as in Theorem 7. In order that $P(X) = \int X d\pi_0$ for all $X \in \mathcal{C}$, it is sufficient that*

$$Y \in \bar{\mathcal{C}}, Y \geq 0 \textit{ and } Y \wedge n \in \mathcal{C} \textit{ for each } n \in N \Rightarrow Y \in \mathcal{C}. \tag{5}$$

Condition (5) trivially holds when $\mathcal{C}$ is closed in the $\mathcal{L}^1(\pi_0)$-norm, and thus Theorem (2.13) of [1] is a corollary of Theorem 8. Further, a suitable strengthening of (5) yields a $\sigma$-additive representation for a real valued prevision.

**Theorem 9** *Let $\mathcal{C}$, $P$ and $\pi_0$ be as in Theorem 7. If*

$$Y \geq 0 \textit{ and } Y \wedge n \in \mathcal{C} \textit{ for each } n \in N \Rightarrow Y \in \mathcal{C}, \tag{6}$$

*then $\pi_0$ is $\sigma$-additive and $P(X) = \int X d\pi_0$ for all $X \in \mathcal{C}$.*

Of course, condition (6) is very strong. However, under (6), the only way to be coherent and real valued is to calculate the integral w.r.t. a $\sigma$-additive probability. This kind of phenomenon is typical of the unbounded case. In fact, if only bounded r.q.'s are concerned, a prevision need not admit a $\sigma$-additive integral representation even if it is defined on its "maximal" domain (which, in this case, is $\mathcal{V}_b$).

Till this point, given a prevision $P$ on $\mathcal{C}$, we have found conditions under which $P$ has an integral representation. Now, we start with a given strong prevision $P$. Then, it is always true that $P$ is the integral on all essentially bounded r.q.'s in its domain.

**Theorem 10** *Let $\pi$ be a probability on $\mathcal{P}(\Omega)$ and*

$$\mathcal{E} = \{X \in \mathcal{V} : -\infty < \operatorname{ess\,inf} X \leq \operatorname{ess\,sup} X < +\infty\}.$$

*If $P : \mathcal{C} \to \bar{R}$ is strongly coherent with $\pi$, then $P(X) = \int X d\pi$ for all $X \in \mathcal{C} \cap \mathcal{E}$.*

*Proof.* Fix $X \in \mathcal{C} \cap \mathcal{E}$, and let $a = \text{ess inf } X$, $b = \text{ess sup } X$. Fixed $\epsilon > 0$, define $A = \{a - \epsilon \le X \le b + \epsilon\}$, so that $\pi(A) = 1$. Let $P'$ be an extension of $P$ to $\mathcal{V}$ strongly coherent with $\pi$. Then, $P(X) = P'(XA) + P'(XA^c) = P'(XA) = \int_A X d\pi = \int X d\pi$, where the second equality is because $P'$ is strongly coherent with $\pi$, and the third depends on $XA \in \mathcal{V}_b$. □

## 3 Coherence and strong coherence for previsions of random elements taking values in a Banach space

So far, the notion of coherence has been investigated with reference to r.q.'s, i.e., real r.e.'s. In this section, we deal with the extension of such a notion to r.e.'s taking values in a general Banach space $\mathcal{Y}$. Indeed, a definition of coherence, for the prevision of r.e.'s with values in spaces different from $R$, is necessary at least for theoretical reasons: to make complete the theory of coherence, in analogy with the usual (i.e., Kolmogorovian) theory of probability, where Banach valued r.e.'s play a role. But, a general notion of coherence can be useful for more practical reasons, too. For instance, it can find applications in mathematical finance; cf. [4].

To our knowledge, the only specific reference is [1]; we resume and extend the work done in that paper, by studying mainly the problem of integral representation of previsions.

In Section 1, we have shown that it is necessary, in order to get the extension theorem and the monotonicity property, that a prevision can take non real values. The natural ordering on $R$ suggests the adoption of $\bar{R}$, instead of $R$, as the range of a prevision. When the r.e.'s take values in a general Banach space $\mathcal{Y}$, no natural ordering is available that can suggest the value of $P$ on some $X$, when every element of $\mathcal{Y}$ is inadequate. Consequently, we will confine ourselves to previsions taking values in $\mathcal{Y}$. Moreover, we give both the definition of coherence and that of strong coherence; our main concern is *strong* coherence, however, for the reasons explained in Section 1.

Let $\mathcal{Y}^*$ denote the dual space of the Banach space $\mathcal{Y}$, i.e., the space of all real valued, continuous, linear functionals on $\mathcal{Y}$. Suppose $\mathcal{Y} = R$ and $P$ does not assume infinite values. For the sake of simplicity, suppose also that $\mathcal{C} \supset \mathcal{P}(\Omega)$. Then, the definition of strong coherence can be restated as (cf. Definition 1): $P$ is a strong prevision if the restriction $\pi$ of $P$ to $\mathcal{P}(\Omega)$ is a probability, and

$$\text{ess inf} \sum_{i=1}^{n} f_i(X_i) \le \sum_{i=1}^{n} f_i(P(X_i)) \le \text{ess sup} \sum_{i=1}^{n} f_i(X_i)$$

for every $n \ge 1, X_1, \ldots, X_n$ in $\mathcal{C}$ and continuous linear functionals $f_1, \ldots, f_n$. In other words, strong coherence of $P$ on r.q.'s means (essential) internality w.r.t. the result of linear operations on finite families of r.q.'s. According to us, this property has to be preserved when passing from r.q.'s to more general r.e.'s.

In the sequel, a function $X : \Omega \to \mathcal{Y}$ is called an *$\mathcal{Y}$-random element* ($\mathcal{Y}$-r.e.), $\mathcal{C}$ is any family of $\mathcal{Y}$-r.e.'s, and $\mathcal{I}$ denotes the class

$$\mathcal{I} = \{yA : y \in \mathcal{Y}, A \subset \Omega\},$$

where $yA$ denotes the $\mathcal{Y}$-r.e. taking value $y$ on $A$ and 0 on $A^c$. Moreover, given any probability $\pi$ on a field, $\mathcal{L}^1_{\mathcal{Y}}(\pi)$ denotes the class of all $\pi$-integrable $\mathcal{Y}$-r.e.'s.

**Definition 3 (coherence on a class of $\mathcal{Y}$-r.e.'s)** $P : \mathcal{C} \to \mathcal{Y}$ is said to be *coherent* if

$$\inf \sum_{i=1}^{n} f_i(X_i) \leq \sum_{i=1}^{n} f_i(P(X_i)) \leq \sup \sum_{i=1}^{n} f_i(X_i) \tag{7}$$

for all $f_1, \ldots, f_n$ in $\mathcal{Y}^*$, $X_1, \ldots, X_n$ in $\mathcal{C}$ and $n \geq 1$. A coherent $P$ is called a *prevision.*

**Definition 4 (strong coherence on a class of $\mathcal{Y}$-r.e.'s)** Let $P : \mathcal{C} \to \mathcal{Y}$ and $\pi$ a probability on $\mathcal{P}(\Omega)$. In case

$$P(yA) = y\pi(A) \text{ for every } yA \text{ in } \mathcal{C} \cap \mathcal{I}, \tag{8}$$

define $Q$ on $\mathcal{C} \cup \mathcal{I}$ as $Q(X) = P(X)$ for $X$ in $\mathcal{C}$ and $Q(yA) = y\pi(A)$ for $yA$ in $\mathcal{I}$. Then, $P$ is said to be *strongly coherent with* $\pi$ if (8) holds together with

$$\text{ess inf} \sum_{i=1}^{n} f_i(X_i) \leq \sum_{i=1}^{n} f_i(Q(X_i)) \leq \text{ess sup} \sum_{i=1}^{n} f_i(X_i)$$

for all $f_1, \ldots, f_n$ in $\mathcal{Y}^*$, $X_1, \ldots, X_n$ in $\mathcal{C} \cup \mathcal{I}$ and $n \geq 1$. Moreover, $P$ is said to be a *strong prevision* if it is strongly coherent with some probability $\pi$ on $\mathcal{P}(\Omega)$.

Clearly, a strong prevision is also a prevision. Moreover, a prevision $P$ is linear, in the sense that $P(aX + bY) = aP(X) + bP(Y)$ whenever $X, Y, aX + bY$ are in $\mathcal{C}$ and $a, b$ are real numbers. In fact, if $P$ is a prevision, for every $f \in \mathcal{Y}^*$ one has

$$f(P(aX + bY) - aP(X) - bP(Y)) = 0, \tag{9}$$

by applying Definition 3 with: $n = 3$, $f_1 = f$, $f_2 = -af$, $f_3 = -bf$, $X_1 = aX + bY$, $X_2 = X$ and $X_3 = Y$. Since (9) holds for every $f$ in $\mathcal{Y}^*$, one obtains $P(aX + bY) = aP(X) + bP(Y)$.

When $\mathcal{Y} = R$, Definition 4 (Definition 3) reduces to Definition 1 (Definition 2) provided that, in the latter, the range of $P$ is assumed to be a subset of $R$. Because of this restriction on the range of $P$, no general extension theorem, of the type of Theorem 2, is available. This remark gets us to think that no general theory can be based on the previous definitions. On the other hand, it is true that the integral w.r.t. some probability $\pi$ is the only prevision (necessarily strong), if the domain $\mathcal{C}$ is sufficiently large.

In particular, Theorem (3.2) of [1] covers the case $\mathcal{C} = \mathcal{L}^1_{\mathcal{Y}}(\pi)$. To state it, suppose $\pi$ is a probability on a field $\mathcal{F}$ and $P : \mathcal{L}^1_{\mathcal{Y}}(\pi) \to \mathcal{Y}$. Then, $P(X) = \int X d\pi$ *for all* $X \in \mathcal{L}^1_{\mathcal{Y}}(\pi)$ *if and only if* $P$ *is linear and*

$$\inf f(X) \leq f(P(X)) \leq \sup f(X) \textit{ for every } X \in \mathcal{L}^1_{\mathcal{Y}}(\pi) \textit{ and } f \in \mathcal{Y}^*, \tag{10}$$

$$\|P(yA)\| = \pi(A) \textit{ for every } A \textit{ in } \mathcal{F} \textit{ and for some } y \textit{ in } \mathcal{Y} \textit{ with } \|y\| = 1. \quad (11)$$

Since a prevision is linear and meets (10) by definition, it follows that the only prevision on $\mathcal{L}^1_{\mathcal{Y}}(\pi)$ satisfying (11) is the integral w.r.t. $\pi$.

One more consequence of Theorem (3.2) is that $P : \mathcal{L}^1_{\mathcal{Y}}(\pi) \to \mathcal{Y}$ is strongly coherent with $\pi$ if and only if it is the integral w.r.t. $\pi$. In fact, if $P$ is strongly coherent with $\pi$, then $P$ is linear and satisfies (10) and (11) by definition. Conversely, suppose that $P(X) = \int X d\pi$ for all $X \in \mathcal{L}^1_{\mathcal{Y}}(\pi)$. Let $\pi'$ be a probability on $\mathcal{P}(\Omega)$ extending $\pi$, and let $P'(X) = \int X d\pi'$ for $X$ in $\mathcal{L}^1_{\mathcal{Y}}(\pi')$. Then, $P'(yA) = \int yA d\pi' = y\pi'(A)$ for every $yA \in \mathcal{I}$. Moreover, given $f_1, \ldots, f_n$ in $\mathcal{Y}^*$ and $X_1, \ldots, X_n$ in $\mathcal{L}^1_{\mathcal{Y}}(\pi')$, one obtains

$$\sum_{i=1}^{n} f_i(P'(X_i)) = \sum_{i=1}^{n} f_i\left(\int X_i d\pi'\right) = \int \sum_{i=1}^{n} f_i(X_i) d\pi'.$$

Thus, $P'$ is strongly coherent with $\pi'$, and in particular $P$ is strongly coherent with $\pi$.

At least in the case $\mathcal{C} = \mathcal{L}^1_{\mathcal{Y}}(\pi)$, thus, coherence and strong coherence admit a nice characterization. In particular, condition (10) plays the role of the usual positivity condition and has a clear geometrical meaning. Indeed, (10) can be equivalently stated as

$$P(X) \text{ belongs to the closed convex hull of the range of } X, \text{ for each } X \in \mathcal{L}^1_{\mathcal{Y}}(\pi). \quad (12)$$

In fact, by a standard separation theorem (cf. [11], V.2.10), (10) implies (12), while the converse is straightforward.

There is another case in which the theory of coherence and strong coherence for $\mathcal{Y}$-r.e.'s is quite simple. Precisely, this happens when $\mathcal{C} \subset \mathcal{K}$, where $\mathcal{K}$ is the set of all $\mathcal{Y}$-r.e.'s whose range has a compact closure. If $\mathcal{Y} = R$, then $\mathcal{K}$ coincides with the class $\mathcal{V}_b$ of all bounded r.q.'s. As next Theorem 11 shows, the theory of coherence and strong coherence on $\mathcal{K}$ parallels the classical de Finetti's theory reminded in Section 1. In particular, when $\mathcal{C} \subset \mathcal{K}$, the notions of coherence and strong coherence coincide.

**Theorem 11** *Let $\mathcal{C} \subset \mathcal{K}$ and $P : \mathcal{C} \to \mathcal{Y}$. Then, $P$ is a prevision if and only if there is a probability $\pi$ on $\mathcal{P}(\Omega)$ such that $P(X) = \int X d\pi$ for every $X \in \mathcal{C}$. In particular, $P$ is a prevision if and only if it is a strong prevision.*

In order to prove Theorem 11 (and the subsequent results), we need a lemma. Given any class $\mathcal{C}$ of $\mathcal{Y}$-r.e.'s, let $\tilde{\mathcal{C}}$ denote the linear space generated by the r.q.'s of the form $f \circ X$, for all $f \in \mathcal{Y}^*$ and $X \in \mathcal{C}$. Further, given a prevision $P$ on $\mathcal{C}$, let

$$\tilde{P}\left(\sum_{i=1}^{n} f_i \circ X_i\right) = \sum_{i=1}^{n} f_i\left(P(X_i)\right) \text{ for every } n \in N, f_1, \ldots, f_n \text{ in } \mathcal{Y}^* \text{ and } X_1, \ldots, X_n \text{ in } \mathcal{C}.$$

**Lemma 1** *If $P$ is a prevision on $\mathcal{C}$, then $\tilde{P}$ is well defined and it is a real valued prevision on $\tilde{\mathcal{C}}$.*

*Proof.* Suppose $\sum_{i=1}^{n} f_i \circ X_i = \sum_{j=1}^{m} g_j \circ Y_j$ for some $f_1, \dots, f_n, g_1, \dots, g_m \in \mathcal{Y}^*$ and $X_1, \dots, X_n, Y_1, \dots, Y_m \in \mathcal{C}$. Then, coherence of $P$ implies

$$0 = \inf \left( \sum_{i=1}^{n} f_i \circ X_i - \sum_{j=1}^{m} g_j \circ Y_j \right) \leq$$

$$\leq \sum_{i=1}^{n} f_i \left(P(X_i)\right) - \sum_{j=1}^{m} g_j \left(P(Y_j)\right) \leq \sup \left( \sum_{i=1}^{n} f_i \circ X_i - \sum_{j=1}^{m} g_j \circ Y_j \right) = 0.$$

Hence, $\sum_{i=1}^{n} f_i \left(P(X_i)\right) = \sum_{j=1}^{m} g_j \left(P(Y_j)\right)$, that is, $\tilde{P}$ is well defined. By definition, $\tilde{P}$ is real valued. Finally, since $\tilde{\mathcal{C}}$ is a linear space and $\tilde{P}$ is linear and such that $\inf \phi \leq \tilde{P}(\phi) \leq \sup \phi$ for each $\phi \in \tilde{\mathcal{C}}$, it follows that $\tilde{P}$ is a prevision. □

*Proof. (of Theorem 11.)* The "if" part is straightforward. We prove the "only if" part. Let $P$ be a prevision. Since $\mathcal{C} \subset \mathcal{K}$, each element of $L := \{f \circ X : X \in \mathcal{C}, f \in \mathcal{Y}^*\}$ is a bounded r.q.. Indeed, $| f \circ X | \leq \|f\| \, \|X\| \leq \alpha$ for some suitable $\alpha$. By Lemma 1, $\tilde{P}(f \circ X) = f(P(X))$ is well defined and coherent on $L$. Hence, there is a probability $\pi$ on $\mathcal{P}(\Omega)$ such that

$$f(P(X)) = \tilde{P}(f \circ X) = \int (f \circ X) d\pi = f\left(\int X d\pi\right) \quad \text{for all } f \in \mathcal{Y}^* \text{ and } X \in \mathcal{C},$$

where the last equality depends on the fact that, since $X$ is in $\mathcal{K}$, $X$ is $\pi$-integrable. Hence, $P(X) = \int X d\pi$ for all $X$ in $\mathcal{C}$. □

One consequence of Theorem 11 is that, if $\mathcal{C} \subset \mathcal{K}$, then $P : \mathcal{C} \to \mathcal{Y}$ is a prevision if and only if $P$ can be extended as a linear function $P' : \mathcal{K} \to \mathcal{Y}$ such that, for each $X \in \mathcal{K}$, $P'(X)$ belongs to the closed convex hull of the range of $X$.

We close the paper with three more results. The first one is the analogous, for $\mathcal{Y}$-r.e.'s, of Theorem 10.

**Theorem 12** *Let $\pi$ be a probability on $\mathcal{P}(\Omega)$ and*

$$\mathcal{E}_1 = \{X : X \textit{ is an } \mathcal{Y}\textit{-r.e. and } \pi(X \in K) = 1 \textit{ for some compact set } K \subset \mathcal{Y}\}.$$

*If $P : \mathcal{C} \to \mathcal{Y}$ is strongly coherent with $\pi$, then $P(X) = \int X d\pi$ for all $X \in \mathcal{C} \cap \mathcal{E}_1$.*

*Proof.* Given $f \in \mathcal{Y}^*$, let $\mathcal{C}_f = \{f \circ X : X \in \mathcal{C}\}$. By Lemma 1, $\tilde{P}(f \circ X) = f(P(X))$ is well defined on $\mathcal{C}_f$. Moreover, a direct calculation shows that $\tilde{P}$ is strongly coherent with $\pi$. Now, fix $X \in \mathcal{C} \cap \mathcal{E}_1$. Since $X \in \mathcal{E}_1$, $X$ is $\pi$-integrable, and since $f \circ X \in \mathcal{C}_f \cap \mathcal{E}$, Theorem 10 yields $f(P(X)) = \tilde{P}(f \circ X) = \int (f \circ X) \, d\pi = f\left(\int X d\pi\right)$. Since $f \in \mathcal{Y}^*$ is arbitrary, this concludes the proof. □

Finally, given a prevision $P$ on $\mathcal{C}$, let $\tilde{P}$ be the prevision on $\tilde{\mathcal{C}}$ involved in Lemma 1 and, as in Section 2, let

$$\mathcal{F}_0 = \{A \subset \Omega : \forall \epsilon > 0, \exists \phi, \psi \in \tilde{\mathcal{C}} \text{ with } \phi \leq A \leq \psi \text{ and } \tilde{P}(\psi - \phi) < \epsilon\}.$$

By Theorem 5, if $\tilde{\mathcal{C}}$ is a vector lattice including the constants, then $\mathcal{F}_0$ is a field and there is a unique probability $\pi_0$ on $\mathcal{F}_0$ such that $\tilde{P}(\phi) = \int \phi d\pi_0$ for every $\phi \in \tilde{\mathcal{C}} \cap \mathcal{V}_b$. By using the material in Subsection 2.3, it is possible to improve the quoted Theorem (3.2) of [1]. This is done, in two slightly different ways, by next Theorems 13 and 14.

**Theorem 13** *Suppose $\mathcal{C}$ includes a non zero constant $\mathcal{Y}$-r.e., $P : \mathcal{C} \to \mathcal{Y}$ is a prevision, and $\tilde{\mathcal{C}}$ is a lattice. Then, $P(X) = \int X d\pi_0$ for every $X \in \mathcal{C} \cap \mathcal{L}^1_{\mathcal{Y}}(\pi_0)$ provided*

$$\phi \in \tilde{\mathcal{C}} \text{ whenever: } \phi \wedge n \in \tilde{\mathcal{C}} \text{ for each } n \in N, \phi \geq 0, \text{ and } \phi \text{ is in the closure of } \tilde{\mathcal{C}} \text{ in the } \mathcal{L}^1(\pi_0)\text{-norm.} \quad (13)$$

*Proof.* Since $\tilde{\mathcal{C}}$ is a lattice and $\mathcal{C}$ includes a non zero constant $\mathcal{Y}$-r.e., $\tilde{\mathcal{C}}$ is a vector lattice containing the constants. By Lemma 1, $\tilde{P}$ is a real valued prevision on $\tilde{\mathcal{C}}$. By (13), condition (5) of Theorem 8 holds with $\tilde{\mathcal{C}}$ in the place of $\mathcal{C}$. It follows that $\tilde{P}(\phi) = \int \phi d\pi_0$ for every $\phi \in \tilde{\mathcal{C}}$. Hence, given $X \in \mathcal{C} \cap \mathcal{L}^1_{\mathcal{Y}}(\pi_0)$, one has

$$f(P(X)) = \tilde{P}(f \circ X) = \int (f \circ X) d\pi_0 = f\left(\int X d\pi_0\right) \text{ for every } f \in \mathcal{Y}^*,$$

and thus $P(X) = \int X d\pi_0$. □

**Theorem 14** *Let $\mathcal{C}$ be a linear space including all constant $\mathcal{Y}$-r.e.'s, $P : \mathcal{C} \to \mathcal{Y}$ a prevision, and $\pi$ any probability on $\mathcal{P}(\Omega)$ such that $\tilde{P}(\phi) = \int \phi d\pi$ for all $\phi \in \tilde{\mathcal{C}} \cap \mathcal{V}_b$. (One such $\pi$ exists by Lemma 1). Suppose that, for each fixed $f \in \mathcal{Y}^*$,*

*(i)* $\mathcal{C}_f = \{f \circ X : X \in \mathcal{C}\}$ *is a lattice;*
*(ii)* $\phi \in \mathcal{C}_f$ *whenever:* $\phi \wedge n \in \mathcal{C}_f$ *for each* $n \in N$, $\phi \geq 0$, *and* $\phi$ *is in the closure of* $\mathcal{C}_f$ *in the* $\mathcal{L}^1(\pi)$*-norm.*

*Then, $P(X) = \int X d\pi$ for every $X \in \mathcal{C} \cap \mathcal{L}^1_{\mathcal{Y}}(\pi)$.*

*Proof.* Fix $f \in \mathcal{Y}^*$, $f \neq 0$. By (i) and the assumptions on $\mathcal{C}$, $\mathcal{C}_f$ is a vector lattice including the constants. By Lemma 1, $\tilde{P}(f \circ X) = f(P(X))$ is a real valued prevision on $\mathcal{C}_f$. Further, $\tilde{P}(\phi) = \int \phi d\pi$ for all $\phi \in \mathcal{C}_f \cap \mathcal{V}_b$. Hence, $\mathcal{C}_f \subset \mathcal{L}^1(\pi)$ and, by (ii), $\tilde{P}(\phi) = \int \phi d\pi$ for all $\phi \in \mathcal{C}_f$; see [2], Theorem 3. Thus, given $X \in \mathcal{C} \cap \mathcal{L}^1_{\mathcal{Y}}(\pi)$, one has $f(P(X)) = \tilde{P}(f \circ X) = \int (f \circ X) d\pi = f\left(\int X d\pi\right)$. Since $f$ is an arbitrary non zero element of $\mathcal{Y}^*$, this concludes the proof. □

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