The Journal of Fourier Analysis and Applications

Volume 6, Issue 5, 2000

A Littlewood-Paley Inequality for the Carleson Operator

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Communicated by T. Körner

ABSTRACT. The Carleson operator is closely related to the maximal partial sum operator for Fourier series. We study generalizations of this operator in one and several variables.

1. Introduction

Let $\mathbb{T} = [0, 2\mathbb{T}]$ and for $f \in L^1(\mathbb{T})$ let $S_n f(x)$ denote the partial sums in the Fourier series for f. Carleson [1] proved that if $f \in L^2(\mathbb{T})$, then $S_n f(x)$ converges to $f(x)$ almost everywhere. Hunt [4] extended this result and proved that if $p > 1$ and $f \in L^p(\mathbb{T})$, then the above convergence also holds. The proof of the convergence is based on the following fact. Set

$$Nf(x) = \sup_n |S_n f(x)| , \quad x \in \mathbb{T}, \quad f \in L^1(\mathbb{T}) .$$

Then the maximal partial sum operator N is bounded on $L^p(\mathbb{T})$, $1 < p < \infty$.

The Carleson operator C is closely related to N and is defined by the formula

$$Cf(x) = \sup_{\xi \in \mathbb{R}} \left| \int_{\mathbb{T}} \frac{1}{x-t} e^{-i\xi t} f(t)\, dt \right| , \quad x \in \mathbb{T}, \quad f \in L^1(\mathbb{T}) ,$$

where the integral is taken in the principal value sense. It is proved in [4] that C is bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$ and the boundedness of N is a consequence of this result.

An alternative proof of convergence almost everywhere of Fourier series was obtained by Fefferman [2].

Sjölin [7] considered an analog of the operator C in several variables. This is obtained by replacing the kernel $1/x$ by a Calderón-Zygmund kernel k. Let $s \geq 2$ be an integer and assume that

Math Subject Classifications. 42A50, 42B20, 42B25.

Keywords and Phrases. Fourier series, maximal partial sum operator, maximal singular integrals, Littlewood-Paley inequality, L^p estimates.

Acknowledgements and Notes. The authors have received support from the European Commission via the TMR network "Harmonic Analysis."

ISSN 1069-5869

k is a complex-valued function on $\mathbb{R}^s \setminus \{0\}$, which is homogeneous of degree $-s$ and belongs to $C^{s+1}(\mathbb{R}^s \setminus \{0\})$. Also assume that

$$\int_{S^{s-1}} k(x) d\,S(x) = 0\,,$$

where $d\,S$ denotes the area measure on the unit sphere S^{s-1}. Set

$$Cf(x) = \sup_{\xi \in \mathbb{R}^s} \left| \int_{\mathbb{T}^s} k(x-t) e^{-i\xi \cdot t} f(t) dt \right|, \quad x \in \mathbb{T}^s, \quad f \in L^1(\mathbb{T}^s)\,,$$

where the integral is taken in the principal value sense. It is proved in [7] that the operator C is bounded on $L^p(\mathbb{T}^s)$ for $1 < p < \infty$.

The proof in the case of several variables is different from the one-dimensional proof in [4] at some points. The greatest difficulty in the case $s \geq 2$ lies in the proof of the inequality needed for the change of pairs, in which one estimates an expression of the form

$$\int_{\omega} k(x-t) \left(1 - e^{i\xi \cdot (x-t)}\right) h(t) dt\,,$$

where ω is a cube in $\mathbb{R}^s$, $x \in \omega$, and $h \in L^1(\omega)$. In the one variable case $k(t) = t^{-1}$ and

$$k(t) \left(1 - e^{i\xi \cdot t}\right) = t^{-1} \left(1 - e^{i\xi t}\right)$$

is a C^∞ function, which makes the estimate of the above integral easy (see [4, p. 252]). In the case when k is a Calderón-Zygmund kernel, the function $k(t)\,(1 - e^{i\xi \cdot t})$ has a singularity at the origin and we need a new idea to get the desired estimate (see [7, p. 72–75]).

The operators C can also be defined in a similar way for $f \in L^p(\mathbb{R}^s)$, $1 < p < \infty$, $s \geq 1$. By use of the homogeneity of the kernels $1/x$ and $k(x)$ it is proved in [7, p. 78] that the corresponding operators are bounded on $L^p(\mathbb{R}^s)$, $1 < p < \infty$.

Here we shall consider some generalizations of the above operators. To formulate the problem let us first introduce some notation. Assume that $\psi \in C_0^\infty(\mathbb{R})$ is even and that $\operatorname{supp} \psi \subset \{x;\, 1/2 \leq |x| \leq 2\}$ and also that

$$\sum_{-\infty}^{\infty} \psi\left(2^j x\right) = 1 \text{ for } x \neq 0\,.$$

Set $\varphi(x) = \psi(x)/x$ so that

$$\varphi\left(2^j x\right) = \frac{1}{2^j x} \psi\left(2^j x\right)\,.$$

We also set $\varphi_j(x) = 2^j \varphi(2^j x)$ and then have

$$\varphi_j(x) = \frac{1}{x} \psi\left(2^j x\right)$$

and $\operatorname{supp} \varphi_j \subset \{x;\, 2^{-j-1} \leq |x| \leq 2^{-j+1}\}$. It is also clear that

$$\sum_{-\infty}^{\infty} \varphi_j(x) = \frac{1}{x}, \quad x \neq 0\,,$$

and

$$\sum_{0}^{\infty} \varphi_j(x) = \frac{1}{x}, \quad 0 < |x| < 1 .$$

We let $(r_j)_0^\infty$ denote the Rademacher functions and set

$$k_s(x) = \sum_{0}^{\infty} r_j(s)\, \varphi_j(x), \quad x \in \mathbb{R}, \quad s \in [0, 1] .$$

We then ask the question if it is possible to replace the kernel $1/x$ in the definition of the operator C by the kernel k_s. We shall prove that this question has a positive answer. More than that we shall prove a more general result.

It is clear that k_s satisfies the following conditions:

$$k \in C^2(\mathbb{R} \setminus \{0\}) ,$$

$$|k(x)| \le C \frac{1}{|x|} , \tag{1.1}$$

$$\left|k'(x)\right| \le C \frac{1}{x^2} , \tag{1.2}$$

$$\left|k''(x)\right| \le C \frac{1}{|x|^3} , \tag{1.3}$$

$$k \text{ is odd} , \tag{1.4}$$

and

$$\operatorname{supp} k \subset [-2, 2] . \tag{1.5}$$

Here the constants C can be taken to be independent of s. We can now formulate our first theorem.

Theorem 1.

Assume that $k \in C^2(\mathbb{R} \setminus \{0\})$ and that k satisfies conditions (1.1) *through* (1.5). *Set*

$$Sf(x) = \sup_{\xi \in \mathbb{R}} \left| \int_{\mathbb{T}} k(x - t)\, e^{-i\xi t} f(t)\, dt \right| , \quad x \in \mathbb{T}, \quad f \in L^1(\mathbb{T}) .$$

Then

$$\|Sf\|_p \le C_p \|f\|_p, 1 < p < \infty ,$$

where $\| \quad \|_p$ *denotes the norm in* $L^p(\mathbb{T})$.

We remark that one cannot directly apply Hunt's proof in [4] to prove this theorem. This is because the function $k(t)\,(1 - e^{i\xi t})$ is not necessarily continuous at the origin. However, if one uses the modifications in Sjölin [7] mentioned above, a proof can be obtained.

As a corollary we prove the following Littlewood-Paley type inequality for the Carleson operator.

Corollary 1.

Let n be a measurable real-valued function on $\mathbb{T}$. *Set*

$$C_j f(x) = \int_{\mathbb{T}} \varphi_j(x - t)\, e^{-in(x)t} f(t)\, dt, \quad x \in \mathbb{T}, j = 0, 1, 2, \ldots .$$

Then

$$\left\| \left(\sum_{0}^{\infty} |C_j f|^2 \right)^{1/2} \right\|_p \le C_p \|f\|_p, \, 1 < p < \infty \,,$$

where the norms are taken in $L^p(\mathbb{T})$, *and* C_p *is independent of the function n.*

Theorem 1 has the following analog in several variables.

Theorem 2.

Let $s \ge 2$ *and assume that* $\Omega \in C^{s+1}(\mathbb{R}^s \setminus \{0\})$ *and that* Ω *is homogeneous of degree* 0 *and also that*

$$\int_{S^{s-1}} \Omega(x) d\, S(x) = 0 \,.$$

Let $\ell \in C^{s+1}(0, \infty)$ *and assume that*

$$|\ell(r)| \le C \frac{1}{r^s} \,,$$

$$|\ell'(r)| \le C \frac{1}{r^{s+1}} \,,$$

$$\vdots$$

$$\left|\ell^{(s+1)}(r)\right| \le C \frac{1}{r^{2s+1}} \,,$$

and also that $\ell(r) = 0$ *for* $r > 2$. *Set* $k(x) = \Omega(x)\ell(|x|)$ *for* $x \in \mathbb{R}^s \setminus \{0\}$ *and*

$$Sf(x) = \sup_{\xi \in \mathbb{R}^s} \left| \int_{\mathbb{T}^s} k(x-t)\, e^{-i\xi \cdot t} f(t)\, dt \right|, \quad x \in \mathbb{T}^s, \quad f \in L^1\left(\mathbb{T}^s\right) \,.$$

Then

$$\|Sf\|_p \le C_p \|f\|_p, \quad 1 < p < \infty \,,$$

where the norms are taken in $L^p(\mathbb{T}^s)$.

We shall also prove weighted estimates with respect to A_p weights (Theorem 4) and moreover vector valued estimates (Theorem 5). This last result has an application in [5].

2. Proofs of Theorems 1 and 2 and Corollaries

We shall now prove Theorem 1. We shall first use some standard Calderón-Zygmund theory, but for completeness we give some details.

Assume that k satisfies (1.1) through (1.5). It is easy to see that

$$\sup_{\substack{r>0 \\ \xi \in \mathbb{R}}} \left| \int_{-r}^{r} e^{i\xi x} k(x)\, dx \right| \le C \,,$$

where the integral is taken in the principal value sense. See Stein [8, p. 36–37]. Then set

$$k_\varepsilon(x) = \begin{cases} k(x), & |x| > \varepsilon \\ 0, & |x| \le \varepsilon \end{cases}, \qquad \varepsilon > 0 \,,$$

and $T_\varepsilon f = k_\varepsilon * f$. We then have $k_\varepsilon \in L^2$ and $|\hat{k}_\varepsilon| \le C$. One also has

$$\int_{|x|\ge 2|y|} |k_\varepsilon(x-y) - k_\varepsilon(x)|\, dx \le C, \quad y \ne 0\,,$$

and

$$\int_{|x|\ge 2|y|} |k(x-y) - k(x)|\, dx \le C, \quad y \ne 0\,.$$

It follows from [8, p. 35] that

$$\|T_\varepsilon f\|_p \le C_p \|f\|_p, \quad 1 < p < \infty\,.$$

Also $\lim_{\varepsilon\to 0} T_\varepsilon f = Tf$ exists in L^p norm if $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Hence, $\|Tf\|_p \le C_p \|f\|_p$, $1 < p < \infty$.

The proof shows that $C_p = \mathcal{O}(p)$ as $p \to \infty$.

We shall now study the corresponding maximal operator.

Claim.

Set

$$T^* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|\,.$$

Then T^* is of weak type $(1,1)$ and strong type (p,p), $1 < p < \infty$, with constant $C_p = \mathcal{O}(1/(p-1))$ as $p \to 1$, and $C_p = \mathcal{O}(p)$ as $p \to \infty$. It follows that $\lim_{\varepsilon\to 0} T_\varepsilon f(x)$ exists almost everywhere if $f \in L^p(\mathbb{R})$, $1 \le p < \infty$.

Proof of Claim. Choose $\varphi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset (-1,1)$, $\int \varphi\, dx = 1$ and $\varphi \ge 0$. Set

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon), \quad \varepsilon > 0\,.$$

First assume $|x| \le 2\varepsilon$. We have

$$\begin{aligned} k * \varphi_\varepsilon(x) &= \int_{|y|\le 3\varepsilon} k(y)\varphi_\varepsilon(x-y)\, dy \\ &= \int_{|y|\le 3\varepsilon} k(y)\,(\varphi_\varepsilon(x-y) - \varphi_\varepsilon(x))\, dy \end{aligned}$$

and

$$|\varphi_\varepsilon(x-y) - \varphi_\varepsilon(x)| = \left|\frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right) - \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon} - \frac{y}{\varepsilon}\right)\right| \le C\frac{|y|}{\varepsilon^2}\,.$$

It follows that

$$|k * \varphi_\varepsilon(x)| \le \int_{|y|\le 3\varepsilon} C\frac{1}{|y|}\frac{|y|}{\varepsilon^2}\, dy = C\frac{1}{\varepsilon}\,.$$

Hence, $|k * \varphi_\varepsilon(x) - k_\varepsilon(x)| \le C\frac{1}{\varepsilon}$ for $|x| \le 2\varepsilon$.

Now, assume $|x| > 2\varepsilon$. We have

$$|k * \varphi_\varepsilon(x) - k_\varepsilon(x)| = \left| \int_{|y|\le\varepsilon} k(x-y)\varphi_\varepsilon(y)\,dy - k(x) \right|$$
$$= \left| \int_{|y|\le\varepsilon} (k(x-y) - k(x))\,\varphi_\varepsilon(y)\,dy \right|$$
$$\le C\frac{1}{x^2} \int_{|y|\le\varepsilon} |y|\varphi_\varepsilon(y)\,dy \le C\frac{\varepsilon}{x^2}\,.$$

Set

$$\psi(x) = \begin{cases} 1, & |x| \le 1 \\ \frac{1}{x^2}, & |x| > 1 \end{cases}.$$

Then

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right) = \begin{cases} \frac{1}{\varepsilon}, & |x| \le \varepsilon \\ \frac{1}{\varepsilon}\frac{\varepsilon^2}{x^2} = \frac{\varepsilon}{x^2}, & |x| > \varepsilon \end{cases}.$$

It follows that

$$|k * \varphi_\varepsilon(x) - k_\varepsilon(x)| \le C\psi_\varepsilon(x)$$

and hence

$$|(k * \varphi_\varepsilon - k_\varepsilon) * f| \le C\psi_\varepsilon * |f| \le C\,Mf\,,$$

where Mf denotes the Hardy-Littlewood maximal function of f. Thus,

$$|T_\varepsilon f| \le C\,Mf + |(k * \varphi_\varepsilon) * f|$$

if $1 < p < \infty$ and $f \in L^p$. We have

$$(k_\delta * \varphi_\varepsilon) * f = \varphi_\varepsilon * (k_\delta * f)$$

and letting $\delta \to 0$ we obtain

$$(k * \varphi_\varepsilon) * f = \varphi_\varepsilon * (Tf)\,.$$

It follows that

$$T^* f \le C\,Mf + C\,M(Tf)\,. \tag{2.1}$$

Hence

$$\left\|T^* f\right\|_p \le C_p \|f\|_p, \quad 1 < p < \infty\,,$$

where $C_p = \mathcal{O}(p)$ as $p \to \infty$.

A weak (1, 1) estimate for T^* follows as in [8, p. 43–45]. Interpolation then gives

$$C_p = \mathcal{O}\left(\frac{1}{p-1}\right), \quad p \to 1\,.$$

The claim is proved. □

Assume ω is a bounded interval, $f \in L^\infty(\omega)$, $f = 0$ outside ω and $\|f\|_\infty \le 1$. We have

$$\left(\int_\omega |T^* f|^p\,dx\right)^{1/p} \le Cp\left(\int_\omega |f|^p dx\right)^{1/p}, \quad p \ge 2\,,$$

and invoking the formula

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

we are able to prove that

$$\int_\omega e^{aT^*f}\,dx \le C|\omega|\,,$$

if $a > 0$ is small enough. It follows that

$$\left|\{x \in \omega;\, T^*f(x) > \lambda\}\right| \le Ce^{-a\lambda}|\omega|, \quad \lambda > 0\,,$$

and for a general function $f \in L^\infty(\omega)$ we obtain

$$\left|\{x \in \omega;\, T^*f(x) > \lambda\}\right| \le Ce^{-a\lambda/\|f\|_\infty}|\omega|, \quad \lambda > 0\,.$$

To prove Theorem 1 we can now use Hunt's proof in [4] with modifications according to [7, p. 69–75]. The proof in [7] is carried out in dimension $s \ge 2$, but it is easily modified to work also in dimension $s = 1$. For instance, on [7, p. 73], a kernel H is defined by the formula

$$H(t) = C_0 J_{s+1/2}(|t|)\,|t|^{-s-1/2}, \quad t \in \mathbb{R}^s \setminus \{0\}\,,$$

where $J_{s+1/2}$ denotes a Bessel function. In our case $s = 1$ we simply take H as the Fejér kernel on the line.

We shall now use Theorem 1 to prove Corollary 1.

Proof of Corollary 1. We set $k_s(x) = \sum_0^N r_j(s)\varphi_j(x)$, where N is a positive integer and φ_j is defined as in the introduction. Also set

$$A_s f(x) = \int_{\mathbb{T}} k_s(x-t)e^{-in(x)t}f(t)\,dt, \quad x \in \mathbb{T}, \quad s \in [0,1]\,.$$

The kernels k_s satisfy the conditions (1.1) through (1.5) uniformly in s and N and it follows from the proof of Theorem 1 that

$$\left(\int_{\mathbb{T}} |A_s f(x)|^p\,dx\right)^{1/p} \le C_p \left(\int_{\mathbb{T}} |f(x)|^p\,dx\right)^{1/p}, \quad 1 < p < \infty\,,$$

where C_p is independent of s and N. We have

$$\begin{aligned} A_s f(x) &= \sum_0^N r_j(s) \int_{\mathbb{T}} \varphi_j(x-t)e^{-in(x)t}f(t)\,dt \\ &= \sum_0^N r_j(s) C_j f(x)\,, \end{aligned}$$

and invoking a well-known inequality for the Rademacher functions we obtain

$$\left\|\left(\sum_0^N |C_j f|^2\right)^{1/2}\right\|_p \le C_p\|f\|_p, \quad 1 < p < \infty\,.$$

Letting $N \to \infty$ we obtain Corollary 1. □

Let K denote a subset of the non-negative integers. Then the kernel

$$k(x) = \sum_{j \in K} \varphi_j(x)$$

satisfies the conditions in Theorem 1. We remark that the corresponding operator S appears in the proof of Lemma 3 in Prestini [5].

The following corollaries deal with maximal operators obtained by considering sharp cut-offs and smooth cut-offs, respectively.

Corollary 2.

Let k satisfy the conditions in Theorem 1. Set

$$S^* f(x) = \sup_{\substack{\varepsilon > 0 \\ \xi \in \mathbb{R}}} \left| \int_{\{t \in \mathbb{T};\, |x-t| > \varepsilon\}} k(x-t) e^{-i\xi t} f(t)\, dt \right|, \quad x \in \mathbb{T}, \quad f \in L^1(\mathbb{T}).$$

Then S^* is bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$.

Proof. It follows from (2.1) that

$$S^* f \leq C\, Mf + C\, M(Sf)$$

and the boundedness of S^* is a consequence of this estimate. □

Corollary 3.

Set

$$\widetilde{S} f(x) = \sup_{\substack{j_0 \geq 0 \\ \xi \in \mathbb{R}}} \left| \int_{\mathbb{T}} \left(\sum_{j=0}^{j_0} r_j(s) \varphi_j(x-t) \right) e^{-i\xi t} f(t)\, dt \right|, \quad x \in \mathbb{T}, \quad f \in L^1(\mathbb{T}).$$

Then $\widetilde{S}$ is a bounded operator on $L^p(\mathbb{T})$ for $1 < p < \infty$.

Proof. Set $k(x) = \sum_0^\infty r_j(s) \varphi_j(x)$ and

$$k_\varepsilon(x) = \begin{cases} k(x), & |x| > \varepsilon \\ 0, & |x| \leq \varepsilon \end{cases} \quad \text{for } \varepsilon > 0.$$

Fix j_0 and set $\varepsilon = 2^{-j_0}$. Then $k_\varepsilon(x) = \sum_0^{j_0} r_j(s) \varphi_j(x)$ for $|x| > 10\varepsilon$ and

$$\left| k_\varepsilon(x) - \sum_0^{j_0} r_j(s) \varphi_j(x) \right| \leq C \frac{1}{\varepsilon} \quad \text{for } |x| \leq 10\varepsilon.$$

It follows that

$$\widetilde{S} f \leq C\, Mf + C\, S^* f$$

and an application of Corollary 2 gives the boundedness of $\widetilde{S}$. □

Proof of Theorem 2. From the conditions on Ω and ℓ in Theorem 2 it follows that

$$|k(x)| \leq C|x|^{-s},$$
$$|D^\alpha k(x)| \leq C|x|^{-s-1} \quad \text{for } |\alpha| = 1,$$
$$|D^\alpha k(x)| \leq C|x|^{-s-2} \quad \text{for } |\alpha| = 2,$$
$$\vdots$$
$$|D^\alpha k(x)| \leq C|x|^{-2s-1} \quad \text{for } |\alpha| = s+1.$$

Using these estimates we can then prove Theorem 2 in the same way as Theorem 1. We omit the details. □

Theorem 3.

Let k satisfy the conditions in Theorem 1 or Theorem 2. For $1 < p < \infty$ and $f \in L^p(\mathbb{R}^s)$ set

$$Sf(x) = \sup_{\xi \in \mathbb{R}^s} \left| \int_{\mathbb{R}^s} k(x-t)e^{-i\xi \cdot t} f(t)\,dt \right|, \quad x \in \mathbb{R}^s.$$

Then S is a bounded operator on $L^p(\mathbb{R}^s)$ for $1 < p < \infty$.

Proof. The proof is simple since k has compact support. We give the proof for $s = 1$ and remark that the proof for $s \geq 2$ can be obtained in the same way.

We first observe that if $f \in L^p(\mathbb{R})$ and f has support in an interval of length 1, then it follows from Theorem 1 that

$$\left(\int_{\mathbb{R}} |Sf|^p dx \right)^{1/p} \leq C_p \left(\int_{\mathbb{R}} |f|^p dx \right)^{1/p} \tag{2.2}$$

if $1 < p < \infty$. For a general $f \in L^p(\mathbb{R})$ write $f = \sum_{-\infty}^{\infty} f_j$, where f_j has support in the interval $[j, j+1]$. It is clear that

$$Sf \leq \sum_{-\infty}^{\infty} Sf_j \quad \text{and}$$
$$|Sf|^p \leq C_p \sum_{-\infty}^{\infty} |Sf_j|^p$$

and the boundedness of S follows if we invoke (2.2). □

3. Weighted and Vector-Valued Inequalities

In this section let the operator S be defined as in Theorem 3. We shall use weight functions w which belong to the Muckenhoupt classes A_p. For the definition of A_p, see García-Cuerva and Rubio de Francia [3, p. 396].

Theorem 4.

Assume $1 < p < \infty$ and that $w \in A_p$. Then

$$\int_{\mathbb{R}^s} |Sf|^p w\,dx \leq C_{p,w} \int_{\mathbb{R}^s} |f|^p w\,dx.$$

Proof. A proof can be obtained by use of the proof of [6, Theorem 2.1 on p. 32]. □

Theorem 5.

Assume $1 < p < \infty$ *and* $1 < q < \infty$. *Then*

$$\left\| \left(\sum_j |Sf_j|^q \right)^{1/q} \right\|_p \leq C_{p,q} \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_p ,$$

where the norms are taken in $L^p(\mathbb{R}^s)$.

Proof. The inequality follows from Theorem 6.4 on p. 519–520 in [3]. □

The estimate of Theorem 5, in the case $q = 2$, $1 < p \leq 2$, is used in the proof of Lemma 3 in [5].

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Received November 4, 1999

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