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Some Essential Properties of $Q_p(\partial\Delta)$-Spaces

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ABSTRACT. *For $p \in (-\infty, \infty)$, let $Q_p(\partial\Delta)$ be the space of all complex-valued functions f on the unit circle $\partial\Delta$ satisfying*

$$\sup_{I\subset\partial\Delta} |I|^{-p} \int_I \int_I \frac{|f(z) - f(w)|^2}{|z - w|^{2-p}} |dz||dw| < \infty ,$$

where the supremum is taken over all subarcs $I \subset \partial\Delta$ with the arclength $|I|$. In this paper, we consider some essential properties of $Q_p(\partial\Delta)$. We first show that if $p > 1$, then $Q_p(\partial\Delta) = BMO(\partial\Delta)$, the space of complex-valued functions with bounded mean oscillation on $\partial\Delta$. Second, we prove that a function belongs to $Q_p(\partial\Delta)$ if and only if it is Möbius bounded in the Sobolev space $\mathcal{L}_p^2(\partial\Delta)$. Finally, a characterization of $Q_p(\partial\Delta)$ is given via wavelets.

1. Introduction

Throughout this paper, suppose that Δ, $\bar{\Delta}$, and $\partial\Delta$ are the open unit disk, the closed unit disk, and the unit circle in the finite complex plane $\mathbb{C}$. For $p \in (-\infty, \infty)$, let $Q_p(\partial\Delta)$ be the space of all Lebesgue measurable functions $f : \partial\Delta \to \mathbb{C}$ with

$$\|f\|_{Q_p(\partial\Delta)} = \sup_{I\subset\partial\Delta} \left[|I|^{-p} \int_I \int_I \frac{|f(z) - f(w)|^2}{|z - w|^{2-p}} |dz||dw|\right]^{\frac{1}{2}} < \infty , \tag{1.1}$$

where the supremum is taken over all subarcs $I \subset \partial\Delta$ of the arclength $|I|$. Note that if $p = 2$, then $Q_p(\partial\Delta) = BMO(\partial\Delta)$, John–Nirenberg's space of functions having bounded mean oscillation on $\partial\Delta$. A Lebesgue measurable function $f : \partial\Delta \to \mathbb{C}$ is in $BMO(\partial\Delta)$ [8] if and only if

$$\|f\|_{BMO(\partial\Delta)} = \sup_{I\subset\partial\Delta} \left[|I|^{-1} \int_I |f(z) - f_I|^2 \, |dz|\right]^{\frac{1}{2}} < \infty , \tag{1.2}$$

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where the supremum ranges over all subarcs $I \subset \partial\Delta$ and f_I stand for the average of f over I

$$f_I = \frac{1}{|I|}\int_I f(z)|dz| .$$

Recall that the space $Q_p(\partial\Delta)$, $p \in (0,1)$ was introduced in [5] (there it was written as Q_p^r) when Essén and Xiao studied the boundary behavior of the holomorphic Q_p-space [1], which is the set of all holomorphic functions f on Δ obeying

$$\|f\|_{Q_p} = \sup_{w\in\Delta}\left[\iint_\Delta |f'(z)|^2\left[1-|\phi_w(z)|^2\right]^p dxdy\right]^{\frac{1}{2}} < \infty, \quad z = x+iy . \tag{1.3}$$

Here and henceforth,

$$\phi_w(z) = \frac{w-z}{1-\overline{w}z} \tag{1.4}$$

is a Möbius transform sending w to 0, and $dxdy$ ($z = x+iy$) means the two-dimensional Lebesgue measure on Δ. Later on, Poisson extension to Δ, $\bar{\partial}$-equations, and a Fefferman–Stein type decomposition of $Q_p(\partial\Delta)$, $p \in (0,1)$ were established by Nicolau and Xiao in [11]. As a continuation of [5], Janson discussed the dyadic analog of $Q_p(\partial\Delta)$, $p \in (0,1)$ [7].

The major purpose of the present paper is to investigate some essential properties of $Q_p(\partial\Delta)$. First, in Section 2 we show that $Q_p(\partial\Delta)$ is nondecreasing with p, in particular $Q_p(\partial\Delta) = BMO(\partial\Delta)$ or $\mathbb{C}$ when $p > 1$ or $p \le -1$. Next, in Section 3 we reveal that $Q_p(\partial\Delta)$ is a Möbious bounded subspace of the Sobolev space on $\partial\Delta$. Finally, we give a description of $Q_p(\partial\Delta)$ in terms of wavelets.

Throughout this paper, the letters C and c denote different positive constants which are not necessarily the same from line to line. Moreover, $A \approx B$ means that there are two constants C and c independent of both A and B to ensure $cA \le B \le CA$. Also, for an $r \in (0,\infty)$ and a subarc I, rI represents the subarc with the same center as I and with the length $r|I|$.

2. Monotonicity

In this section, we focus on the monotonicity of $Q_p(\partial\Delta)$ and discover that the case $p \in (0,1]$ is of independent interest.

Theorem 1.

Let $p \in (-\infty,\infty)$. Then $Q_p(\partial\Delta)$ is nondecreasing with p. In particular,
(i) *If $p \in (-\infty,-1]$, then $Q_p(\partial\Delta) = \mathbb{C}$.*
(ii) *If $-1 < p_1 \neq p_2 \le 1$, then $Q_{p_1}(\partial\Delta) \neq Q_{p_2}(\partial\Delta)$ and $Q_1(\partial\Delta) \neq BMO(\partial\Delta)$.*
(iii) *If $p \in (1,\infty)$, then $Q_p(\partial\Delta) = BMO(\partial\Delta)$.*

Proof. Let $p_1 < p_2$. If $f \in Q_{p_1}(\partial\Delta)$, then for any subarc $I \subset \partial\Delta$,

$$\begin{aligned}\int_I\int_I \frac{|f(z)-f(w)|^2}{|z-w|^{2-p_2}}|dz||dw| &= \int_I\int_I \frac{|f(z)-f(w)|^2}{|z-w|^{2-p_1}}|z-w|^{(p_2-p_1)}|dz||dw| \\ &\le |I|^{p_2-p_1}\int_I\int_I \frac{|f(z)-f(w)|^2}{|z-w|^{2-p_1}}|dz||dw| \\ &\le |I|^{p_2}\|f\|^2_{Q_{p_1}(\partial\Delta)} ,\end{aligned}$$

namely, $f \in Q_{p_2}(\partial\Delta)$. So, $Q_{p_1}(\partial\Delta) \subset Q_{p_2}(\partial\Delta)$.

(i) Let $f \in Q_p(\partial\Delta)$, $p \le -1$ with Fourier series

$$f(z) \sim \sum_n a_n z^n, \quad z \in \partial\Delta .$$

If f is not a constant a.e. on $\partial\Delta$, then there would exist some $a_n \ne 0$ (where $n \ne 0$). It is clear that for any $z \in \partial\Delta$,

$$a_n z^n = \frac{1}{2\pi}\int_{\partial\Delta} f(zw)\,(\bar{w})^n\,|dw| .$$

Put $f_w(z) = f(zw)$. An application of Minkowski's inequality to the last equation implies

$$|a_n|\,\|z^n\|_{Q_p(\partial\Delta)} \le \frac{1}{2\pi}\int_{\partial\Delta} \|f_w\|_{Q_p(\partial\Delta)}\,|dw| \le \|f\|_{Q_p(\partial\Delta)} .$$

Thus, z^n is in $Q_p(\partial\Delta)$, $p \in (-\infty, -1]$. However, there is a small neighborhood $I(1,r) = \{z \in \partial\Delta : |z-1| < r\}$ such that

$$\left|z^n - w^n\right| \ge \frac{|z-w|}{2}, \quad z, w \in I(1,r) ,$$

and

$$\begin{aligned}\left\|z^n\right\|^2_{Q_p(\partial\Delta)} &\ge \frac{1}{(2r)^p}\int_{I(1,r)}\int_{I(1,r)} \frac{|z^n - w^n|^2}{|z-w|^{2-p}}|dz||dw| \\ &\ge \frac{1}{4(2r)^p}\int_{I(1,r)}\int_{I(1,r)} |z-w|^p |dz||dw| \\ &= \infty ,\end{aligned}$$

a contradiction. Hence, f must be a constant a.e. on $\partial\Delta$.

(ii) Consider the following lacunary Fourier series

$$f(z) = \sum_{n=0}^{\infty} a_n z^{2^n}, \quad z \in \partial\Delta .$$

Case 1: $p \in (-1, 1)$. This condition leads to:

$$f \in Q_p(\partial\Delta) \Longleftrightarrow \sum_{n=0}^{\infty} 2^{(1-p)n}\,|a_n|^2 < \infty . \tag{2.1}$$

In fact, if $p \in (-1, 0]$, then $f \in Q_p(\partial\Delta)$ is equivalent to

$$\int_{\partial\Delta}\int_{\partial\Delta} \frac{|f(z) - f(w)|^2}{|z-w|^{2-p}}|dz||dw| < \infty .$$

Further, an application of Parseval's formula to this integral gives (2.1). Also, if $p \in (0, 1)$, then both [1, Theorem 6] and [5, Theorem 2.1] imply (2.1).

Case 2: $p = 1$. If $f \in Q_p(\partial\Delta)$, then

$$\begin{aligned}\infty &> \|f\|^2_{Q_p(\partial\Delta)} \\ &\ge c\int_{\partial\Delta} |w-1|^{-1}\left[\int_{\partial\Delta} |f(zw) - f(z)|^2 |dz|\right]|dw| \\ &\ge c\sum_{n=1}^{\infty} |a_n|^2 \int_0^{\pi} \left(\sin\frac{t}{2}\right)^{-1}\left(\sin 2^{n-1}t\right)^2 dt \\ &\approx \sum_{n=0}^{\infty} n\,|a_n|^2 .\end{aligned} \tag{2.2}$$

In the last estimate we have used a basic fact that for any integer $n \geq 0$,

$$\int_0^{\pi} \left(\sin \frac{t}{2}\right)^{-1} \left(\sin \frac{nt}{2}\right)^2 dt \approx \log(n+1) . \tag{2.3}$$

Case 3: $BMO(\partial\Delta)$. It is well known (cf. [12, p. 178]) that

$$f \in BMO(\partial\Delta) \Longleftrightarrow \sum_{n=0}^{\infty} |a_n|^2 < \infty . \tag{2.4}$$

The above discussion is enough to illuminate (ii). For instance, if

$$f_1(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{(n+1)}, \quad z \in \partial\Delta ,$$

then $f_1 \in BMO(\partial\Delta) \setminus Q_1(\partial\Delta)$ follows from (2.4) and (2.2).

(iii) We take account of the following two cases.

Case 1: $p \in (1, 2]$. At the moment, it follows from the previous argument that $Q_p(\partial\Delta) \subset BMO(\partial\Delta)$. On the other hand, if $f \in BMO(\partial\Delta)$, then with the help of the translation invariance of $BMO(\partial\Delta)$, we get

$$\begin{aligned}
\int_I \int_I \frac{|f(z) - f(w)|^2}{|z-w|^{2-p}} |dz||dw| &\leq C \int_{|t|<|I|} \left[\int_I \left|f\left(ze^{it}\right) - f(z)\right|^2 |dz|\right] \left|\sin \frac{t}{2}\right|^{p-2} dt \\
&\leq C \int_{|t|<|I|} \left[\int_{3I} |f(z) - f_{3I}|^2 |dz|\right] \left|\sin \frac{t}{2}\right|^{p-2} dt \\
&\leq C\|f\|^2_{BMO(\partial\Delta)} |I|^p .
\end{aligned}$$

Thus, $f \in Q_p(\partial\Delta)$ and consequently $Q_p(\partial\Delta) = BMO(\partial\Delta)$.

Case 2: $p \in (2, \infty)$. In this case, $BMO(\partial\Delta) \subset Q_p(\partial\Delta)$ is already known. Now let $f \in Q_p(\partial\Delta)$. Then an elementary geometric analysis gives

$$\begin{aligned}
\int_I \int_I |f(z) - f(w)|^2 |dz||dw| &\leq \sum_{k=1}^{\infty} \iint_{2^{-k}|I|<|z-w|\leq 2^{1-k}|I|} |f(z) - f(w)|^2 |dz||dw| \\
&\leq C \sum_{k=1}^{\infty} \left(\frac{|I|}{2^k}\right)^{2-p} \iint_{|z-w|\leq 2^{1-k}|I|} \frac{|f(z) - f(w)|^2}{|z-w|^{2-p}} |dz||dw| \\
&\leq C \sum_{k=1}^{\infty} \left(\frac{|I|}{2^k}\right)^{2-p} 2^k \left(\frac{|I|}{2^{k-1}}\right)^p \\
&\leq C|I|^2 \sum_{k=1}^{\infty} 2^{-k} ,
\end{aligned}$$

that is to say, $f \in BMO(\partial\Delta)$ and hence $Q_p(\partial\Delta) \subset BMO(\partial\Delta)$. Finally, $Q_p(\partial\Delta) = BMO(\partial\Delta)$ yields. □

Remark 1. The case 1 of (iii) was pointed out in [7] as well. In addition, $Q_1(\partial\Delta)$ contains all functions $f : \partial\Delta \to \mathbb{C}$ obeying

$$|f(z) - f(w)| \leq C \left(\log \frac{2}{|z-w|}\right)^{-1}, \quad z, w \in \partial\Delta .$$

This shows that for $\alpha \in (0, 1)$, all Lip_α functions lie in $Q_1(\partial\Delta)$. But $L^\infty(\partial\Delta)$ is not a subspace of $Q_1(\partial\Delta)$. For example,

$$f_2(z) = \sum_{n=0} 2^{-n} z^{2^n}, \quad z \in \partial\Delta,$$

belongs to $L^\infty(\partial\Delta) \setminus Q_1(\partial\Delta)$ (cf. (2.2) as well as (2.3)).

Since $BMO(\partial\Delta)$ is a Banach space (provided we identify functions which differ a.e. by a constant), we naturally have the following:

Corollary 1.
Let $p \in (-1, \infty)$. Then $Q_p(\partial\Delta)$ is complete with respect to (1.1).

Proof. Let $\{f_n\}$ be a Cauchy sequence in $Q_p(\partial\Delta)$. By Theorem 1, $Q_p(\partial\Delta)$ embeds $BMO(\partial\Delta)$ with the inclusion map bounded. Hence, $\{f_n\}$ is a Cauchy sequence in $BMO(\partial\Delta)$ as well, and $f_n \to f$ in $BMO(\partial\Delta)$ for some f. It follows easily from Fatou's lemma that for every integer $k \geq 1$,

$$\|f - f_k\|_{Q_p(\partial\Delta)} \leq \limsup_{n\to\infty} \|f_n - f_k\|_{Q_p(\partial\Delta)},$$

which implies $f_k \to f$ in $Q_p(\partial\Delta)$. □

3. Connection with the Sobolev Space

From Sections 1 and 2 it turns out that $Q_p(\partial\Delta)$ is closely related to the Sobolev space on $\partial\Delta$. This section clarifies this deep relation.

For $p \in (-\infty, \infty)$, denote by $\mathcal{L}_p^2(\partial\Delta)$ the Sobolev space on $\partial\Delta$, of all Lebesgue measurable functions $f : \partial\Delta \to \mathbb{C}$ for which

$$\|f\|_{\mathcal{L}_p^2(\partial\Delta)} = \left[\int_{\partial\Delta}\int_{\partial\Delta} \frac{|f(z) - f(w)|^2}{|z - w|^{2-p}} |dz||dw|\right]^{\frac{1}{2}} < \infty. \tag{3.1}$$

It is clear that $L^2(\partial\Delta)$ is a subspace of $\mathcal{L}_p^2(\partial\Delta)$, $p > 1$. However, a similar way to show Theorem 1 produces that $\mathcal{L}_p^2(\partial\Delta) = \mathbb{C}$ when $p \in (-\infty, -1]$ and $\mathcal{L}_p^2(\partial\Delta) = L^2(\partial\Delta)$ when $p \in (1, 2]$.

By (1.1) and (3.1) it follows that $Q_p(\partial\Delta)$ is a subspace of $\mathcal{L}_p^2(\partial\Delta)$. Moreover, if $p \in (-\infty, 0]$, then $Q_p(\partial\Delta) = \mathcal{L}_p^2(\partial\Delta)$. Thus, $Q_0(\partial\Delta)$ has the following Möbius boundedness:

$$\|f\|_{Q_0(\partial\Delta)} = \|f \circ \phi_w\|_{\mathcal{L}_0^2(\partial\Delta)}, \quad w \in \Delta.$$

This fact draws our attention to the case $p \in (0, \infty)$. As a matter of fact, we find the following:

Theorem 2.
Let $p \in (0, \infty)$ and let $f \in \mathcal{L}_p^2(\partial\Delta)$. Then $f \in Q_p(\partial\Delta)$ if and only if

$$|||f|||_{Q_p(\partial\Delta)} = \sup_{w\in\Delta} \|f \circ \phi_w\|_{\mathcal{L}_p^2(\partial\Delta)} < \infty. \tag{3.2}$$

Proof. First of all, with the help of (1.4), we establish an identity:

$$\begin{aligned}
\|f \circ \phi_w\|^2_{\mathcal{L}_p^2(\partial\Delta)} &= \int_{\partial\Delta}\int_{\partial\Delta} \frac{|f \circ \phi_w(u) - f \circ \phi_w(v)|^2}{|u - v|^{2-p}} |du||dv| \\
&= \int_{\partial\Delta}\int_{\partial\Delta} \frac{|f(u) - f(v)|^2}{|u - v|^{2-p}} \left(\frac{1 - |w|^2}{|1 - \bar{w}u|\,|1 - \bar{w}v|}\right)^p |du||dv| \\
&= (2\pi)^p \int_{\partial\Delta}\int_{\partial\Delta} \frac{|f(u) - f(v)|^2}{|u - v|^{2-p}} [P_w(u)P_w(v)]^{\frac{p}{2}} |du||dv|,
\end{aligned} \tag{3.3}$$

where

$$P_w(u) = \frac{1-|w|^2}{2\pi\,|1-\bar{w}u|^2}$$

is the Poisson kernel.

Next, we verify the sufficiency. Suppose $|||f|||_{Q_p(\partial\Delta)} < \infty$. Arbitrarily pick a subarc I of $\partial\Delta$. If $I \neq \partial\Delta$, then we choose a point $w \in \Delta \setminus \{0\}$ such that $w/|w|$ and $2\pi(1-|w|)$ are the center and the arclength of I, respectively. If $I = \partial\Delta$, then we take $w = 0$. With such a w, as well as the following inequality:

$$\cos t \geq 1 - \frac{t^2}{2}, \quad t \in (-\infty, \infty)\,,$$

we get that for $u \in I$,

$$P_w(u) \geq \frac{c}{1-|w|} \approx \frac{1}{|I|}\,. \tag{3.4}$$

Applying (3.4) to (3.3), we obtain $\|f\|_{Q_p(\partial\Delta)} \leq C|||f|||_{Q_p(\partial\Delta)} < \infty$.

Finally, we return to the necessity. Let $f \in Q_p(\partial\Delta)$ with $\|f\|_{Q_p(\partial\Delta)} < \infty$. To each point $w \in \partial\Delta \setminus \{0\}$ we associate the subarc I_w with center $w/|w|$ and arclength $2\pi(1-|w|)$. For $w = 0$, we set $I_w = \partial\Delta$. Also, set

$$I^n = 2^n I_w, \quad n = 0, 1, \ldots, N-1\,,$$

where N is the smallest integer such that $2^N|I_w| \geq 2\pi$. Then set $I^N = \partial\Delta$.

Through the help of the elementary inequality:

$$\cos t \leq 1 - \frac{2t^2}{\pi^2}, \quad t \in [-\pi, \pi]\,,$$

we know that for every point $u \in \partial\Delta$,

$$P_w(u) \leq \frac{C}{1-|w|}\,. \tag{3.5}$$

Furthermore, for $u \in \partial\Delta \setminus I^n$,

$$P_w(u) \leq \frac{C}{2^{2n}|w|\,|I_w|}\,.$$

From now on, we may assume that $|w| \geq 1/2$, otherwise, the result is obviously true. Therefore, if $u \in I^{n+1} \setminus I^n$, we have

$$P_w(u) \leq \frac{C}{2^{2n}\,|I_w|}\,. \tag{3.6}$$

With the above notations, we break $\|f \circ \phi_w\|^2_{\mathcal{L}^2_p(\partial\Delta)}$ of (3.3) into two parts.

$$\begin{aligned}
\frac{\|f\circ\phi_w\|^2_{\mathcal{L}^2_p(\partial\Delta)}}{(2\pi)^{\frac{p}{2}}} &= \int_{\partial\Delta}\left(\int_{I_w} + \sum_{n=0}^{N-1}\int_{I^{n+1}\setminus I^n}\right)\frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}\,[P_w(u)P_w(v)]^{\frac{p}{2}}\,|du||dv| \\
&= \int_{\partial\Delta}\int_{I_w}\{\ldots\} + \sum_{n=0}^{N-1}\int_{\partial\Delta}\int_{I^{n+1}\setminus I^n}\{\ldots\} \\
&= A + B\,.
\end{aligned}$$

By Theorem 1, (3.5), (3.6), and the identity:

$$\frac{1}{|I|}\int_I |f(z)-a|^2|dz| = \frac{1}{|I|}\int_I |f(z)-f_I|^2\,|dz| + |f_I - a|^2\,, \quad a\in\mathbb{C}\,,$$

we have

$$\begin{aligned}
A &= \left(\int_{I_w}\int_{I_w} + \sum_{n=0}^{N-1}\int_{I^{n+1}\setminus I^n}\int_{I_w}\right)\{\ldots\}\\
&\le C\|f\|^2_{Q_p(\partial\Delta)} + C\sum_{n=1}^{N-1}\frac{1}{(2^{2n}|I_w|)^p}\int_{I^{n+1}\setminus I^n}\int_{I_w}\frac{|f(u)-f(v)|^2}{|u-v|^{2-p}}|du||dv|\\
&\le C\|f\|^2_{Q_p(\partial\Delta)} + C\sum_{n=1}^{N-1}\frac{1}{(2^{2n}|I_w|)^2}\int_{I^{n+1}\setminus I^n}\int_{I_w}|f(u)-f(v)|^2|du||dv|\\
&\le C\|f\|^2_{Q_p(\partial\Delta)} + C\sum_{n=1}^{N-1}\frac{1}{(2^{2n}|I_w|)^2}\int_{I^{n+1}\setminus I^n}\int_{I_w}\left[\left|f(u)-f_{I_w}\right|^2 + \left|f(v)-f_{I_w}\right|^2\right]|du||dv|\\
&\le C\|f\|^2_{Q_p(\partial\Delta)} + C\left(\sum_{n=1}^{\infty}\frac{1}{2^n}\right)\|f\|^2_{BMO(\partial\Delta)} + C\left(\sum_{n=1}^{\infty}\frac{n^2}{2^n}\right)\|f\|^2_{BMO(\partial\Delta)}\\
&\le C\|f\|^2_{Q_p(\partial\Delta)}\,.
\end{aligned}$$

Concerning B, in the same manner as handling A, we can establish

$$\begin{aligned}
B &= \left(\sum_{n=0}^{N-1}\int_{I_w}\int_{I^{n+1}\setminus I^n} + \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\int_{I^{n+1}\setminus I^n}\int_{I^{m+1}\setminus I^m}\right)\{\ldots\}\\
&\le C\|f\|^2_{Q_p(\partial\Delta)} + \left(\sum_{m=0}^{N-1}\int_{I^1\setminus I_w}\int_{I^{m+1}\setminus I^m} + \sum_{n=1}^{N-1}\sum_{m=0}^{N-1}\int_{I^{n+1}\setminus I^n}\int_{I^{m+1}\setminus I^m}\right)\{\ldots\}\\
&= C\|f\|^2_{Q_p(\partial\Delta)} + \left[\sum_{m=0}^{N-1}\int_{I^1\setminus I_w}\int_{I^{m+1}\setminus I^m} + \sum_{n=1}^{N-1}\left(\sum_{m<n}+\sum_{m\ge n}\right)\int_{I^{n+1}\setminus I^n}\int_{I^{m+1}\setminus I^m}\right]\{\ldots\}\\
&\le C\|f\|^2_{Q_p(\partial\Delta)} + C\left(\sum_{n=1}^{\infty}\frac{n^2}{2^n} + \sum_{n=1}^{\infty}\frac{1}{2^{pn}}\right)\|f\|^2_{BMO(\partial\Delta)}\\
&\le C\|f\|^2_{Q_p(\partial\Delta)}\,.
\end{aligned}$$

Combining the estimations of A and B, we finally reach $|||f|||_{Q_p(\partial\Delta)} < \infty$, which concludes the proof. □

It is very interesting to know that $BMO(\partial\Delta)$ is the Möbius bounded subspace of $\mathcal{L}^2_p(\partial\Delta)$, $p > 1$ (in particular $L^2(\partial\Delta)$). This is probably a new discovery of $BMO(\partial\Delta)$. Observing that $L^2_0(\partial\Delta)$ and $BMO(\partial\Delta)$ are Möbius invariant, we obtain the following.

Corollary 2.

Let $p \in (0,\infty)$. Then $Q_p(\partial\Delta)$ is a Möbius invariant space in the sense of that $|||f|||_{Q_p(\partial\Delta)} = |||f\circ\phi_w|||_{Q_p(\partial\Delta)}$ for any $f \in Q_p(\partial\Delta)$ and $w \in \Delta$.

Proof. It follows easily from Theorem 2. □

Moreover, we would like to point out that a motive behind Theorem 2 and Corollary 2 is the corresponding holomorphic case. Note that $Q_1 = BMOA$ (taking $p = 1$ in (1.3)) and $Q_1(\partial\Delta) \neq$

$BMO(\partial\Delta)$. Now suppose that

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,\quad z\in\bar{\Delta},$$

is a member of the Hardy space H^2. Using Parseval's formula (cf. (2.2) and (2.3)), we see that $f\in\mathcal{L}_1^2(\partial\Delta)$ if and only if

$$\sum_{n=0}^{\infty}|a_n|^2\log(n+1)<\infty,$$

which, as Essén showed us in a private communication [4], is equivalent to

$$\iint_{\Delta}\left|f'(z)\right|^2\mu(|z|)dxdy<\infty,\quad z=x+iy,$$

where

$$\mu(r)=\int_0^{\log\frac{1}{r^2}}|\log s|ds\approx\left(1-r^2\right)\log\frac{1}{1-r^2},\quad r\to1.$$

This formula has not been solved until now, see [14] and its references. These observations tell us that $f\in Q_1(\partial\Delta)$ if and only if

$$\sup_{w\in\Delta}\iint_{\Delta}\left|f'(z)\right|^2\mu\left(|\phi_w(z)|\right)dxdy<\infty,\quad z=x+iy. \tag{3.7}$$

However, (3.7) is different from (1.3) in the case $p=1$. Hence, we have the following:

Remark 2. $BMOA$ does not equal the holomorphic extension of $Q_1(\partial\Delta)$ to Δ.

4. Representation via Wavelets

This section is devoted to discussing expansion of $Q_p(\partial\Delta)$-functions in a series of Haar or wavelet basis.

We start with defining the dyadic $Q_p(\partial\Delta)$ space. Following [7] and using the map: $t\to e^{2\pi it}$, we identify $\partial\Delta$ with the unit interval $[0,1)$, where subintervals may wrap around 0. Meanwhile, a subarc of $\partial\Delta$ corresponds to a subinterval of $[0,1)$. A dyadic interval is an interval of the type: $[m2^{-n},(m+1)2^{-n})$. Denote by $\mathcal{D}$ the set of all dyadic subintervals of $\partial\Delta$ (of course, including $\partial\Delta$ itself). For each $p\in(-\infty,\infty)$, $Q_p^d(\partial\Delta)$, the dyadic counterpart of $Q_p(\partial\Delta)$, is defined by the set of all Lebesgue measurable functions $f:\partial\Delta\to\mathbb{C}$ with

$$\|f\|_{Q_p^d(\partial\Delta)}=\sup_{I\in\mathcal{D}}\left[|I|^{-p}\int_I\int_I\frac{|f(z)-f(w)|^2}{|z-w|^{2-p}}|dz||dw|\right]^{\frac{1}{2}}<\infty. \tag{4.1}$$

Also, $BMO^d(\partial\Delta)$ (defined via replacing the supremum of (1.2) by one taken over all intervals $I\in\mathcal{D}$) stands for the dyadic counterpart of $BMO(\partial\Delta)$ [6]. As in Theorem 1, it is not hard to figure out that $Q_p^d(\partial\Delta)$ is nondecreasing with p, and that $Q_p^d(\partial\Delta)=\mathbb{C}$ whenever $p\in(-\infty,-1]$, as well as $Q_p^d(\partial\Delta)=BMO^d(\partial\Delta)$ whenever $p\in(1,\infty)$. Of course, $Q_p(\partial\Delta)\subsetneq Q_p^d(\partial\Delta)$. A close relation between both (for which the case $p\in(0,1)$ is due to Janson) is delivered by the following:

Theorem 3.
Let $p\in(0,\infty)$. Then $Q_p(\partial\Delta)=Q_p^d(\partial\Delta)\cap BMO(\partial\Delta)$.

Proof. If $p \in (0,1)$, then the proof can be found by [7, Theorem 8]. In fact, Janson's proof is valid for the case $p = 1$ as well. As to $p \in (1,\infty)$, Theorem 3 follows from Theorem 1. □

Janson's demonstration for the case $p \in (0,1)$ of Theorem 3 is based on the local analysis on $Q_p(\partial\Delta)$. It is more helpful to recall his notations. For each interval $I \subset \partial\Delta$ and for each integer $n \geq 0$, denote by $\mathcal{D}_n(I)$ the set of the 2^n subintervals of I with length $2^{-n}|I|$ obtained by n successive bipartition of I. Further, for a Lebesgue measurable function $f: I \to \mathbb{C}$, put

$$R_{f,p}(I) = \sum_{n=0}^{\infty} 2^{-pn} \sum_{J \in \mathcal{D}_n(I)} |J|^{-1} \int_J |f(z) - f_J|^2 |dz| . \tag{4.2}$$

With the aid of (4.2), we have the following conclusion which is due to Janson in the case $p \in (0,1)$.

Lemma 1.

Let $p \in (0,\infty)$ and let $f \in L^2(\partial\Delta)$. Then
(i) *$f \in Q_p^d(\partial\Delta)$ if and only if $\sup_{I\in\mathcal{D}} R_{f,p}(I) < \infty$.*
(ii) *$f \in Q_p(\partial\Delta)$ if and only if $\sup_{I\subset\partial\Delta} R_{f,p}(I) < \infty$, where the supremum is taken over all subarcs I of $\partial\Delta$. In particular, for any subarc $I \subset \partial\Delta$,*

$$|I|^{-p} \int_I \int_I \frac{|f(z) - f(w)|^2}{|z-w|^{2-p}} |dz||dw| \leq C R_{f,p}(I) .$$

Proof. It suffices to verify (ii). If $p \in (0,1)$, then both Lemma 3 and the estimate (13) in [7] indicate the truth of (ii) right now. Although Janson's proof is ready for the case $p \in (0,1)$, it applies to the case $p = 1$. In addition, if $p \in (1,\infty)$, then from the convergence:

$$\sum_{n=0}^{\infty} 2^{-pn} \sum_{J\in\mathcal{D}_n(I)} 1 = \sum_{n=0}^{\infty} 2^{-(p-1)n} < \infty \tag{4.3}$$

it derives that $f \in BMO(\partial\Delta) \Leftrightarrow \sup_{I\subset\partial\Delta} R_{f,p}(I) < \infty$. Since the equivalence: $f \in Q_p(\partial\Delta) \Leftrightarrow f \in BMO(\partial\Delta)$ is known (cf. Theorem 1), the desired assertion yields. □

Let us now take $f \in L^2(\partial\Delta)$ with

$$f = \sum_{\omega\in\mathcal{D}} c(\omega) h_\omega , \tag{4.4}$$

where $\{h_\omega\}_{\omega\in\mathcal{D}}$ is Haar basis on $\partial\Delta$ and

$$c(\omega) = \int_{\partial\Delta} f(z)\overline{h_\omega(z)} |dz| .$$

Carleson [2] pointed out that $f \in BMO^d(\partial\Delta)$ if and only if

$$\sup_{\sigma\in\mathcal{D}} |\sigma|^{-1} \sum_{\omega\subset\sigma} |c(\omega)|^2 < \infty . \tag{4.5}$$

In order to extend this to $Q_p^d(\partial\Delta)$, we need to introduce a formula similar to (4.2). More precisely, for every $I \in \mathcal{D}$ and $f \in L^2(\partial\Delta)$ with (4.4) let

$$S_{f,p}(I) = \sum_{n=0}^{\infty} 2^{-pn} \sum_{\sigma\in\mathcal{D}_n(I)} |\sigma|^{-1} \sum_{\omega\subset\sigma} |c(\omega)|^2 . \tag{4.6}$$

This definition is employed to produce a $Q_p^d(\partial\Delta)$-analog of $BMO^d(\partial\Delta)$.

Theorem 4.
Let $p \in (0,\infty)$ and let $f \in L^2(\partial\Delta)$ with (4.4). Then $f \in Q_p^d(\partial\Delta)$ if and only if

$$|||f|||_{S_p} = \sup_{I\in\mathcal{D}} S_{f,p}(I) < \infty . \tag{4.7}$$

Proof. Because (4.6) and (4.7) rely only upon the dyadic intervals in $\mathcal{D}$, Theorem 4 follows readily from Lemma 1 (i) and the fact that for any $I \in \mathcal{D}$ and $\sigma \in \mathcal{D}_n(I)$,

$$\int_\sigma |f(z) - f_\sigma|^2\,|dz| \approx \sum_{\omega\subset\sigma} |c(\omega)|^2 . \qquad \square$$

Combining Theorem 3 with Theorem 4, we can obtain a characterization of $Q_p(\partial\Delta)$ in terms of $BMO(\partial\Delta)$ and Haar basis $\{h_\omega\}_{\omega\in\mathcal{D}}$. Nevertheless, Haar basis does not possess good smoothness. To further represent $BMO(\partial\Delta)$-functions, Carleson [2] used a modified Haar basis which has some smoothness ($Lip1$ actually), but has no the orthonormal property. Here, it is worth mentioning that Wojtaszczyk [15] chose the orthonormal Franklin system to expand $BMO(\partial\Delta)$-functions. After that, Strömberg [13] modified the Franklin system (later, Lemarié and Meyer [9] and Daubechies [3] consulted other approaches) and finally constructed the so-called orthonormal wavelet basis.

In the sequel, we adapt notations in [10, Section 5.6] (or [16, Sections 2.5 and 8.4]). Suppose $\{1\} \cup \{\psi_{j,k}\}$ ($j = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots, 2^j - 1$) is an orthonormal (periodic Meyer) wavelet basis on $\partial\Delta$ which satisfies the 1-regular condition. For convenience, write the shorter notation $\psi_{j,k}$ as ψ_λ. For every $\lambda = (j,k)$, denote by $I(\lambda)$ the dyadic interval $\{t : 2^j t - k \in [0,1)\}$.

We shall consider functions $f \in L^2(\partial\Delta)$ with the form:

$$f = \sum_\lambda a(\lambda)\psi_\lambda \tag{4.8}$$

where

$$a(\lambda) = (f, \psi_\lambda) = \int_{\partial\Delta} f(z)\overline{\psi_\lambda(z)}|dz| .$$

Like (4.6), for each $I \in \mathcal{D}$ and $f \in L^2(\partial\Delta)$ with (4.8) let

$$T_{f,p}(I) = \sum_{n=0}^{\infty} 2^{-pn} \sum_{J\in\mathcal{D}_n(I)} |J|^{-1} \sum_{I(\lambda)\subset J} |a(\lambda)|^2 . \tag{4.9}$$

Theorem 5.
Let $p \in (0,\infty)$ and let $f \in L^2(\partial\Delta)$ with (4.8). Then $f \in Q_p(\partial\Delta)$ if and only if

$$|||f|||_{T_p} = \sup_{I\in\mathcal{D}} T_{f,p}(I) < \infty . \tag{4.10}$$

Proof. Note that in the case $p > 1$ [cf. (4.3)], (4.10) holds if and only if (4.11) holds, where

$$\sup_{I\in\mathcal{D}} |I|^{-1} \sum_{I(\lambda)\subset I} |a(\lambda)|^2 < \infty . \tag{4.11}$$

In the meantime, $f \in BMO(\partial\Delta)$ if and only if (4.11) is true (cf. [2] and [10, Section 5.6]). So, from our Theorem 1 it turns out that Theorem 5 is valid for $p > 1$. Therefore, it remains to take an account of the case $p \in (0,1]$.

In what is going on, p is always restricted to be in $(0,1]$. However, the proof presented here is actually suitable for $p \in (0,2)$ and hence also for the $BMO(\partial\Delta)$-case. To begin with, we should notice that the support of the wavelet ψ_λ is contained in the interval $mI(\lambda)$, where $m > 0$ is a constant independent of any $I(\lambda)$.

Next, we check the necessity. Let f belong to $Q_p(\partial\Delta)$. Suppose $I \in \mathcal{D}$ and $n = 0, 1, 2, \ldots$. For $J \in \mathcal{D}_n(I)$, we split

$$f = f_{mJ} + (f - f_{mJ})\chi_{mJ} + (f - f_{mJ})\chi_{\partial\Delta\setminus mJ} = f_1 + f_2 + f_3 ,$$

where χ_E is the characteristic function of the set $E \subset \partial\Delta$. By the geometric construction of the support of the wavelets, $(f_3, \psi_\lambda) = 0$ if $I(\lambda) \subset J$. On the other hand, the integral of ψ_λ over $\partial\Delta$ is zero. So $(f, \psi_\lambda) = (f_2, \psi_\lambda)$, furthermore,

$$\sum_{I(\lambda)\subset J} |(f, \psi_\lambda)|^2 \le \sum_\lambda |(f_2, \psi_\lambda)|^2 = |mJ|^{-1} \int_{mJ}\int_{mJ} |f(z) - f(w)|^2 |dz||dw| .$$

This gives that for $J \in \mathcal{D}_n(I)$,

$$|J|^{-1} \sum_{I(\lambda)\subset J} |a(\lambda)|^2 \le (|J||mJ|)^{-1} \int_{mJ}\int_{mJ} |f(z) - f(w)|^2 |dz||dw| .$$

In a completely similar fashion to arguing the inequality (12) of [7], we obtain

$$T_{f,p}(I) \le C|mI|^{-p} \int_{mI}\int_{mI} \frac{|f(z) - f(w)|^2}{|z - w|^{2-p}} |dz||dw| ,$$

which forces (4.10) to come out [owing to $f \in Q_p(\partial\Delta)$].

Conversely, let us claim the sufficiency. Suppose that (4.10) holds. For a given interval $I \subset \partial\Delta$, we define an integer q by $2^{-q} \le |I| < 2^{-q+1}$. At first, we consider small intervals of size $2^{-j} \le 2^{-q}$ and then large intervals for which $2^{-j} > 2^{-q}$. The wavelets corresponding to the small intervals are themselves of two kinds — their supports either meet I or do not meet I. If a small interval $I(\lambda)$ is such that $mI(\lambda)$ intersects with I, then $I(\lambda)$ is certainly included in MI, where $M > 1$ is a constant depending only on m. We write $f = f_1 + f_2$ according to the small and large intervals, then f_1 splits $f_{11} + f_{12}$ and $f_{12} = 0$ on I, whereas f_{11} involves the small intervals $I(\lambda)$ contained in MI. Thus, for any $J \in \mathcal{D}_n(I)$,

$$\int_J \left|f_{11}(z) - (f_{11})_J\right|^2 |dz| \le C \sum_{I(\lambda)\subset MJ} |a(\lambda)|^2 .$$

Consequently,

$$R_{f_1,p}(I) = R_{f_{11},p}(I) \le CT_{f,p}(I) .$$

By Lemma 1 (ii),

$$\int_I\int_I \frac{|f_1(z) - f_1(w)|^2}{|z - w|^{2-p}} |dz||dw| \le C|I|^p . \tag{4.12}$$

We turn to the large intervals and their subseries f_2. Now, we use the fact that the wavelets are "flat" and that, moreover, for a given size 2^{-j} of the large dyadic interval $I(\lambda)$, only M wavelets ψ_λ (expanding f_2) are not identically zero on I (because the support of ψ_λ is a subset of $mI(\lambda)$). For each of the remaining M wavelets ψ_λ, we have

$$|\psi_\lambda(z) - \psi_\lambda(w)| \le C2^{\frac{3j}{2}}|z - w|, \quad z, w \in I ,$$

due to the regularity of the wavelets. Thus,

$$\int_I\int_I \frac{|\psi_\lambda(z) - \psi_\lambda(w)|^2}{|z - w|^{2-p}} |dz||dw| \le C2^{3j}|I|^{p+2} .$$

Since the corresponding wavelet coefficients $|a(\lambda)|$ are bounded above $2^{-j/2}$, Minkowski's inequality deduces

$$\begin{aligned}\left[\int_I\int_I \frac{|f_2(z)-f_2(w)|^2}{|z-w|^{2-p}}|dz||dw|\right]^{\frac{1}{2}} &\le \sum_{j<q}|a(\lambda)|\left[\int_I\int_I \frac{|\psi_\lambda(z)-\psi_\lambda(w)|^2}{|z-w|^{2-p}}|dz||dw|\right]^{\frac{1}{2}}\\ &\le C\sum_{j<q}2^j|I|^{1+\frac{p}{2}}\\ &\le C|I|^{\frac{p}{2}}. \qquad (4.13)\end{aligned}$$

Combining (4.12) and (4.13) we obtain

$$\int_I\int_I \frac{|f(z)-f(w)|^2}{|z-w|^{2-p}}|dz||dw| \le C|I|^p .$$

In other words, $f \in Q_p(\partial\Delta)$. This completes the proof. □

Let U be a mapping with $U(h_\omega) = \psi_\lambda$ (for $\omega = I(\lambda) \in \mathcal{D}$). Then Theorems 4 and 5 tell us that the mapping U can be extended to an isomorphism between $Q_p(\partial\Delta)$ and $Q_p^d(\partial\Delta)$.

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