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ON DETERMINATION OF THE NATURAL FREQUENCY OF TRANSVERSE AND LONGITUDINAL VIBRATIONS OF A CRACKED BEAM. PART 1. ANALYTICAL APPROACH

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We consider a perfectly elastic cantilever beam with a crack. The influence of the crack depth and location on the vibration mode of the beam is simulated by a local change in the cross section. This problem is solved by the Rayleigh method. Fairly simple formulas for the determination of the natural frequency of the first mode of bending and longitudinal vibrations of a cantilever beam with an open and closing crack are derived.

For a long time, substantial research efforts have been focused on the development of relatively simple and reliable nondestructive methods of damage inspection for critical and expensive structural elements in spacecraft and aircraft [1–3], nuclear-power-plant equipment [4, 5], turbines [6–8], bridge structures [9–11], food conduits [12], building structures [13], offshore-drilling platforms [14–16], etc. One of the earliest works on vibration diagnostics dates back to 1944 [17]. This long-standing interest in the problem can be attributed to the ever-pressing requirements of safer and longer life of machinery and structures, on the one hand, and to the fact that it has been rather difficult to obtain a simple and reliable criterion of damage evaluation.

The vibration diagnostics is the most promising method because it is inexpensive and simple to use. This method is based on a relationship between the fatigue-crack parameters, i.e., depth and location, and certain changes in the dynamic characteristics of a structure as a whole and its elements. In the review of works dealing with vibration indicators of fatigue damage, Krawczuk and Ostachowicz [18] demonstrated that the majority of those works had been dedicated to the study of natural frequencies and modes of vibration of a body with a crack.

Many researchers simulate a so-called open crack [6, 19–22] and assume the weakened section to have the same stiffness in both half-cycles of vibrations. The simulation of a closing crack is a more complex problem because in this case the model must include the changes in the stiffness of the weakened section [23, 24]. However, even for an open crack the analytical solutions are rather cumbersome and require the use of numerical methods and are, therefore, of limited applicability for engineering practice.

The simulation of a crack involves fracture-mechanics approaches [21, 25, 26] or assumed stress distributions in the vicinity of the crack tip [22, 27]. Meanwhile, the linear-vibration theory and the theory of elasticity allow one to come up with fairly simple and quite accurate (for engineering applications) expressions for the determination of the natural frequencies of bending and longitudinal vibrations for a beam with a single-edge open crack as well as with a single- and double-edge closing crack.

The objective of the present work is to develop an analytical approach that would yield engineering expressions for the determination of the natural frequency of the first mode of transverse and longitudinal vibrations of a cantilever beam with an open or closing edge crack.

Transverse Vibrations. Consider an ideal elastic cantilever crack-free beam of length L. We determine its natural frequency using the Rayleigh method [28]. For this purpose, the shape of the deflection curve of the beam in vibration is to be set a priori. A fairly good approximation to an actual deflection curve shape can be achieved by subjecting a horizontally positioned beam to a static force that simulates the beam's own mass. We will simulate the mass of the beam by a uniformly distributed load q, and the mass of the end mass of the beam by a concentrated force P.

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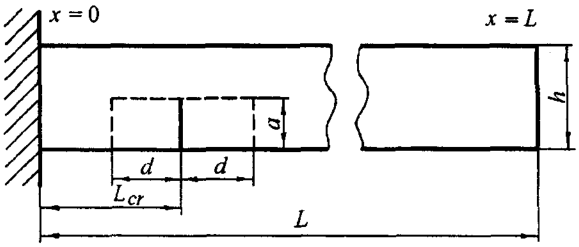

Fig. 1. Schematic representation of a cracked cantilever beam.

From the solution to a differential equation for the beam deflection curve [29] we derive an equation of the elastic line of the cantilever beam subjected to a uniformly distributed load and a concentrated force:

$$y(x) = y_p(x) + y_q(x), \tag{1}$$

$$y_p(x) = \frac{y_p}{2L^3}(3Lx^2 - x^3), \tag{2}$$

$$y_q(x) = \frac{y_q}{3L^4}(6L^2x^2 - 4Lx^3 + x^4), \tag{3}$$

where $y_p = \frac{PL^3}{3EI}$ and $y_q = \frac{qL^4}{8EI}$ are the maximum deflections of the beam end ($x = L$) by the action of the concentrated force P and the uniformly distributed load q, respectively; E is the elastic modulus of the material, and I is the second moment of the cross-sectional area.

To determine the maximum kinetic energy (K) and maximum potential energy (Π) of a beam carrying an end mass, we use the following expressions [28]:

$$K = \frac{1}{2}p^2\left[\int_0^L \rho F y^2(x)dx + m(y_p + y_q)^2\right], \tag{4}$$

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2y}{dx^2}\right)^2 dx, \tag{5}$$

where p is the angular frequency, ρ is the density of the beam material, m is the mass, and F is the cross-sectional area of the beam.

From Eqs. (4) and (5) we obtain a general expression for the natural frequency of vibrations of a beam without a crack:

$$f^t = \frac{p}{2\pi} = \frac{1}{2\pi}\left\{\frac{\int_0^L EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^L \rho F y^2(x)dx + m(y_p + y_q)^2}\right\}^{1/2}. \tag{6}$$

Since the substitution of Eq. (1) into (6) yields a cumbersome expression, in the general case we shall determine the natural frequency of the beam vibration numerically.

Consider now a cracked cantilever beam (Fig. 1). A crack of depth a lies at a distance L_{cr} from the base. We assume that during the half-cycle of vibration when the crack is open, in the volume of the material given by $0.5\pi abd$ (b is the width of the beam) the elastic strain energy is lost. This is similar to what Griffith [30] assumed when determining the conditions of fracture of a plate with an elliptical crack in uniform tension.

The loss of the elastic strain energy in some volume of the beam is equivalent to the removal of this volume of the material. In this case, in the half cycle of the crack opening the beam compliance will increase and the deflection curve will change its shape. To simplify the problem, let us represent this removed volume in the form of a parallelepiped of size $a \times 2d \times b$ (Grabowski [31] considered the volume $a \times a \times b$).

Applying the method of initial parameters [29], we obtain the following equation for an elastic line for the second segment ($L_{cr} \le x \le L$) of the beam in the general case (i.e., including the action of the concentrated force and distributed load):

$$y_{cr}(x) = y_{pcr}(x) + y_{qcr}(x), \tag{7}$$

$$y_{pcr}(x) = \frac{y_p}{2L^3}\left[3Lx^2 - x^3 - (\beta - 1)(\chi_p^3 - \chi_m^3 - 3\lambda_p \chi_p^2 + 3\lambda_m \chi_m^2)\right], \tag{8}$$

$$y_{qcr}(x) = \frac{y_q}{2L^4}\Big\{6L^2x^2 - 4Lx^3 + x^4 - 8\beta dx^3$$

$$- (\beta - 1)\Big[- 24dL_{cr}x^2 - 6\lambda_m^2x^2 + 6\lambda_p^2x^2 - 12\lambda_m x^2(L_{cr} + d) + 12\lambda_p x^2(L_{cr} - d)$$

$$+ 4\lambda_m x^3 - 4\lambda_p x^3 + 4\lambda_p \chi_p^3 - 4\lambda_m \chi_m^3 - 6\lambda_p^2 \chi_p^2 + 6\lambda_m^2 \chi_m^2 - \chi_p^4 + \chi_m^4\Big]\Big\}, \tag{9}$$

where $\beta = I/I_{cr} = 2h^3/[h^3 + (h - 2a)^3]$, I_{cr} is the moment of inertia of the weakened section of the beam, $\lambda_m = L - L_{cr} - d$, $\lambda_p = L - L_{cr} + d$, $\chi_m = x - L_{cr} - d$, and $\chi_p = x - L_{cr} + d$. It is clear that with $\beta = 1$ expression (7) is reduced to the form (1). The respective maximum deflections of the end of the cracked beam ($x = L$) caused by the concentrated force and distributed load are given by

$$y_{pcr} = \frac{y_p}{L^3}\left\{L^3 + 2d(\beta - 1)\left[3(L - L_{cr})^2 + d^2\right]\right\},$$

$$y_{qcr} = \frac{y_q}{3L^4}\left\{3L^4 - 8\beta dL^3 + 8d(\beta - 1)\left[4L^3 - 3LL_{cr}(3L - L_{cr}) - 2dL(3L_{cr} - 4d)\right]\right\}.$$

For the first segment of the beam ($x \le L_{cr}$) we use the equation for an elastic line (1). Then, the maximum kinetic and potential energies of the cracked beam will be represented by the following respective equations:

$$K_{cr} = \frac{1}{2}p^2\left\{\rho F\left[\int_0^{L_{cr}} y^2(x)dx + \int_{L_{cr}}^{L} y_{cr}^2(x)dx\right] + m(y_{pcr} + y_{qcr})^2\right\}, \tag{10}$$

$$\Pi_{cr} = \frac{1}{2}EI\left[\int_0^{L_{cr}} \left(\frac{d^2y}{dx^2}\right)^2 dx + \int_{L_{cr}}^{L} \left(\frac{d^2y_{cr}}{dx^2}\right)^2 dx\right]. \tag{11}$$

Expressions (10) and (11) are valid for the half cycle of vibration when the crack is open. For the half cycle of vibration when the crack is closed and the beam works as a continuous one, we can apply expressions (4) and (5). To determine the natural frequency of vibrations of a beam with a closing crack, let us use the respective values of the maximum energies over the whole cycle of vibration. Then, the general expression for the natural frequency of the beam with a closing crack and with a mass at its end will assume the following form:

$$f_e^t = \frac{1}{2\pi}\left(\frac{EI\left[\int_0^{L}\left(\frac{d^2y}{dx^2}\right)^2 dx + \int_0^{L_{cr}}\left(\frac{d^2y}{dx^2}\right)^2 dx + \int_{L_{cr}}^{L}\left(\frac{d^2y_{cr}}{dx^2}\right)^2 dx\right]}{\rho F\left[\int_0^{L} y^2dx + \int_0^{L_{cr}} y^2dx + \int_{L_{cr}}^{L} y_{cr}^2dx\right] + m\left[(y_p + y_q)^2 + (y_{pcr} + y_{qcr})^2\right]}\right)^{1/2}. \tag{12}$$

The above approach, which yielded expression (12), will be referred to as the energy-based one.

Another approach to the determination of the natural frequency of a cracked beam (we call it the frequency-based approach) is based on the evaluation of the period of natural vibrations of a bilinear system as a sum of half-periods of vibration of two linear systems [32, 33] and is represented by the formula

$$f_1^t = \frac{2f^t f_{cr}^t}{f^t + f_{cr}^t}, \tag{13}$$

where f^t is given by formula (6) and f_{cr}^t by the expression

$$f_{cr}^t = \frac{1}{2\pi}\left(\frac{EI\left[\int_0^{L_{cr}}\left(\frac{d^2y}{dx^2}\right)^2 dx + \int_{L_{cr}}^{L}\left(\frac{d^2y_{cr}}{dx^2}\right)^2 dx\right]}{\rho F\left[\int_0^{L_{cr}} y^2(x)dx + \int_{L_{cr}}^{L} y_{cr}^2(x)dx\right] + m(y_{pcr} + y_{qcr})^2}\right)^{1/2}. \tag{14}$$

Expression (13) determines the vibration frequency of a beam with a single-edge closing crack. For a beam having two closing cracks at its opposite sides, this expression can be rearranged to give

$$f_2^t = \frac{2f_{cr1}^t f_{cr2}^t}{f_{cr1}^t + f_{cr2}^t}. \tag{15}$$

We find the values of f_{cr1}^t and f_{cr2}^t from expression (14), where all the "cr" subscripts are replaced by the "cr1" and "cr2" subscripts, respectively.

In order to simulate a single-edge open crack, it is sufficient to equate the f_{cr1}^t and f_{cr2}^t values in (15). Then, the natural frequency of a beam with a single-edge open crack will be given by

$$f_0^t = f_{cr}^t. \tag{16}$$

Note that in this case the energy- and frequency-based approaches yield the same expression for the frequency.

When we take into account the deflection caused by the concentrated force only, expressions (6) and (14) become much simpler:

$$f_p^t = \frac{1}{2\pi L^2}\left(\frac{420EIL}{33\rho FL + 140m}\right)^{1/2}, \tag{17}$$

$$f_{pcr}^t = \frac{1}{2\pi L^2}\left(\frac{420EIL}{33\rho FL(1 + A) + 140m(1 + B)}\right)^{1/2}, \tag{18}$$

where

$$A = \frac{140}{33L^7}\Big\{4d^2(\beta - 1)^2\left[3(L^5 - L_{cr}^5) + (3d^2 - 15LL_{cr})(L^3 - L_{cr}^3) + (30L^2L_{cr}^2 - 9d^2LL_{cr} + d^4)(L - L_{cr})\right]$$

$$+ 0.1d(\beta - 1)\left[3(11L^6 + L_{cr}^6) + 15L^2L_{cr}^2(3L^2 + L_{cr}^2) - 3LL_{cr}(26L^4 + 6L_{cr}^4) + 5d^2(3L^4 - 4LL_{cr}^3 + L_{cr}^4)\right]\Big\},$$

$$B = \frac{4d}{L^6}\left\{L^3(\beta - 1)\left[3(L - L_{cr})^2 + d^2\right] + d(\beta - 1)^2\left[3(L - L_{cr})^2 + d^2\right]^2\right\}.$$

Then, replacing f^t and f_{cr}^t in (13) with f_p^t and f_{pcr}^t, we arrive at a fairly simple formula for the determination of the natural vibration frequency of a cantilever beam with a closing single-edge crack:

$$f_{p1}^t = \frac{2 f_p^t f_{pcr}^t}{f_p^t + f_{pcr}^t}. \tag{19}$$

For a beam with an open crack, expression (19) can be simplified to give

$$f_{op}^t = f_{pcr}^t. \tag{20}$$

Let us discuss another approach to the determination of the natural-frequency variation for a cantilever beam with a closing crack. We relate this variation to that of the beam compliance. Denoting the compliance of a cracked and crack-free beam by C and C_{cr}, respectively, we can represent the relationship between them by the expression

$$C_{cr} = C(1 + \alpha_{cr}^t). \tag{21}$$

When we take into consideration the action of the concentrated force alone, we can readily derive a formula for α_{cr}^t from Eqs. (2), (8), and (21):

$$\alpha_{cr}^t = \frac{\beta - 1}{L^3}\left[(L - L_{cr} + d)^3 - (L - L_{cr} - d)^3\right]. \tag{22}$$

Formula (13) can be rearranged to give

$$\frac{f_c^t}{f_p^t} = \frac{2}{1 + (1 + \alpha_{cr}^t)^{1/2}}. \tag{23}$$

For the case with an open crack, we can represent expression (23) as follows:

$$\frac{f_{oc}^t}{f_p^t} = \frac{1}{(1 + \alpha_{cr}^t)^{1/2}}. \tag{24}$$

Longitudinal Vibrations. Consider two variants of the determination of the natural frequency of the first-mode longitudinal vibrations of a cracked cantilever beam. They differ merely in the involved distribution functions of displacements along the beam. In these cases, the form and size of the removed volume and all the designations are the same as those for the bending vibrations.

In the first variant (a simpler one), the distribution functions of displacements along the cracked and crack-free beams are assumed to be the same as those in the case with the action of a force applied to the beam end, P:

$$u(x) = \frac{Px}{EF}, \tag{25}$$

$$u_{cr}(x) = \frac{Px}{EF}\left[1 + \frac{2ad}{L(h - a)}\right]. \tag{26}$$

According to this assumption, the maximum kinetic and potential energies of a crack-free and cracked beam will be given by the following respective formulas:

$$K = p^2 \frac{P^2 L^2}{6E^2 F^2}(\rho LF + 3m), \quad K_{cr1} = p^2 \frac{P^2}{6E^2 F^2}\left(L + \frac{2ad}{h - a}\right)^2 (\rho LF + 3m),$$

$$\Pi = \frac{P^2 L}{2EF}, \quad \Pi_{cr1} = \frac{P^2}{2EF}\left(L + \frac{2ad}{h-a}\right).$$

The energy- and frequency-based approaches yield the following expressions for the natural frequency of longitudinal vibrations of a cantilever beam with a closing crack and with a mass at its end:

$$f_{e1}^{l} = \frac{1}{2\pi}\left(\frac{3EF\left(L + \dfrac{ad}{h-a}\right)}{(\rho FL + 3m)\left[L^2 + \dfrac{2Lad}{h-a} + \dfrac{2a^2d^2}{(h-a)^2}\right]}\right)^{1/2}, \tag{27}$$

$$f_1^{l} = \frac{2f^{l} f_{cr1}^{l}}{f^{l} + f_{cr1}^{l}}, \tag{28}$$

where

$$f^{l} = \frac{1}{2\pi L}\left(\frac{3EFL}{\rho FL + 3m}\right)^{1/2}, \tag{29}$$

$$f_{cr1}^{l} = \frac{1}{2\pi L}\left(\frac{3EFL}{(\rho FL + 3m)\left[1 + \dfrac{2ad}{L(h-a)}\right]}\right)^{1/2}. \tag{30}$$

By analogy with the bending vibrations, in the second variant a cracked beam is assumed to consist of two segments. The displacement of the first segment ($x \le L_{cr}$) is given by formula (23), while that of the second one ($L_{cr} \le x \le L$) is given by

$$u_{cr2}(x) = \frac{P(x - 2d)}{EF} + \frac{2Pd}{EF_1}, \tag{31}$$

where $F_1 = d(h - a)$ is the area of the weakened section.

Then, the maximum kinetic energy of a cracked beam (the potential energy remaining as in the first variant) is evaluated by

$$K_{cr2} = p^2\left\{\frac{\rho P^2}{6E^2F}\left[L^3 + 6d(L^2 - L_{cr}^2)\frac{a}{h-a} + 12d^2(L - L_{cr})\frac{a^2}{(h-a)^2}\right] + \frac{mP^2}{2E^2F^2}\left(L + \frac{2ad}{h-a}\right)^2\right\}.$$

Using this expression, we derive the following formulas for the natural frequency of longitudinal vibrations of a cantilever beam with a closing crack by the energy- and frequency-based approaches, respectively:

$$f_{e2}^{l} = \frac{1}{2\pi}\left(\frac{3EF\left(L + \dfrac{ad}{h-a}\right)}{\rho F\left[L^3 + \dfrac{3ad(L^2 - L_{cr}^2)}{h-a} + \dfrac{6a^2d^2(L - L_{cr})}{(h-a)^2}\right] + 3m\left[L^2 + \dfrac{2adL}{h-a} + \dfrac{2a^2d^2}{(h-a)^2}\right]}\right)^{1/2}, \tag{32}$$

$$f_2^{l} = \frac{2f^{l} f_{cr2}^{l}}{f^{l} + f_{cr2}^{l}}, \tag{33}$$

where

$$f^{l}_{crn} = \frac{1}{2\pi}\left(\frac{3EF\left(L + \frac{ad}{h-a}\right)}{\rho F\left[L^3 + \frac{6ad(L^2 - L^2_{cr})}{h-a} + \frac{12a^2d^2(L - L_{cr})}{(h-a)^2}\right] + 3m\left(L + \frac{2ad}{h-a}\right)^2}\right)^{1/2}. \tag{34}$$

A formula for the determination of the relative variation of the longitudinal vibration frequency of a beam with a closing crack in terms of its compliance will have the form

$$\frac{f^l_c}{f^l} = \frac{2}{1 + (1 + \alpha^l_{cr})^{1/2}}, \tag{35}$$

where

$$\alpha^l_{cr} = \frac{2ad}{L(h-a)}. \tag{36}$$

By analogy with the approach used for transverse vibrations, it is easy to derive formulas for the frequency of longitudinal vibrations of a beam with an open crack.

Conclusions. We have derived several variants of fairly simple formulas for the evaluation of the natural frequency of the first-mode bending and longitudinal vibrations of a cracked cantilever beam. However, before using these formulas we must establish a relationship between the crack depth a and parameter d. This, as well as the experimental verification of the applicability of the present formulas, will be the objective of our next report.

The approach suggested in this work can also be used for deriving formulas for the frequencies of higher-mode vibrations.

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