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Riesz Bounds of Wilson Bases Generated by B-Splines

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ABSTRACT. In this paper, we are concerned with biorthogonal Wilson bases having B–splines as well as powers of sinc functions as window functions. We prove properties of B–splines and exponential Euler splines and use these properties to estimate the Riesz bounds of the Wilson bases.

1. Introduction

Gabor frames $\{g(x-an)e^{2\pi ibmx} : m, n \in \mathbb{Z}\}$ $(a, b \in \mathbb{R}_+)$ have found wide applications in digital signal processing, in particular in time–frequency localization of signals (cf. [13]). However, by the Balian–Low theorem, Riesz bases of the above form have necessarily bad localization properties in time or frequency. See [11, p. 108] and the references therein. Therefore, Wilson [19] introduced orthonormal bases that avoid the Balian–Low phenomenon by considering functions having two peaks in frequency domain. Wilsons's suggestion was simplified to a constructive approach in [12].

More general constructions are the orthonormal local trigonometric bases proposed in [9] and [15]. Here the concept of folding operators plays a significant role (cf. [1]). In contrast to Wilson bases, local Fourier bases require the basic assumption that only immediate neighboring windows are allowed to overlap. According to [7], we call this assumption the *two–overlapping condition.* On the other hand, local trigonometric bases can also be constructed using a nonuniform partition of the real axis.

Based on an extension of the folding concept, biorthogonal local Fourier bases were examined in [7, 3]. The consideration of biorthogonal Wilson bases was addressed in [8, 6] and, for special Gaussian windows, in [10].

In this paper, we are concerned with biorthogonal Wilson bases. In Section 2, we provide a simple approach to basic material concerning biorthogonal Wilson bases which differs from [8, 6] and from the approach to orthonormal Wilson bases in [12]. The approach is based on the *connection of the folding concept with the Zak transform* and was suggested by Bittner [4].

Based on the results in Section 2, we show that certain Wilson systems with cardinal B–splines and their Fourier transforms as window functions form Riesz bases and estimate their Riesz bounds.

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For this, we have to prove properties of cardinal B–splines and exponential Euler splines which may also be interesting in other contexts.

2. Biorthogonal Wilson Bases

Based on the orthonormal bases $\{c_k : k \in \mathbb{N}_0\}$ and $\{s_k : k \in \mathbb{N}\}$ of $L^2([0, 1/2])$ given by

$$c_0(x) := \sqrt{2}\,,\ c_k(x) := 2\cos(2\pi kx)\,,\ s_k(x) := 2\sin(2\pi kx) \quad (k \in \mathbb{N})\,,$$

we follow [14] and introduce the functions

$$\psi_k^j(x) = \begin{cases} \sqrt{2}g(x - j/2) & k = 0, j \in \mathbb{Z} \text{ even}\,, \\ 2\,g(x - j/2)\cos(2\pi kx) & k \in \mathbb{N}, j \in \mathbb{Z} \text{ even}\,, \\ 2\,g(x - j/2)\sin(2\pi kx) & k \in \mathbb{N}, j \in \mathbb{Z} \text{ odd}\,, \end{cases} \tag{2.1}$$

where $g \in L^2(\mathbb{R})$ denotes a window function. We are interested in properties of Wilson systems

$$\mathcal{B}_g := \left\{\psi_k^{2j} : j \in \mathbb{Z}, k \in \mathbb{N}_0\right\} \cup \left\{\psi_k^{2j+1} : j \in \mathbb{Z}, k \in \mathbb{N}\right\}\,. \tag{2.2}$$

Clearly, a similar approach is possible with respect to intervals other than $[0, 1/2]$ and with respect to the other orthonormal bases of $L^2([0, 1/2])$ usually involved in the construction of local Fourier bases. See [1].

If $\operatorname{supp} g \subseteq [-1/4, 3/4]$, then the functions ψ_k^j satisfy a two–overlapping condition and we consider a special case of local Fourier bases.

To define a folding operator for arbitrary $g \in L^2(\mathbb{R})$ similar to the folding operator known from local Fourier bases (cf. [7, 3]), we apply the Zak transform.

The *Zak transform* $Z : L^2(\mathbb{R}) \to L^2(\mathbb{T}^2) := L^2([0, 1]^2)$ is the unitary linear operator, which maps the orthonormal basis $\{E_{jk}(x) := e^{2\pi i jx}\mathbf{1}_{[0,1]}(x - k) : j, k \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ to the orthonormal basis $\{e_{jk}(s, t) := e^{2\pi i js}e^{2\pi ikt} : j, k \in \mathbb{Z}\}$ of $L^2(\mathbb{T}^2)$, i.e.,

$$Z\left(E_{jk}\right) = e_{jk} \quad (j, k \in \mathbb{Z})$$

(cf. [14, p. 406]). Here $\mathbf{1}_I$ denotes the characteristic function of the interval I. For $f \in L^2(\mathbb{R})$, the Zak transform is given by

$$Zf(s, t) = \sum_{k\in\mathbb{Z}} f(s + k)e^{2\pi ikt} \qquad \text{a.e. on } \mathbb{T}^2\,. \tag{2.3}$$

Furthermore, we have

$$Zf(s + 1, t) = e^{-2\pi it}Zf(s, t)\,,\ Zf(s, t + 1) = Zf(s, t) \qquad \text{a.e. on } \mathbb{T}^2\,. \tag{2.4}$$

Let the *Fourier transform* $\hat{f} \in L^2(\mathbb{R})$ of a function $f \in L^2(\mathbb{R})$ be defined by

$$\hat{f}(v) := \int_{\mathbb{R}} f(x)e^{-2\pi ixv}\,\mathrm{d}x \qquad \text{a.e.}$$

The Zak transforms of $f \in L^2(\mathbb{R})$ and $\hat{f} \in L^2(\mathbb{R})$ are related by

$$Z\hat{f}(s, t) = e^{-2\pi ist}Zf(t, -s) \quad \left((s, t) \in \mathbb{T}^2\right) \qquad \text{a.e. on } \mathbb{T}^2 \tag{2.5}$$

(cf. [2]). Let $I_j := [j/2, (j+1)/2]$. By (2.3) and (2.4), it is easy to check that

$$Z\left(\mathbf{1}_{I_{2j}} c_k\right)(s,t) = \begin{cases} c_k(s) e^{2\pi i j t} & s \in [0, 1/2)\,, \\ 0 & s \in [-1/2, 0)\,, \end{cases}$$

$$Z\left(\mathbf{1}_{I_{2j+1}} s_k\right)(s,t) = \begin{cases} 0 & s \in [0, 1/2)\,, \\ s_k(s) e^{2\pi i (j+1) t} & s \in [-1/2, 0) \end{cases}$$

and that

$$Z\left(\psi_k^{2j}\right)(s,t) = c_k(s) e^{2\pi i j t} Zg(s,t)\,,$$

$$Z\left(\psi_k^{2j+1}\right)(s,t) = -s_k(s) e^{2\pi i (j+1) t}(-Zg(s+1/2,t))\,.$$

This can be rewritten as

$$\begin{pmatrix} Z\psi_k^{2j}(s,t) \\ Z\psi_k^{2j}(-s,t) \end{pmatrix} = M_g^*(s,t) \begin{pmatrix} Z\left(\mathbf{1}_{I_{2j}} c_k\right)(s,t) \\ Z\left(\mathbf{1}_{I_{2j}} c_k\right)(-s,t) \end{pmatrix} \qquad ((s,t) \in [0,1/2] \times \mathbb{T}) \tag{2.6}$$

$$\begin{pmatrix} Z\psi_k^{2j+1}(s,t) \\ Z\psi_k^{2j+1}(-s,t) \end{pmatrix} = M_g^*(s,t) \begin{pmatrix} Z\left(\mathbf{1}_{I_{2j+1}} s_k\right)(s,t) \\ Z\left(\mathbf{1}_{I_{2j+1}} s_k\right)(-s,t) \end{pmatrix} \qquad ((s,t) \in [0,1/2] \times \mathbb{T})\,, \tag{2.7}$$

where

$$M_g(s,t) = \begin{pmatrix} \overline{Zg(s,t)} & \overline{Zg(-s,t)} \\ -\overline{Zg(s+1/2,t)} & \overline{Zg(-s+1/2,t)} \end{pmatrix}$$

and $M_g^* = \bar{M}_g^T$.

Let $g \in L^2(\mathbb{R})$ satisfy the decay property

$$|g(x)| \le C(1+|x|)^{-1-\varepsilon} \qquad \text{a.e.} \tag{2.8}$$

for some positive constants C and ε. Then, by (2.3),

$$|(Zg)(s,t)| \le \sum_{k \in Z} |g(s+k)| \le \sum_{k \in Z} (1+|s+k|)^{-1-\varepsilon} < \infty \qquad \text{a.e. on } \mathbb{T}^2$$

and consequently $Zg \in L^\infty(\mathbb{T}^2)$ and $Zg\, Zf \in L^2(\mathbb{T}^2)$ for all $f \in L^2(\mathbb{R})$. Together with (2.6) and (2.7), this motivates the following definition of the *adjoint folding operator* $T_g^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$\begin{pmatrix} Z\left(T_g^* f\right)(s,t) \\ Z\left(T_g^* f\right)(-s,t) \end{pmatrix} = M_g^*(s,t) \begin{pmatrix} Zf(s,t) \\ Zf(-s,t) \end{pmatrix} \qquad \text{a.e. on } [0,1/2] \times \mathbb{T}\,.$$

Clearly, the corresponding *folding operator* $T_g : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by

$$\begin{pmatrix} Z\left(T_g f\right)(s,t) \\ Z\left(T_g f\right)(-s,t) \end{pmatrix} = M_g(s,t) \begin{pmatrix} Zf(s,t) \\ Zf(-s,t) \end{pmatrix} \qquad \text{a.e. on } [0,1/2] \times \mathbb{T}\,.$$

In particular, we see by (2.6) and (2.7) that

$$Z\psi_k^{2j} = ZT_g^*\left(\mathbf{1}_{I_{2j}} c_k\right)\,, \quad Z\psi_k^{2j+1} = ZT_g^*\left(\mathbf{1}_{I_{2j+1}} s_k\right)\,. \tag{2.9}$$

In the "two–overlapping" setting, the folding operator T_g coincides with the usual folding operator for local Fourier bases on the equally partitioned real axis [7, 3].

In Section 3, we examine window functions $g \in L^2(\mathbb{R})$ which are symmetric with respect to $1/4$, i.e.,

$$g(x) = \overline{g(1/2 - x)} \qquad \text{a.e.} \tag{2.10}$$

For these window functions, we have

$$Zg(s,t) = \sum_{k\in\mathbb{Z}} g(s+k)e^{2\pi ikt} = \sum_{k\in\mathbb{Z}} \overline{g(1/2-s-k)}e^{-2\pi i(-k)t} = \overline{Zg(1/2-s,t)} \qquad \text{a.e. on } \mathbb{T}^2$$

so that M_g has the simpler form

$$M_g(s,t) = \begin{pmatrix} \overline{Zg(s,t)} & \overline{Zg(-s,t)} \\ -Zg(-s,t) & Zg(s,t) \end{pmatrix} . \tag{2.11}$$

With the above folding concept at hand, we consider (2.2).

Remember that a set of functions $\{u_k \in L^2(\mathbb{R}) : k \in \mathbb{Z}\}$ is called a *Riesz basis* of $L^2(\mathbb{R})$, if $L^2(\mathbb{R})$ is the closure of all finite linear combinations of the functions u_k $(k \in \mathbb{Z})$ there exist constants $0 < A \le B < \infty$ so that

$$A \ \|\{c_k\}\|_{l^2}^2 \le \left\| \sum_{k\in\mathbb{Z}} c_k u_k \right\|_{L^2(\mathbb{R})}^2 \le B \ \|\{c_k\}\|_{l^2}^2 \quad \left(\{c_k\}_{k\in\mathbb{Z}} \in l^2\right) .$$

The best possible constants A and B are the *Riesz bounds.*

Further, $\{u_k \in L^2(\mathbb{R}) : k \in \mathbb{Z}\}$ is an orthonormal basis if and only if $A = B = 1$. Riesz bases are precisely those that are images, under invertible bounded linear operators on $L^2(\mathbb{R})$, of orthonormal bases. Every function $f \in L^2(\mathbb{R})$ can be reconstructed a.e. from the values $(f, u_k)_{L^2(\mathbb{R})}$ $(k \in \mathbb{Z})$, by

$$f = \sum_{k\in\mathbb{Z}} (f, u_k)\, \tilde{u}_k ,$$

where $\{\tilde{u}_k : k \in \mathbb{Z}\}$ denotes the dual basis of $\{u_k : k \in \mathbb{Z}\}$. The convergence of the above sum is determined by the quotient $\frac{B-A}{B+A} = \frac{B/A-1}{B/A+1}$ which should be small (cf. [11, p. 62]).

For our Wilson systems $\mathcal{B}_g$, we can establish the following:

Theorem 1.

Let $g \in L^2(\mathbb{R})$ with property (2.8). Then, for $\mathcal{B}_g$ given by (2.1) and (2.2), the following statements are equivalent:

i) The Wilson system $\mathcal{B}_g$ is a Riesz basis with Riesz bounds A, B.

ii) The matrix $M_g(s,t)$ is nonsingular a.e. on $[0, 1/2] \times \mathbb{T}$ and there exist constants $0 < A \le B < \infty$ so that

$$A \le \left\| M_g(s,t)^{-1} \right\|_2^{-2} , \quad \| M_g(s,t) \|_2^2 \le B \quad \text{a.e. on } [0,1/2] \times \mathbb{T} , \tag{2.12}$$

where A, B are the best possible constants fulfilling these inequalities, i.e., $A = \operatorname{ess\,inf}_{(s,t)\in[0,1/2]\times\mathbb{T}} \|M_g(s,t)^{-1}\|_2^{-2}$ and $B = \operatorname{ess\,sup}_{(s,t)\in[0,1/2]\times\mathbb{T}} \|M_g(s,t)\|_2^2$. Here $\|\cdot\|_2$ denotes the spectral norm.

For a proof see [5].

If $g \in L^2(\mathbb{R})$ satisfies the symmetry property (2.10), then we have by (2.11) that

$$M_g^*(s,t) M_g(s,t) = \begin{pmatrix} D_g(s,t) & 0 \\ 0 & D_g(s,t) \end{pmatrix} ,$$

where

$$D_g(s,t) := |Zg(s,t)|^2 + |Zg(-s,t)|^2 .$$

Hence, (2.12) can be rewritten as

$$A \le D_g(s,t) \le B \quad \text{a.e. on } [0,1/2] \times \mathbb{T} . \tag{2.13}$$

Note that conditions on the window function $g \in L^2(\mathbb{R})$ so that the (dual) Wilson set associated with the Gabor frame $\{g(x-an)e^{2\pi i bmx} : m, n \in \mathbb{Z}\}$ $(a, b \in \mathbb{R}_+)$ again forms a frame were established in [6]. If these conditions are fulfilled, the authors can directly relate the frame bounds of the Wilson system to the frame bounds of the associated Gabor system. It turns out that in the case of critical sampling considered in our paper, the conditions on $g \in L^2(\mathbb{R})$ are always fulfilled if g satisfies the symmetry property (2.10). Consequently, in this case, we can apply well–known results on the bounds of Gabor frames which confirm (2.13) from another point of view.

Finally, one can also follow [12] to obtain (2.13).

3. B-Splines and their Fourier Transforms as Window Functions

The *cardinal B–splines N_m of order* m are defined by

$$N_1 := \frac{1}{2}\left(1_{[0,1)} + 1_{(0,1]}\right), \quad N_{m+1} := N_m * N_1 \quad (m \in \mathbb{N}),$$

where $*$ denotes the convolution in $L^2(\mathbb{R})$. The *centered cardinal B–splines M_m of order* m are given by

$$M_m(x) := N_m(x + m/2) . \tag{3.1}$$

Note that $\mathrm{supp}(N_m) = [0, m]$ and that N_m is symmetric with respect to $m/2$, i.e., $N_m(m/2 - x) = N_m(m/2 + x)$. The Fourier transform of M_m is given by

$$\hat{M}_m(v) = (\mathrm{sinc}(v))^m , \tag{3.2}$$

where

$$\mathrm{sinc}(v) := \begin{cases} 1 & v = 0 , \\ \frac{\sin(\pi v)}{\pi v} & \text{otherwise} . \end{cases}$$

Moreover, B–splines fulfill the *two–scale relation*

$$N_m(x) = 2^{1-m} \sum_{k=0}^{m} \binom{m}{k} N_m(2x - k) . \tag{3.3}$$

We begin with the consideration of the two–overlapping case, i.e., we set $g(x) := M_m(a(x - 1/4))$ $(a \ge m)$. To determine the Riesz bounds of the corresponding Wilson bases, we have to apply the following lemma which seems to be clear at first glance.

Lemma 1.

For $m \ge 2$, the cardinal B–splines have the following properties:

i) N_m' *is monotone increasing on* $[0, \frac{m+1}{4}]$,

ii) $N_m'(x) \le N_m'\left(\frac{m}{2} - x\right)$ *for all* $x \in [0, \frac{m}{4}]$.

Proof. We prove the assertion by induction on m, where we mainly apply that the derivatives of cardinal B-splines fulfill (cf. [18])

$$N'_{m+1}(x) = N_m(x) - N_m(x-1) = \int_{x-1}^{x} N'_m(t)\, dt\,. \tag{3.4}$$

For the "hat function" N_2, the assertion is obvious.

Assume now that i) and ii) hold for $k \leq m$. First, we show that N'_{m+1} is monotone increasing on $[0, \frac{m+2}{4}]$. By induction hypothesis i), we have for $t \in [0, \frac{m+1}{4}]$ that

$$N'_m(t) - N'_m(t-1) \geq 0\,.$$

Let $t \in [\frac{m+1}{4}, \frac{m+2}{4}]$ so that $t-1 \in [\frac{m-3}{4}, \frac{m-2}{4}]$ and $\frac{m}{2} - t \in [\frac{m-2}{4}, \frac{m-1}{4}]$. Then we obtain by assumption i) that

$$N'_m(t) - N'_m(t-1) \geq N'_m(t) - N'_m\left(\frac{m}{2} - t\right)$$

and further, since by induction hypothesis ii) for $t \in [\frac{m}{4}, \frac{m}{2}]$

$$N'_m\left(\frac{m}{2} - t\right) \leq N'_m(t)\,,$$

that

$$N'_m(t) - N'_m(t-1) \geq 0\,.$$

Thus, we get for $0 \leq x \leq y \leq \frac{m+2}{4}$ that

$$\begin{aligned} \int_0^x N'_m(t) - N'_m(t-1)\, dt &\leq \int_0^y N'_m(t) - N'_m(t-1)\, dt\,, \\ N_m(x) - N_m(x-1) &\leq N_m(y) - N_m(y-1)\,, \end{aligned}$$

which yields assertion i) by (3.4).

Next, we prove ii). We distinguish between the cases $x \in [0, \frac{1}{2}]$, $x \in [\frac{1}{2}, \frac{m-1}{4}]$ and $x \in [\frac{m-1}{4}, \frac{m+1}{4}]$.

Let $x \in [0, \frac{1}{2}]$. Then we obtain by (3.4) and since $N'_m(t) = -N'_m(m-t)$ that

$$\begin{aligned} N'_{m+1}\left(\frac{m+1}{2} - x\right) &= \int_{\frac{m+1}{2}-x-1}^{\frac{m+1}{2}-x} N'_m(t)\, dt = \int_{\frac{m-1}{2}-x}^{\frac{m}{2}} N'_m(t)\, dt + \int_{\frac{m}{2}}^{\frac{m+1}{2}-x} N'_m(t)\, dt \\ &= \int_{\frac{m}{2}-\frac{1}{2}-x}^{\frac{m}{2}-\frac{1}{2}+x} N'_m(t)\, dt \end{aligned}$$

and further by assumption ii) and i) that

$$N'_{m+1}\left(\frac{m+1}{2} - x\right) \geq \int_{-x+\frac{1}{2}}^{x+\frac{1}{2}} N'_m(t)\, dt \geq \int_0^x N'_m(t)\, dt = N'_{m+1}(x)\,.$$

Let $x \in [\frac{1}{2}, \frac{m-1}{4}]$. By (3.4) and ii), we obtain for $x \in [\frac{1}{2}, \frac{m}{4} - \frac{1}{2}]$ that

$$N'_{m+1}\left(\frac{m+1}{2} - x\right) = \int\limits_{\frac{m-1}{2}-x}^{\frac{m+1}{2}-x} N'_m(t)\,\mathrm{d}t \geq \int\limits_{x-\frac{1}{2}}^{x+\frac{1}{2}} N'_m(t)\,\mathrm{d}t$$

and similarly for $x \in (\frac{m}{4} - \frac{1}{2}, \frac{m-1}{4}]$ that

$$\begin{aligned} N'_{m+1}\left(\frac{m+1}{2} - x\right) &= \int\limits_{\frac{m-1}{2}-x}^{x+\frac{1}{2}} N'_m(t)\,\mathrm{d}t + \int\limits_{x+\frac{1}{2}}^{\frac{m+1}{2}-x} N'_m(t)\,\mathrm{d}t \\ &\geq \int\limits_{\frac{m-1}{2}-x}^{x+\frac{1}{2}} N'_m(t)\,\mathrm{d}t + \int\limits_{x-\frac{1}{2}}^{\frac{m-1}{2}-x} N'_m(t)\,\mathrm{d}t = \int\limits_{x-\frac{1}{2}}^{x+\frac{1}{2}} N'_m(t)\,\mathrm{d}t\,. \end{aligned}$$

Now assumption i) implies that

$$N'_{m+1}\left(\frac{m+1}{2} - x\right) \geq \int\limits_{x-\frac{1}{2}}^{x+\frac{1}{2}} N'_m(t)\,\mathrm{d}t \geq \int\limits_{x-1}^{x} N'_m(t)\,\mathrm{d}t = N'_{m+1}(x)\,.$$

Finally, let $x \in [\frac{m-1}{4}, \frac{m+1}{4}]$. By (3.4) we obtain

$$\begin{aligned} N'_{m+1}\left(\frac{m+1}{2} - x\right) - N'_{m+1}(x) &= \int\limits_{\frac{m-1}{2}-x}^{\frac{m+1}{2}-x} N'_m(t)\,\mathrm{d}t - \int\limits_{x-1}^{x} N'_m(t)\,\mathrm{d}t \\ &= \int\limits_{x}^{\frac{m+1}{2}-x} N'_m(t)\,\mathrm{d}t - \int\limits_{x-1}^{\frac{m-1}{2}-x} N'_m(t)\,\mathrm{d}t \\ &= \int\limits_{x}^{\frac{m+1}{4}} N'_m(t)\,\mathrm{d}t + \int\limits_{\frac{m+1}{4}}^{\frac{m+1}{2}-x} N'_m(t)\,\mathrm{d}t \\ &\quad - \int\limits_{x-1}^{\frac{m-3}{4}} N'_m(t)\,\mathrm{d}t - \int\limits_{\frac{m-3}{4}}^{\frac{m-1}{2}-x} N'_m(t)\,\mathrm{d}t\,. \end{aligned}$$

By induction hypothesis i), we have

$$\int\limits_{x}^{\frac{m+1}{4}} N'_m(t)\,\mathrm{d}t \geq \int\limits_{x-1}^{\frac{m-3}{4}} N'_m(t)\,\mathrm{d}t\,,$$

while assumptions ii) and i) yield

$$\int\limits_{\frac{m+1}{4}}^{\frac{m+1}{2}-x} N'_m(t)\,\mathrm{d}t \geq \int\limits_{x-\frac{1}{2}}^{\frac{m-1}{4}} N'_m(t)\,\mathrm{d}t \geq \int\limits_{\frac{m-3}{4}}^{\frac{m-1}{2}-x} N'_m(t)\,\mathrm{d}t\,.$$

Thus, we get assertion ii) for $x \in [\frac{m-1}{4}, \frac{m+1}{4}]$. This completes the proof. $\square$

Theorem 2.

Let $g(x) := M_m(a(x-1/4))$ $(m \geq 2)$. Then $\mathcal{B}_g$ is not a Riesz basis for $a \geq 2m$, while it constitutes a Riesz basis for $m \leq a < 2m$ with Riesz bounds $A_m = 2\,M_m^2(a/4)$ and $B_m = M_m^2(0)$.

Proof. Since $\mathrm{supp}(g) \subseteq [-1/4, 3/4]$ and $M_m(x) = M_m(-x)$, we obtain by (2.3) for $a \geq m$ that

$$D_g(s,t) = D_g(s) = M_m^2(as) + M_m^2(a(1/2-s)) \quad (s \in [0, 1/2]) .$$

We show that the function $D_g(s)$ attains its minimum on $[0, 1/4]$ at $s = 1/4$ and its maximum at $s = 0$. To this end we calculate the derivative

$$D_g'(s) = 2a\left(M_m(as)M_m'(as) - M_m(a(1/2-s))M_m'(a(1/2-s))\right) .$$

By Lemma 1, we have $M_m'(as) \leq M_m'(a(1/2-s)) \leq 0$ for $s \in [0, 1/4]$.

Since further $M_m(as) \geq M_m(a(1/2-s)) \geq 0$ for $s \in [0, 1/4]$, we conclude that $D_g'(s) \leq 0$ for $s \in [0, 1/4]$. Consequently, we obtain by Theorem 1 for $m \leq a < 2m$ that $A_m = 2M_m^2(a/4) > 0$ and $B_m = 2M_m^2(0) < \infty$. For $a \geq 2m$, we see that $M_m(a/4) = 0$ so that $\mathcal{B}_g$ is not a Riesz basis. $\square$

To see how $C_m := B_m/A_m$ increases with m when $a = m$, we consider the following table:

m	2	6	10	22	26	30	34	38
C_{m+1}/C_m	1.778	2.2580	2.2623	2.2640	2.2641	2.2641	2.2642	2.2642

Indeed the above computations can be confirmed by using the *saddle point method* for the asymptotic estimation of $M_m(mb)$ when $m \to \infty$: Let $b \in [0, \frac{1}{2}]$. By (3.2), we have

$$M_m(mb) = \lim_{A\to\infty} \int_{-A}^{A} (\mathrm{sinc}(t))^m e^{2\pi imbt}\,dt$$

and since the integrand is entire, by Cauchy's theorem

$$M_m(mb) = \frac{1}{\pi} \lim_{A\to\infty} \int_{\Gamma_A} e^{m\left(2ibz + \mathrm{Ln}\left(\frac{e^{iz}-e^{-iz}}{2iz}\right)\right)} j\,dz = \frac{1}{\pi} \lim_{A\to\infty} \int_{\Gamma_A} e^{mf(z)}\,dz ,$$

where Γ_A is an integration path from $-A$ to A which has to be specified. Now

$$f'(z) = i\left(\coth(iz) - \frac{1}{iz} + 2b\right)$$

so that $z_0 := iw$ is a saddle point of $\mathrm{Re} f(z)$, where $w = w_b \in \mathbb{R}$ is the unique solution of

$$\coth w - \frac{1}{w} = 2b . \tag{3.5}$$

Then

$$f(z_0) = -2bw + \ln\left(\frac{e^w - e^{-w}}{2w}\right) , \quad f''(z_0) = \frac{1}{(\sinh w)^2} - \frac{1}{w^2} .$$

As integration path passing through the saddle point we can choose the broken line through the points $-A, -\varepsilon + iw, \varepsilon + iw, A$ $\quad (\varepsilon := m^{-2/5})$ which approximates the path $\mathrm{Im} f = 0$ of steepest descent near the saddle point. Now we obtain by the well-known procedure of the saddle point method (cf. [17, p. 486]) that

$$M_m(mb) \sim \sqrt{\frac{2}{\pi m}}\, e^{-2bwm} \left(\frac{\sinh w}{w}\right)^m \left(\frac{1}{w^2} - \frac{1}{(\sinh w)^2}\right)^{-\frac{1}{2}} \quad \text{for } m \to \infty\,.$$

For $b = 0$, we see by (3.5) that $w_0 = 0$ and for $b = \frac{1}{4}$ that $w_{\frac{1}{4}} = 1.796755$. Thus,

$$C_{m+1}/C_m = \frac{M_{m+1}^2(0)\, M_m^2(m/4)}{M_m^2(0)\, M_{m+1}^2(m/4)} \sim \left(\frac{\sinh w_{\frac{1}{4}}}{w_{\frac{1}{4}}}\, e^{-\frac{w_{\frac{1}{4}}}{2}}\right)^{-2} \approx 2.264327$$

which is an excellent agreement with the above computations.

As preparation for the next result we start with the definition of exponential Euler splines. The *exponential Euler splines* ϕ_m ($m \in \mathbb{N}$) (on the unit circle) are defined by [18]

$$\phi_m(s,t) = \sum_{k\in\mathbb{Z}} M_m(s-k)e^{2\pi ikt} \quad (s \in \mathbb{R}, t \in (-1/2, 1/2])\,. \tag{3.6}$$

The following theorem summarizes results about exponential Euler splines stated in [20].

Theorem 3.

The exponential Euler splines ϕ_m ($m \geq 2$) satisfy:

i) *Let $s, t \in [0, 1/2]$ be fixed. Then $|\phi_m(s,t)| \leq |\phi_{m-1}(s,t)|$.*

ii) *Let $s \in [0, 1]$ be fixed. Then $|\phi_m(s,t)|$ decreases for $t \in [0, 1/2]$.*
Furthermore, $(s,t) = (1/2, 1/2)$ is the unique root of ϕ_m on $[0,1] \times [0, 1/2]$.

iii) *Let $t \in [0, 1/2]$ be fixed. Then $|\phi_m(s,t)|$ decreases for $s \in [0, 1/2]$ and increases for $s \in [1/2, 1]$.*

iv) *B–splines form a partition of unity, i.e., $\phi_m(s, 0) = 1$ for $s \in [0, 1]$.*

v) *The function*

$$U_m(s) := \phi_m(s, 1/2) = \sum_{k\in\mathbb{Z}} (-1)^k M_m(s-k)$$

decreases on $[0, 1]$, where $U_m(0) > 0$, and satisfies the additional properties:

$$\begin{aligned} U_m(1-s) &= -U_m(s)\,, \\ U_m'(-s+1/2) &= U_m'(s+1/2) = -2\,U_{m-1}(s) \quad (m > 2)\,, \\ U_m''(s) &= -4\,U_{m-2}(s) \quad (m > 3)\,. \end{aligned}$$

Now we can formulate our next result.

Theorem 4.

Let $g(x) := M_m(x - 1/4)$ ($m \geq 2$). Then $\mathcal{B}_g$ constitutes a Riesz basis with upper Riesz bound $B = 2$ and lower Riesz bound $A = A_m$. The latter bound can be estimated by

$$U_m^2(0)/2 \leq A_m \leq \min\left\{U_m^2(0),\, U_{m-1}^2(0)/2\right\}\,,$$

i.e., for even m by

$$2\left(1 - 2^{-m}\right)^2 \left(\frac{2}{\pi}\right)^{2m} \leq A_m \leq 4\left(\frac{1 - 2^{-m}}{1 - 2^{1-m}}\right)^2 \left(\frac{2}{\pi}\right)^{2m}\,,$$

and for odd m by

$$2\left(1-2^{-m-1}\right)^2\left(\frac{2}{\pi}\right)^{2(m+1)} \leq A_m \leq \frac{\pi^4}{8}\left(\frac{1-2^{1-m}}{1-2^{2-m}}\right)^2\left(\frac{2}{\pi}\right)^{2(m+1)} .$$

Proof. By (2.3), (3.6), and since M_m is even, we obtain

$$D_g(s,t) = |\phi_m(1/4-s,t)|^2 + |\phi_m(1/4+s,t)|^2 \quad ((s,t) \in [0,1/2] \times \mathbb{T}) .$$

By Theorem 3 ii), the above function attains its minimum at $t = 1/2$ and its maximum at $t = 0$. Thus, we conclude by Theorem 1 that we have to look for $A_m = \min\{D_g(s,1/2) : s \in [0,1/4]\}$ and $B_m = \max\{D_g(s,0) : s \in [0,1/4]\}$. By Theorem 3 iv), we see immediately that $B_m = B = 2$. Following Theorem 3 v), we rewrite A_m in the form

$$A_m = \min\left\{U_m^2(s) + U_m^2(1/2-s) : s \in [0,1/4]\right\} .$$

By straightforward computation we obtain that $A_2 = 1/2$ and $A_3 = 1/4$. In the following, let $m > 3$. We define the linear function

$$h_m(s) := -2U_m(0)\,s + U_m(0)$$

passing through the points $(0, U_m(0))$ and $(1/2, U_m(1/2)) = (1/2, 0)$. Since we have by Theorem 3 v) that $U_m''(s) \leq 0$ for $s \in [0,1/2]$, the function U_m is concave on $[0,1/2]$. Thus, $h_m(s) \leq U_m(s)$ for $s \in [0,1/2]$. On the other hand, we see by Theorem 3 v) that

$$h_{m-1}(s) = -2\,U_{m-1}(0)s + U_{m-1}(0) = U_m'(1/2)s + U_{m-1}(0)$$

so that $U_m(s) \leq h_{m-1}(s)$ for $s \in [0,1/2]$.

Now it is easy to check that $\min\{h_m^2(s) + h_m^2(1/2-s) : s \in [0,1/4]\} = U_m^2(0)/2$. Consequently,

$$U_m^2(0)/2 \leq A_m \leq \min\left\{U_m^2(0), U_{m-1}^2(0)/2\right\} . \tag{3.7}$$

By [16], we have that

$$U_{2m}(0) = \frac{2^{2m}\left(2^{2m}-1\right)}{(2m)!}\,|B_{2m}|$$

and further, since the *Bernoulli numbers* B_{2m} can be estimated by

$$\frac{2(2m)!}{(2\pi)^{2m}} < |B_{2m}| < \frac{2(2m)!}{(2\pi)^{2m}}\,\frac{2^{2m}}{2^{2m}-2} ,$$

that

$$\frac{2(2^{2m}-1)}{\pi^{2m}} < U_{2m}(0) < \frac{2\left(2^{2m}-1\right)}{\pi^{2m}}\,\frac{2^{2m}}{2^{2m}-2} .$$

By Theorem 3 i), it follows $U_{2m+2}(0) \leq U_{2m+1}(0) \leq U_{2m}(0)$ so that

$$\frac{2\left(2^{2m+2}-1\right)}{\pi^{2m+2}} < U_{2m+1}(0) < \frac{2\left(2^{2m}-1\right)}{\pi^{2m}}\,\frac{2^{2m}}{2^{2m}-2} .$$

Together with (3.7) this yields the desired estimates for A_m. □

Note that for $m \to \infty$,

$$C_{m+1}/C_m \approx (A_m/A_{m+2})^{\frac{1}{2}} \sim (\pi/2)^2 \approx 2.467401 .$$

Finally, we consider Wilson bases with powers of sinc–functions as window functions. Again, we prepare our result by proving some properties of B–splines.

Lemma 2.

Let $m \geq 2$ and

$$V_m(x) := \sum_{k\in\mathbb{Z}}(-1)^k M_m(x-2k)\,.$$

Then, for odd $m \in \mathbb{N}$,

$$V_m(1/2) = 2^{(m-3)/2}U_m(0)$$

and for even $m \in \mathbb{N}$,

$$\begin{aligned} V_m(0) &= 2^{(m-2)/2}U_m(0)\,, \\ 2^{(m-4)/2}U_m(0) &\leq V_m(1/2) \leq 2^{(m-2)/2}U_m(0)\,. \end{aligned}$$

Proof. Due to the two–scale relation (3.3) we obtain

$$U_m(0) = \sum_{j\in\mathbb{Z}}(-1)^j\left(2^{1-m}\sum_{k=0}^{m}\binom{m}{k}M_m\left(2j+\frac{m}{2}-k\right)\right)\,. \tag{3.8}$$

Let $m \in \mathbb{N}$ be odd. Then (3.8) can be rewritten as

$$U_m(0) = 2^{1-m}\sum_{l=(-m+1)/2}^{(m+1)/2}\binom{m}{\frac{m-1}{2}+l}\sum_{j\in\mathbb{Z}}(-1)^j M_m\left(2j+\frac{1}{2}-l\right)\,.$$

Since M_m is even and

$$\binom{m}{\frac{m-1}{2}+2r+1} = \binom{m}{\frac{m-1}{2}-2r}\,,$$

we obtain by splitting the above sum into even and odd $l \in \mathbb{N}$ that

$$\begin{aligned} U_{2m}(0) &= 2^{2-m}\sum_{l=\lfloor(-m+3)/4\rfloor}^{\lfloor(m+1)/4\rfloor}\binom{m}{\frac{m-1}{2}+2l}\sum_{j\in\mathbb{Z}}(-1)^j M_m\left(2j+\frac{1}{2}-2l\right) \\ &= 2^{2-m}\,V_m\left(\frac{1}{2}\right)\sum_{l=\lfloor(-m+3)/4\rfloor}^{\lfloor(m+1)/4\rfloor}(-1)^l\binom{m}{\frac{m-1}{2}+2l}\,, \end{aligned}$$

where $\lfloor x\rfloor$ denotes the integer part of x, i.e., $\lfloor x\rfloor \leq x < \lfloor x\rfloor + 1$. The last sum S_o has the form

$$S_o = \begin{cases} \left|\sum_{k=0}^{(m-1)/2}\binom{m}{2k} - 2\sum_{k=0}^{(m-5)/4}\binom{m}{4k+2}\right| & m \equiv 1 \bmod 8 \text{ or } m \equiv 5 \bmod 8\,, \\ \left|\sum_{k=0}^{(m-1)/2}\binom{m}{2k+1} - 2\sum_{k=0}^{(m-3)/4}\binom{m}{4k+3}\right| & m \equiv 3 \bmod 8 \text{ or } m \equiv 7 \bmod 8\,. \end{cases}$$

Using the formulas in [21, p. 17], we obtain that $S_o = 2^{(m-1)/2}$ and consequently $U_m(0) = 2^{(3-m)/2}V_m(1/2)$.

For the rest of the proof let $m \in \mathbb{N}$ be even. Then (3.8) can be rewritten as

$$U_m(0) = 2^{1-m}\sum_{l=-m/2}^{m/2}\binom{m}{\frac{m}{2}+l}\sum_{j\in\mathbb{Z}}(-1)^j M_m(2j-l)\,.$$

Since M_m is even, we have for $l = 2r + 1$ that

$$\sum_{j\in\mathbb{Z}}(-1)^j M_m(2j-2r-1) = (-1)^r \sum_{k\in\mathbb{Z}}(-1)^k M_m(2k-1) = 0$$

so that

$$U_m(0) = 2^{1-m}\sum_{k\in\mathbb{Z}}(-1)^k M_m(2k) \sum_{l=-\lfloor\frac{m}{4}\rfloor}^{\lfloor\frac{m}{4}\rfloor}(-1)^l\binom{m}{\frac{m}{2}+2l}\,.$$

The last sum S_e has the form

$$S_e = \begin{cases} \left|\sum\limits_{k=0}^{m/2}\binom{m}{2k} - 2\sum\limits_{k=0}^{m/4}\binom{m}{4k}\right| & m \equiv 0 \bmod 4\,, \\ \left|\sum\limits_{k=0}^{(m-2)/2}\binom{m}{2k+1} - 2\sum\limits_{k=0}^{(m-2)/4}\binom{m}{4k+1}\right| & m \equiv 2 \bmod 4\,. \end{cases}$$

Using [21, p. 17] again, we see that $S_e = 2^{m/2}$. Hence $U_m(0) = 2^{(2-m)/2}V_m(0)$.

To prove the last assertion we consider $V_m(x)$. Obviously, $V_2(x) = M_2(x) = 1 - x$ for $x \in [0,1]$. Assume that $V_{m-2}(x) > 0$ for $x \in (0,1)$ and $m \geq 4$. By (3.4) and (3.1), it follows $V_m''(x) = -2V_{m-2}(x) < 0$ so that V_m is concave on $(0,1)$. Since further $V_m(0) = 2^{(m-2)/2}U_m(0) > 0$ and $V_m(1) = 0$, we obtain $V_m(x) > 0$ for $x \in (0,1)$. Now concavity of V_m yields

$$V_m(1/2) \geq \frac{1}{2}\left(V_m(0) + V_m(1)\right) = \frac{1}{2}V_m(0)\,.$$

Using that $M_m'(x) = -M_m'(-x)$, we get $V_m'(0) = 0$. Hence, V_m has a local maximum at $x = 0$ and $V_m(1/2) \leq V_m(0)$. This completes the proof. $\square$

Theorem 5.

Let $g(x) := (\mathrm{sinc}(x - 1/4))^m$ $(m \geq 2)$. Then $\mathcal{B}_g$ is a Riesz basis and the Riesz bounds $A = A_m$ and $B = B_m$ can be estimated by

$$0 < A_m \leq \begin{cases} 2^{m-1}U_m^2(0) & m \text{ odd} \\ 2^m U_m^2(0) & m \text{ even} \end{cases} \leq \begin{cases} 4\left(\frac{2\sqrt{2}}{\pi}\right)^{2m-2}\left(\frac{1-2^{1-m}}{1-2^{2-m}}\right)^2 & m \text{ odd}\,, \\ 4\left(\frac{2\sqrt{2}}{\pi}\right)^{2m}\left(\frac{1-2^{-m}}{1-2^{1-m}}\right)^2 & m \text{ even}\,, \end{cases}$$

$$1 + U_m^2(0) \leq B_m \leq 1 + 2U_m(0) + U_m^2(0)\,.$$

Proof. By (2.5) and since $g = (M_m e^{2\pi i\cdot/4})^{\wedge}$, we obtain for $((s,t) \in [0,1/2] \times \mathbb{T})$ that

$$\begin{aligned} D_g(s,t) &= \left|Z\left(M_m e^{2\pi i\cdot/4}\right)(t,-s)\right|^2 + \left|Z\left(M_m e^{2\pi i\cdot/4}\right)(t,s)\right|^2 \\ &= \left|\sum_{k\in\mathbb{Z}} M_m(t+k)e^{2\pi i k(1/4-s)}\right|^2 + \left|\sum_{k\in\mathbb{Z}} M_m(t+k)e^{2\pi i k(1/4+s)}\right|^2\,. \end{aligned}$$

By Theorem 1 and Theorem 3 iii), we have to look for the minimum of D_g in $[0,1/4] \times \{1/2\}$ and for the maximum in $[0,1/4] \times \{0\}$. Concerning the minimum we obtain by $2(|a|^2 + |b|^2) =$

$|a+b|^2 + |a-b|^2$ that

$$\begin{aligned} D_g(s, \frac{1}{2}) &= \left|\sum_{k\in\mathbb{Z}} M_m\left(\frac{1}{2}+k\right) e^{2\pi i k(s-1/4)}\right|^2 + \left|\sum_{k\in\mathbb{Z}} M_m\left(\frac{1}{2}+k\right) e^{2\pi i k(s+1/4)}\right|^2 \\ &= 2\left|\sum_{k\in\mathbb{Z}} M_m\left(\frac{1}{2}+k\right) e^{2\pi i k s}\cos\left(\frac{\pi k}{2}\right)\right|^2 + 2\left|\sum_{k\in\mathbb{Z}} M_m\left(\frac{1}{2}+k\right) e^{2\pi i k s}\sin\left(\frac{\pi k}{2}\right)\right|^2 \\ &= 4\left|\sum_{k\in\mathbb{Z}} M_m\left(\frac{1}{2}+k\right) e^{2\pi i k s}\cos\left(\frac{\pi k}{2}\right)\right|^2 . \end{aligned}$$

For $m \le 10$ it is easy to check by straightforward computation that $D_g(s, 1/2)$ has its minimum at $s = 0$. However, for arbitrary $m \in \mathbb{N}$, we were not able to prove this result. Therefore, $D_g(0, 1/2)$ can only serve as upper bound of the minimum. Applying Lemma 2, we obtain

$$D_g\left(0, \frac{1}{2}\right) = 4\left|\sum_{k\in\mathbb{Z}}(-1)^k M_m\left(\frac{1}{2}+2k\right)\right|^2 \begin{cases} = 2^{m-1}U_m^2(0) & m \text{ odd}, \\ \le 2^m U_m^2(0) & m \text{ even}. \end{cases}$$

By Theorem 3 ii), we see that $D_g(s, 1/2) > 0$. Concerning the maximum we examine

$$\begin{aligned} D_g(s,0) &= \left|\sum_{k\in\mathbb{Z}} M_m(k) e^{2\pi i k s}\right|^2 + \left|\sum_{k\in\mathbb{Z}} M_m(k) e^{2\pi i k(1/2-s)}\right|^2 \\ &= \left(M_m(0) + 2\sum_{k=1}^{\infty} M_m(k)\cos(2\pi k s)\right)^2 + \left(M_m(0) + 2\sum_{k=1}^{\infty}(-1)^k M_m(k)\cos(2\pi k s)\right)^2 \end{aligned}$$

A lower bound for the maximum of $D_g(s, 0)$ is given by

$$D_g(0,0) = 1 + U_m(0)^2 .$$

Regarding that $U_m(0) > 0$, an upper bound for the maximum of $D_g(s, 0)$ can be obtained by $a^2 + b^2 \le (a+b)^2$, $(ab \ge 0)$, namely

$$\begin{aligned} D_g(s,0) &\le \left(2M_m(0) + 4\sum_{k=1}^{\infty} M_m(2k)\cos(2\pi 2ks)\right)^2 \\ &\le 4\left(\sum_{k\in\mathbb{Z}} M_m(2k)\right)^2 = (1 + U_m(0))^2 , \end{aligned}$$

where the last equation follows by Theorem 3 iv) and definition of U_m. This completes the proof. □

Note that for $m \to \infty$, under the assumption that $D_g(s, \frac{1}{2})$ attains its minimum at $s = 0$,

$$C_{m+1}/C_m \approx (A_m/A_{m+2})^{1/2} \sim \left(\frac{\pi}{2\sqrt{2}}\right)^2 \approx 1.233700 .$$

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