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Boundary-Variation Solution of Eigenvalue Problems for Elliptic Operators

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ABSTRACT. We present an algorithm which, based on certain properties of analytic dependence, constructs boundary perturbation expansions of arbitrary order for eigenfunctions of elliptic PDEs. The resulting Taylor series can be evaluated far outside their radii of convergence — by means of appropriate methods of analytic continuation in the domain of complex perturbation parameters. A difficulty associated with calculation of the Taylor coefficients becomes apparent as one considers the issues raised by multiplicity: domain perturbations may remove existing multiple eigenvalues and criteria must therefore be provided to obtain Taylor series expansions for all branches stemming from a given multiple point. The derivation of our algorithm depends on certain properties of joint analyticity (with respect to spatial variables and perturbations) which had not been established before this work. While our proofs, constructions and numerical examples are given for eigenvalue problems for the Laplacian operator in the plane, other elliptic operators can be treated similarly.

1. Introduction

The properties of eigenvalue problems under perturbations have been the subject of comprehensive studies [24, 34], and the area continues to carry great importance to this day [1, 36, 37]. A substantial portion of these investigations relate to properties of *smoothness and analyticity* of eigenvalues and eigenfunctions with respect to perturbations — properties these which may yield estimates of deviations in eigenelements caused by numerical errors, design imperfections, or other departures from simple configurations of a system. However, explicit constructions of high-order *boundary perturbation* expansions for eigenelements have not been attempted before this work. In fact, consideration of low-order expansions has led some authors [29] to conjecture that boundary perturbative methods would only be applicable, as numerical methods, within the very restricted ranges where first or second order theories can be used.

In this article we address this problem in an important example: evaluation of eigenfrequencies of the Laplacian operator under variations of the domain of definition from an exactly solvable

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geometry. In particular, we present explicit algorithms for iterative calculation of all Taylor coefficients, and we show how information about the perturbed eigenelements can be extracted from these series, even far outside their radii of convergence. Our treatment of this problem thus allows for high-order evaluation of eigenvalues, and it is applicable to general situations — including the notoriously difficult cases involving change in multiplicity (compare [39, p. 1627]). The derivation of our algorithm depends on certain properties of *joint analyticity* (with respect to spatial variables and perturbations) which we establish in Section 2 below. While our proofs, constructions, and numerical examples are given for eigenvalue problems for the Laplacian in the plane, other elliptic operators can be treated similarly.

As we said, our constructions depend on certain regularity properties of elliptic operators. The area of elliptic regularity has seen substantial advances over the last 50 years, including results on the properties of *spatial analyticity* of solutions to elliptic equations, see [14, 30]. In particular, it has been shown that solutions to Dirichlet problems for analytic equations can be analytically continued beyond the boundary of their domains of definition provided the Dirichlet data and boundary surfaces are themselves analytic. A complementary contribution to the subject was provided by Calderón [8] and Coifman et al. [10, 11] who studied the (singular) surface potentials associated with Laplace's equation in a two-dimensional Lipschitz domain (see also [12] for higher dimensional analogues). In particular, it follows from that work that the values of the solution at points *away from the boundary* of its domain of definition depend analytically on the given boundary perturbations.

In [5] we considered the question of *joint* analytic dependence on boundary perturbations and spatial variables. (As is well known, joint analyticity does not follow from separate real analyticity properties; consider, for instance, the function $f(x, y) = xy\exp\left[-1/(x^2 + y^2)\right]$). Our interest in such results was motivated by the need to establish the validity of certain recursive formulae we derived for the explicit evaluation of boundary perturbation series. Our theorems, which were established in the context of problems of wave propagation, are not unrelated to the early work of Hadamard and others on boundary-perturbation expansions of Green's functions [20, 3, 13, 15, 16]. The work of these authors, however, does not provide any information on the analyticity with respect to boundary perturbations *at boundary points*. Thus, the joint boundary analyticity properties we established in [5] strengthen earlier results and, in particular, establish the validity of numerical methods based on the recursive formulae mentioned above.

By exploiting these recursions and the underlying analytic structure we developed a class of highly accurate numerical algorithms for the solution of scattering problems (see [6, 7] and the references therein). These methods have been applied to advantage in a number of problems in optics [6], and, due to their remarkable accuracy, have recently permitted to resolve certain important long-standing issues in oceanic scattering [35].

Our general approach to the question of analytic dependence is based on holomorphic formulations of line, surface, and volume integral equations. In [5] we were thus able to establish that, for analytic (one-dimensional) boundaries of two-dimensional domains, the (unique) solutions to various elliptic problems are *jointly holomorphic in the spatial and parameter variables, and that they can be jointly and uniformly continued beyond the bounding curves.* The generalization given in [4] to two-dimensional surfaces in three-dimensional space is not direct, as it requires corresponding jointly analytic extensions of surface potentials. A theorem to this effect in the present context of eigenvalue problems constitutes the main result of Section 2. In this case, additional difficulties in the treatment of the associated potentials arise from the fact that the domains of integration have non-empty boundaries. Indeed, to avoid the collapse of the complex domain of analyticity of potentials as the boundary of the domain of integration is approached (a behavior which is well documented, for instance, in [31, p. 170], [3, 13]) we were led to introduce a concept of tangential analyticity (see equation (2.17) below).

The results of [5] (and subsequent numerical codes) rely heavily on the uniqueness of solution to the corresponding problems which, as we said, allows for the inversion of the differential operators within suitable spaces of holomorphic functions. Thus, the present theory for eigenvalue problems under boundary variations deviates substantially from our previous ones. In addition, the non-uniqueness of solution and the possibility of multiple eigenvalues translate into substantial algorithmic challenges that were not present in our scattering applications, and which certainly do not arise in connection with low-order approximations. As mentioned above, the central difficulty relates to the continuation of multiple eigenvalues of the unperturbed configuration. These eigenvalues may evolve, under shape deformation, as separated, distinct eigenvalues, and this "split" may only become apparent at high orders in their Taylor expansion. Our resolution of this and other algorithmic issues is discussed in Section 3 where the numerical procedure is presented in detail. Numerical results, showing different instances of eigenvalue splitting, finally follow in Section 4.

Our present motivation to generalize the analytic perturbative methods to eigenvalue problems arises, again, from optics: the dependence of cavity eigenvalues and the associated "Q-factors" on boundary perturbations is of primary importance in the design of microscopic lasing cavities (see, e.g., [9, 19, 23, 28, 29, 33, 38]). In fact, the general observation that small to moderate shape changes can have dramatic effects on the properties of conservative and leaky cavities (see, e.g., [18] for a recent experiment) has generated considerable interest in the development of perturbation methods that might shed light on this process [9, 22, 25, 26]. Work to date, however, has only resulted in low- (first and second) order theories and, thus, it is restricted to very small perturbations. These results have prompted the suggestion mentioned above [29] that perturbative methods would only be applicable within this limited range. As we shall show here, however, appropriate uses of analytic function theory can substantially enlarge the domain of applicability of these methods, much as in the scattering calculations of [6, 7].

Naturally, the theoretical and practical relevance of eigenvalue problems goes beyond these optics applications. For generality, it is not appropriate to discuss particular configurations here; applications and engineering generalizations of these methods will be discussed elsewhere. Here we focus on the basic theoretical and numerical issues associated with analytic boundary-perturbation theory as it applies to eigenvalue problems.

2. Analyticity of Eigenfunctions

As mentioned in the introduction, the validity of our analytic perturbation methods hinges on a certain property of analyticity of solutions of differential equations; in the present instance we deal with the corresponding analytic properties of eigenvalues and eigenfunctions associated with the eigenvalue problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega_\delta \\ u = 0 & \text{on } \partial\Omega_\delta \,. \end{cases} \tag{2.1}$$

Here the domain Ω_δ, $\delta \in \mathbb{R}$, is defined as

$$\Omega_\delta = \{(r,\theta) \,:\, r < 1 + \delta f(\theta)\} \tag{2.2}$$

for some 2π-periodic function $f(\theta)$; the analyticity of the eigenvalues $\lambda = \lambda(\delta)$ as functions δ is a well-known classical result [24, 34]:

Theorem 1 (see [34, Sections II.2 and II.6] or [24, Section VII.6]).

Let $f(\theta) \in C^2([0, 2\pi])$ be a 2π-periodic function and let $|\delta_0| < 1/M$, $M = \max|f(\theta)|$. Assume that for $\delta = \delta_0$ there exist m linearly independent solutions of (2.1) with $\lambda = \lambda_0$, that is, λ_0 is an eigenvalue of the Laplacian in Ω_{δ_0} with multiplicity m. Then, there exist functions $\lambda_1(\delta), \lambda_2(\delta), \ldots, \lambda_m(\delta)$ with $\lambda_j(\delta_0) = \lambda_0$ such that

(i) *$\lambda_j(\delta)$ is an eigenvalue for the domain Ω_δ and*

(ii) *for some $\nu_0 > 0$, $\lambda_j(\delta)$ is holomorphic for $|\delta - \delta_0| \leq \nu_0$.*

Regarding the corresponding eigenfunctions, classical results do not provide us with the regularity necessary to justify successive differentiation of the equations in (2.1) with respect to δ which, as we shall see in Section 3, constitute the basis of our numerical algorithm. Indeed, the required high order differentiations of the boundary condition

$$u(1+\delta f(\theta), \theta; \delta) = 0$$

require that u be defined and differentiable to high order in a uniform neighborhood of the domains Ω_δ. Our main result in this connection is the following.

Theorem 2.

Let $f(\theta)$ be an analytic 2π-periodic function and let the domains Ω_δ and the eigenvalues $\lambda_j(\delta)$ be defined as in Theorem 1. Then, the corresponding eigenfunctions can be chosen to depend holomorphically in $(\vec{x}, \delta)$. More precisely, for each j, there exist solutions $\left(u_j(\vec{x}; \delta), \lambda_j(\delta)\right)$ of (2.1) with the property that the functions u_j are analytic in a (complex) neighborhood of $\overline{\Omega}_{\delta_0} \times [\delta_0 - \epsilon_0, \delta_0 + \epsilon_0]$, $0 < \epsilon_0 \le \nu_0$.

Our strategy for proving Theorem 2 relies partially on the possibility of choosing a conformal change of variables that transforms the domains Ω_δ onto the unit disk. This conformal transformation, however, must be judiciously chosen so as to guarantee that it is also *jointly* analytic in the spatial variables and δ. Indeed, we have the following.

Lemma 1 (Analytic dependence of a Riemann mapping).

Let Ω_δ be given by (2.2), $\delta_0 \in \mathbb{R}$ and let $B_R(0) \subset \mathbb{R}^2$ denote the disk of radius R centered at 0. Then, there exists $\epsilon_1 > 0$, a domain D with $\overline{\Omega_\delta} \subset D$ for $\delta \in \mathbb{R}$, $|\delta - \delta_0| \le \epsilon_1$, and functions $U, V : \overline{B_{1+\epsilon_1}(0)} \times \{\delta \in \mathbb{C} : |\delta - \delta_0| \le \epsilon_1\} \to \mathbb{C}$ such that:

(i) *For each fixed $\delta \in \mathbb{R}$ with $|\delta - \delta_0| \le \epsilon_1$, $U(\vec{x}; \delta)$ and $V(\vec{x}; \delta)$ are real-valued.*

(ii) *For each fixed $\delta \in \mathbb{R}$ with $|\delta - \delta_0| \le \epsilon_1$, the function*

$$F(\vec{x}; \delta) = U(\vec{x}; \delta) + iV(\vec{x}; \delta) \tag{2.3}$$

maps $B_1(0)$ and $B_{1+\epsilon_1}(0)$ conformally in the variable $z = x_1 + ix_2$ onto $\Omega_\delta \subset \mathbb{C}$ and onto a neighborhood of $\overline{D} \subset \mathbb{C}$, respectively;

(iii) *$F(\vec{x}; \delta)$ is a jointly analytic function of the complex variables $z = x_1 + ix_2$ and δ; and*

(iv) *throughout its domain of definition*

$$\mathrm{Re}\left(U_{x_1}^2 + V_{x_1}^2\right) \ge c > 0 \quad \textit{for some constant } c\,.$$

In particular, the function

$$S(\vec{x}; \delta) \equiv \left(U_{x_1}^2 + V_{x_1}^2\right)^{1/2} \qquad (\mathrm{Re}(S) > 0) \tag{2.4}$$

is well defined and analytic in $(\vec{x}; \delta)$ for $(\vec{x}; \delta) \in \overline{B_{1+\epsilon_1}(0)} \times \{\delta \in \mathbb{C} : |\delta - \delta_0| \le \epsilon_1\}$.

Proof. The existence of a function satisfying (i) and (ii) is a direct consequence of the Riemann Mapping Theorem and of the analytic continuation results for these mappings that hold for analytic domains (see e.g., [32, Chap. 17] or [27, Chaps. 9 and 10]). However, a more careful analysis is needed to obtain a map that is, in addition, regular in the variable δ (cf. (iii)). To this end, we shall appeal to the equivalence between the Riemann Mapping Problem and the Dirichlet Problem for Laplace's equation [32, Chap. 17] and to the analyticity results for the latter that were derived in [5] (see also [4]).

Following [32, Section 17.25], for each $\delta \in \mathbb{R}$ fixed we consider the functions

$$g(\vec{x}; \delta) = -\log|\vec{x}| + u(\vec{x}; \delta)$$

where u solves

$$\begin{cases} \Delta u(\cdot;\delta) = 0 & \text{in } \Omega_\delta \\ u(\cdot;\delta) = \log|\cdot| & \text{on } \partial\Omega_\delta \,. \end{cases} \tag{2.5}$$

Thus, $g(\cdot;\delta)$ is nothing but (a multiple of) the Green's function for Ω_δ with pole at the origin. In particular, for instance,

$$g(\vec{x};\delta) > 0 \quad \text{for} \quad \vec{x} \in \Omega_\delta \,.$$

Since the domains Ω_δ have analytic boundaries and depend analytically in δ and since the boundary values in (2.5) are themselves jointly analytic functions of $(\vec{x};\delta)$ (in fact, they are independent of δ), it follows from [5] that there exists a number ν_1 and a domain D^1 containing $\overline{\Omega_\delta}$ for $|\delta-\delta_0| \le \nu_1$ such that the solution u is a jointly analytic function of $\vec{x}$ and δ for $(\vec{x};\delta)$ in $\overline{D^1} \times \{\delta \in \mathbb{C} : |\delta-\delta_0| \le \nu_1\}$.

Next, let $\vec{x}_0 \neq 0$ be an arbitrary fixed point belonging to every Ω_δ, $|\delta-\delta_0| \le \nu_1$ ($\delta \in \mathbb{R}$), and let $v(\vec{x};\delta)$ denote the (unique) harmonic function conjugate to $u(\vec{x};\delta)$ in Ω_δ and satisfying $v(\vec{x}_0;\delta) = v_0$ where v_0 is any real constant. Note that the analyticity properties of u imply that there exist a number $\nu_2 \le \nu_1$ and a domain $D^2 \subseteq D^1$, $\overline{\Omega_\delta} \subset D^2$ for $|\delta-\delta_0| \le \nu_2$, such that the function $h(\vec{x};\delta) \equiv u(\vec{x};\delta) + iv(\vec{x};\delta)$ is jointly analytic in $\vec{x}$ and δ for $(\vec{x};\delta)$ in $\overline{D^2} \times \{\delta \in \mathbb{C} : |\delta-\delta_0| \le \nu_2\}$. Thus, the same regularity properties hold for the function

$$R(\vec{x};\delta) = (x_1 + ix_2)\, e^{-h(\vec{x};\delta)} \,. \tag{2.6}$$

On the other hand, the results in [32, Section 17.25] show that, for each $\delta \in \mathbb{R}$, $|\delta-\delta_0| \le \nu_2$, this function maps the domain Ω_δ onto $B_1(0)$ conformally in the variable $z = x_1 + ix_2$. Moreover, since each Ω_δ has an analytic boundary it follows from classical analytic continuation results (see e.g., [32, Chap. 17] or [27, Chaps. 9 and 10]) that there exist $\nu_3 \le \nu_2$ and a domain $D^3 \subseteq D^2$, again with $\overline{\Omega_\delta} \subset D^3$ for $|\delta-\delta_0| \le \nu_3$, such that the function $R(\cdot;\delta)$ in (2.6) defines a conformal transformation from D^3 onto a neighborhood $\mathcal{N}_\delta$ of $\overline{B_1(0)}$. In fact, without loss of generality, we may assume that this last property holds for $\delta \in \mathbb{C}$, $|\delta-\delta_0| < \nu_3$, by slightly reducing ν_3 and the size of the domain D^3 if necessary. Thus, the function R is jointly analytic for $(z;\delta) \in D^3 \times \{\delta \in \mathbb{C} : |\delta-\delta_0| < \nu_3\}$ (since it is analytic in each variable separately [21]) and conformal in z for each δ. Finally, we define

$$F(\cdot;\delta) = R^{-1}(\cdot;\delta) \tag{2.7}$$

in a uniform neighborhood $\mathcal{N}$ of $\overline{B_1(0)}$ with $\overline{\mathcal{N}} \subset \mathcal{N}_\delta$ for $|\delta-\delta_0| \le \nu_3$. Then, F satisfies (i)–(iv) provided we choose $\epsilon_1 < \nu_3$ such that $B_{1+\epsilon_1}(0) \subseteq \mathcal{N}$. $\square$

Using the conformal change of variables F [cf. (2.3)] we may recast the eigenvalue problem (2.1) in the form

$$\begin{cases} \Delta w + \lambda \left[\left(\frac{\partial U}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_1}\right)^2 \right] w = 0 & \text{in } B_1(0) \\ w = 0 & \text{on } \partial B_1(0)\,, \end{cases} \tag{2.8}$$

or

$$w(r,\theta;\delta) = \lambda(\delta) \int_0^1 \int_0^{2\pi} g(r,\rho;\theta-\phi) w(\rho,\phi;\delta) s(\rho,\phi;\delta)^2 \,\rho\, d\phi\, d\rho \tag{2.9}$$

where $g = g(r,\rho;\theta-\phi)$ denotes the Green's function for the unit disk in polar coordinates (see e.g., [17]),

$$\begin{aligned} g(r,\rho;\theta-\phi) = -\frac{1}{2\pi} \Big[&\log\left(r^2 + \rho^2 - 2r\rho\cos(\theta-\phi)\right) \\ &- \log\left((r\rho)^2 + 1 - 2r\rho\cos(\theta-\phi)\right)\Big] \,, \end{aligned} \tag{2.10}$$

and

$$s(r, \theta; \delta) = S(r\cos(\theta)\,,\, r\sin(\theta); \delta) \quad \text{[cf. (2.4)]}\,. \tag{2.11}$$

Equivalently, setting

$$W(r, \theta; \delta) = s(r, \theta; \delta) w(r, \theta; \delta) \tag{2.12}$$

we are to find functions $W(r, \theta; \delta)$ satisfying

$$W(r, \theta; \delta) = \lambda(\delta) \int_0^1 \int_0^{2\pi} k(r, \theta; \rho, \phi; \delta) W(\rho, \phi; \delta)\, \rho\, d\phi\, d\rho = \lambda(\delta)\mathcal{G}(W)(r, \theta; \delta) \tag{2.13}$$

with the kernel k and the operator $\mathcal{G}$ defined by

$$k(r, \theta; \rho, \phi; \delta) = s(r, \theta; \delta) g(r, \rho; \theta - \phi) s(\rho, \phi; \delta) \tag{2.14}$$

and

$$\mathcal{G}(\eta)(r, \theta; \delta) = \int_0^1 \int_0^{2\pi} k(r, \theta; \rho, \phi; \delta)\, \eta(\rho, \phi; \delta)\, \rho\, d\phi\, d\rho\,. \tag{2.15}$$

In operator form, equation (2.13) is

$$\big(I - \lambda(\delta)\mathcal{G}\big)(W) = 0 \tag{2.16}$$

where I denotes the identity map. As we shall show, equation (2.16) can be solved in a space of holomorphic functions. To this end, and for δ_0 as in Theorem 1, we define the (Banach) space *of tangentially analytic* functions

$$\begin{aligned} \mathcal{H}_\epsilon(\delta_0) = \Big\{ \eta = \eta(r, \Theta; \delta) \;:\; & \eta \text{ is } 2\pi\text{-periodic in } \Theta,\ \eta \text{ is continuous for} \\ & r \in [0, 1]\,,\ \big|\mathrm{Im}(\Theta)\big| \le \epsilon\,,\ \delta \in \overline{B_\epsilon(\delta_0)} \subset \mathbb{C} \text{ and, for each } r \in [0, 1]\,, \\ & \eta(r, \cdot; \cdot) \text{ is holomorphic for } \big|\mathrm{Im}(\Theta)\big| < \epsilon\,,\ \delta \in B_\epsilon(\delta_0) \Big\} \end{aligned} \tag{2.17}$$

with norm given by

$$\|\eta\| = \sup_{\substack{r \in [0,1], |\mathrm{Im}(\Theta)| \le \epsilon \\ |\delta - \delta_0| \le \epsilon}} |\eta(r, \Theta; \delta)|\,.$$

As an easy consequence of Lemma 1 we have the following.

Corollary 1.

There exists $\epsilon_2 > 0$ *such that* $s \in \mathcal{H}_{\epsilon_2}(\delta_0)$.

Next we show that the operator $\mathcal{G}$ has a natural extension to the spaces $\mathcal{H}_\epsilon(\delta_0)$.

Lemma 2.

Let ϵ_2 be as in Corollary 1. Then, for any $\epsilon \le \epsilon_2$, $\mathcal{G}$ can be (uniquely) extended to an operator from $\mathcal{H}_\epsilon(\delta_0)$ into itself.

Proof. We shall show that, if $\epsilon \le \epsilon_2$, then for each $\eta \in \mathcal{H}_\epsilon(\delta_0)$ the extension of $\mathcal{G}(\eta)$ is given by

$$\begin{aligned}\mathcal{G}(\eta)(r,\theta+it;\delta) &= \int_0^1\int_0^{2\pi} s(r,\theta+it;\delta)g(r,\rho;\theta-\phi)s(\rho,\phi+it;\delta)\\ &\qquad\times\ \eta(\rho,\phi+it;\delta)\,\rho\,d\phi\,d\rho \\ &= \int_0^1\int_0^{2\pi} k(r,\theta+it;\rho,\phi+it;\delta)\eta(\rho,\phi+it;\delta)\,\rho\,d\phi\,d\rho\end{aligned} \tag{2.18}$$

where $t\in\mathbb{R}$, $|t|\le\epsilon$. Indeed, since $s\in\mathcal{H}_{\epsilon_2}(\delta_0)$ and $\eta\in\mathcal{H}_\epsilon(\delta_0)$ it follows from (2.10) and (2.18) that $\mathcal{G}(\eta)(r,\Theta;\delta)$ is continuous for $r\in[0,1]$, $|\operatorname{Im}(\Theta)|\le\epsilon_2$ and $\delta\in\overline{B_\epsilon(\delta_0)}\subset\mathbb{C}$ and analytic in δ for $\delta\in B_\epsilon(\delta_0)$. On the other hand, since s and η are analytic in $\Theta=\theta+it$ we have

$$(\partial_\theta+i\partial_t)\,s(r,\theta+it;\delta) = \left(\partial_\phi+i\partial_t\right)\eta(\rho,\phi+it;\delta)=0$$

which implies that

$$\begin{aligned}(\partial_\theta+i\partial_t)\,\mathcal{G}(\eta)(r,\theta+it;\delta) &= \int_0^1\int_0^{2\pi}(\partial_\theta+i\partial_t)\big(k(r,\theta+it;\rho,\phi+it;\delta)\\ &\qquad\times\ \eta(\rho,\phi+it;\delta)\big)\,\rho\,d\phi\,d\rho\\ &= -\int_0^1\int_0^{2\pi}\partial_\phi\Big(s(r,\theta+it;\delta)g(r,\rho;\theta-\phi)s(\rho,\phi+it;\delta)\\ &\qquad\times\ \eta(\rho,\phi+it;\delta)\Big)\,\rho\,d\phi\,d\rho=0\end{aligned}$$

that is, $\mathcal{G}(\eta)$ satisfies the Cauchy-Riemann equations in Θ. Thus, $\mathcal{G}(\eta)$ in analytic in Θ, $|\operatorname{Im}(\Theta)|<\epsilon$, and it therefore belongs to $\mathcal{H}_\epsilon(\delta_0)$. □

Our proof of Theorem 2 will be based on the reduction of equation (2.16) to a finite-dimensional (matrix) problem for which analyticity results are well known [34]. To this end, we will need to approximate the operator $\mathcal{G}$ by an operator with a finite-dimensional range.

Lemma 3.

Let k be as in (2.14). *Then, given any $\sigma>0$, there exist $\epsilon_3=\epsilon_3(\sigma,\epsilon_2)>0$, $\epsilon_3\le\epsilon_2$ (with ϵ_2 as in Lemma 2), an integer N, a symmetric matrix $B=\left(b_{ij}\right)\in\mathbb{R}^{N\times N}$ and functions $\beta_j(r,\theta;\delta)$ such that*

(i) $\beta_j\in\mathcal{H}_{\epsilon_2}(\delta_0)$;

(ii) *β_j is real valued when restricted to real arguments;*

(iii) *$\left\{\beta_j(\cdot;\delta)\right\}_{j=1}^N$ is a linearly independent set for $\delta\in\mathbb{C}$, $|\delta-\delta_0|\le\epsilon_2$.*

(iv) *The function*

$$\widehat{k}(r,\rho;\phi,\theta;\delta)\equiv\sum_{i,j=1}^N b_{ij}\beta_i(r,\theta;\delta)\beta_j(\rho,\phi;\delta) \tag{2.19}$$

satisfies

$$\begin{aligned}\max_{\substack{r\in[0,1],|t|\le\epsilon\\ |\delta-\delta_0|\le\epsilon}}\int_0^1\int_0^{2\pi}&\Big|k(r,\theta+it;\rho,\phi+it;\delta)\\ &-\widehat{k}(r,\theta+it;\rho,\phi+it;\delta)\Big|\rho\,d\phi\,d\rho\le\sigma\,.\end{aligned} \tag{2.20}$$

for every $\epsilon \leq \epsilon_3$.

Proof. First note that we can approximate the Green's function g by continuous and symmetric functions g_γ, $\gamma > 0$, defined by

$$g_\gamma(r,\rho;\phi) = l_\gamma(r,\rho;\phi) - m_\gamma(r,\rho;\phi)$$

where

$$l_\gamma(r,\rho;\theta-\phi) = -\frac{1}{2\pi}\begin{cases} \log\left(r^2+\rho^2-2r\rho\cos(\theta-\phi)\right) & \text{if } r^2+\rho^2-2r\rho\cos(\theta-\phi) > \gamma\,, \\ \log(\gamma) & \text{if } r^2+\rho^2-2r\rho\cos(\theta-\phi) \leq \gamma\,, \end{cases}$$

and

$$m_\gamma(r,\rho;\theta-\phi) = -\frac{1}{2\pi}\begin{cases} \log\left((1-r\rho\cos(\theta-\phi))^2 + r^2\rho^2 - (r\rho\cos(\theta-\phi))^2\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } r\rho\cos(\theta-\phi) \leq 1-\gamma, \\ \log\left(\gamma^2 + r^2\rho^2 - (1-\gamma)^2\right) \quad \text{if } r\rho\cos(\theta-\phi) > 1-\gamma\,. \end{cases}$$

Indeed, as is easily checked,

$$\max_{r\in[0,1],\theta\in\mathbb{R}} \int_0^1 \int_0^{2\pi} \left|g(r,\rho;\theta-\phi) - g_\gamma(r,\rho;\theta-\phi)\right| \rho\, d\phi\, d\rho \to 0 \quad \text{as} \quad \gamma \to 0\,.$$

Next, since g_γ is a continuous function of

$$(\vec{x},\vec{y}) = (\vec{x}(r,\theta)\,, \vec{y}(\rho,\phi)) \equiv \Big(r(\cos(\theta),\sin(\theta)), \rho(\cos(\phi),\sin(\phi))\Big)$$

for $(\vec{x},\vec{y}) \in \overline{B_1(0)} \times \overline{B_1(0)}$, we can find polynomials $P_{\ell,\gamma}(\vec{x},\vec{y})$ such that $P_{\ell,\gamma} \to g_\gamma$ uniformly on $\overline{B_1(0)} \times \overline{B_1(0)}$ as $\ell \to \infty$. Moreover, since g_γ is symmetric as a function of $(\vec{x},\vec{y})$, $P_{\ell,\gamma}$ can be chosen to be of the form

$$P_{\ell,\gamma}(\vec{x},\vec{y}) = \sum_{i,j=1}^{N} b_{ij} T_i(\vec{x})\, T_j(\vec{y})$$

where $N = N(\ell)$ and where $\{T_i\}$ are linearly independent polynomials (with real coefficients) and $B = (b_{ij}) \in \mathbb{R}^{N\times N}$ is a symmetric matrix (which depends on ℓ and γ). In particular, given $\sigma > 0$, we can choose γ, ℓ and a number $\epsilon_3 = \epsilon_3(\sigma,\epsilon_2) > 0$, $\epsilon_3 \leq \epsilon_2$, sufficiently small so as to guarantee that

$$\begin{aligned} \max_{r\in[0,1],|t|\leq\epsilon_3} \int_0^1 \int_0^{2\pi} &\left|g(r,\rho;\theta-\phi) - P_{\ell,\gamma}\Big(\vec{x}(r,\theta+it), \vec{y}(\rho,\phi+it)\Big)\right| \rho\, d\phi\, d\rho \\ &\leq \frac{\sigma}{\|s\|^2_{\mathcal{H}_{\epsilon_2}(\delta_0)}} \end{aligned} \tag{2.21}$$

where ϵ_2 is as in Lemma 2. Then, letting

$$\beta_i(r,\theta;\delta) = T_i\,(r\cos(\theta), r\sin(\theta))\, s(r,\theta;\delta)$$

and

$$\begin{aligned} \widehat{k}(r,\rho;\phi,\theta;\delta) &\equiv s(r,\theta;\delta) P_{\ell,\gamma}(\vec{x},\vec{y})\, s(\rho,\phi;\delta) \\ &= \sum_{i,j=1}^{N} b_{ij}\beta_i(r,\theta;\delta)\beta_j(\rho,\phi;\delta) \end{aligned}$$

we clearly have $\beta_j \in \mathcal{H}_{\epsilon_2}(\delta_0)$ satisfying (ii) and (iii). Finally, the estimate (2.20) follows from (2.21). □

Proof of Theorem 2. From Theorem 1 we know that functions $\lambda_l(\delta)$, holomorphic for $|\delta-\delta_0| \le \nu_0$, exist so that equation (2.16) with $\lambda = \lambda_l$ has nontrivial solutions. Since the functions $\lambda_l(\delta)$ are positive for $\delta \in \mathbb{R}$ we may assume, by reducing ν_0 if necessary, that

$$\min_{\substack{|\delta-\delta_0|\le\nu_0 \\ \delta\in\mathbb{C}}} |\lambda_l(\delta)| > 0\,.$$

We shall show that solutions to (2.16) can chosen to belong to a space $\mathcal{H}_\epsilon(\delta_0)$. To this end, fix l and let

$$\sigma = \max_{|\delta-\delta_0|\le\nu_0} \frac{1}{2\,|\lambda_l(\delta)|}\,.$$

For this value of σ, let $\widehat{k}$ denote the kernel in (2.19) satisfying (2.20). Then, defining

$$\widehat{\mathcal{G}}(\eta)(r,\theta+it;\delta) = \int_0^1\int_0^{2\pi} \widehat{k}(r,\theta+it;\rho,\phi+it;\delta)\,\eta(\rho,\phi+it;\delta)\,\rho\,d\phi\,d\rho \tag{2.22}$$

and

$$\mathcal{R}(\eta) = \mathcal{G}(\eta) - \widehat{\mathcal{G}}(\eta)\,, \tag{2.23}$$

we have, from Lemmas 2 and 3, that

$$\max_{|\delta-\delta_0|\le\epsilon} \|\lambda_l(\delta)\mathcal{R}\|_{\mathcal{L}(\mathcal{H}_\epsilon(\delta_0))} \le \frac{1}{2} \quad \text{for every } \epsilon \le \epsilon_4 \tag{2.24}$$

where

$$\epsilon_4 = \epsilon_4(\nu_0,\epsilon_2) \equiv \min(\nu_0,\epsilon_3)$$

and ϵ_3 is as in Lemma 3. Note that, from (2.14), (2.15), (2.19), and (2.22), for $\xi,\eta \in \mathcal{H}_{\epsilon_3}(\delta_0)$ we have

$$\int_0^1\int_0^{2\pi} \mathcal{G}(\eta)(\rho,\phi;\delta)\xi(\rho,\phi;\delta)\,\rho\,d\phi\,d\rho = \int_0^1\int_0^{2\pi} \eta(\rho,\phi;\delta)\mathcal{G}(\xi)(\rho,\phi;\delta)\,\rho\,d\phi\,d\rho$$
$$\int_0^1\int_0^{2\pi} \widehat{\mathcal{G}}(\eta)(\rho,\phi;\delta)\xi(\rho,\phi;\delta)\,\rho\,d\phi\,d\rho = \int_0^1\int_0^{2\pi} \eta(\rho,\phi;\delta)\widehat{\mathcal{G}}(\xi)(\rho,\phi;\delta)\,\rho\,d\phi\,d\rho$$

so that, using (2.23),

$$\int_0^1\int_0^{2\pi} \mathcal{R}(\eta)(\rho,\phi;\delta)\xi(\rho,\phi;\delta)\,\rho\,d\phi\,d\rho = \int_0^1\int_0^{2\pi} \eta(\rho,\phi;\delta)\mathcal{R}(\xi)(\rho,\phi;\delta)\,\rho\,d\phi\,d\rho\,. \tag{2.25}$$

Thus, for $\lambda = \lambda_l$ we may write equation (2.16) in the form

$$\left(I - \lambda_l\mathcal{M}\right)\left(\widetilde{W}\right) = 0 \tag{2.26}$$

where

$$\widetilde{W} = (I - \lambda_l\mathcal{R})^{1/2}(W) \tag{2.27}$$

and

$$\mathcal{M} \equiv (I - \lambda_l \mathcal{R})^{-1/2} \widehat{\mathcal{G}} (I - \lambda_l \mathcal{R})^{-1/2} . \tag{2.28}$$

From (2.22), (2.25), and (2.28)

$$\mathcal{M}(\eta)(r, \theta + it; \delta) = \int_0^1 \int_0^{2\pi} \widehat{m}(r, \theta + it; \rho, \phi + it; \delta)\, \eta(\rho, \phi + it; \delta)\, \rho \, d\phi \, d\rho \tag{2.29}$$

where

$$\widehat{m}(r, \theta; \rho, \phi; \delta) \equiv \sum_{i,j=1}^{N} b_{ij} \mu_i(r, \theta; \delta) \mu_j(\rho, \phi; \delta) \tag{2.30}$$

and

$$\mu_j \equiv (I - \lambda_l \mathcal{R})^{-1/2} (\beta_j) \in \mathcal{H}_\epsilon(\delta_0) , \quad \epsilon \leq \epsilon_4 .$$

In view of equation (2.24), the operator $(I - \lambda_l \mathcal{R})^{-1/2} : \mathcal{H}_\epsilon(\delta_0) \to \mathcal{H}_\epsilon(\delta_0)$, $\epsilon \leq \epsilon_4$, is well defined and, in fact, it can be represented by a Neumann series. Also, from (ii) and (iii) of Lemma 3, the functions μ_j are linearly independent and real valued on real arguments.

Next, let $\{\alpha_j(\cdot; \delta)\}_{j=1}^N \subset \mathcal{H}_\epsilon(\delta_0)$, $\epsilon \leq \epsilon_4$, denote the orthonormal set of functions that results from the Gram-Schmidt process applied to $\{\mu_j(\cdot; \delta)\}_{j=1}^N$, so that

$$\int_0^1 \int_0^{2\pi} \alpha_i(\rho, \phi; \delta) \alpha_j(\rho, \phi; \delta)\, \rho \, d\phi \, d\rho = \begin{cases} 1 & \text{if } i = j , \\ 0 & \text{if } i \neq j . \end{cases} \tag{2.31}$$

Then,

$$\mu_i(r; \theta; \delta) = \sum_{j=1}^{N} c_{ij}(\delta) \alpha_j(r; \theta; \delta)$$

where $c_{ij}(\delta)$ is holomorphic for $|\delta - \delta_0| < \epsilon_4$ and, since μ_i is real for real δ, $c_{ij}(\delta) \in \mathbb{R}$ for $\delta \in \mathbb{R}$. Thus,

$$\begin{aligned} \widehat{m}(r, \theta; \rho, \phi; \delta) &= \sum_{i,j=1}^{N} b_{ij} \mu_i(r, \theta; \delta) \mu_j(\rho, \phi; \delta) \\ &= \sum_{i,j=1}^{N} b_{ij} \sum_{k=1}^{N} c_{ik}(\delta) \alpha_k(r, \theta; \delta) \sum_{n=1}^{N} c_{jn}(\delta) \alpha_n(\rho, \phi; \delta) \\ &= \sum_{k,n=1}^{N} A_{kn}(\delta) \alpha_k(r, \theta; \delta) \alpha_n(\rho, \phi; \delta) \end{aligned} \tag{2.32}$$

where

$$A_{kn}(\delta) = \sum_{i,j=1}^{N} b_{ij} c_{ik}(\delta) c_{jn}(\delta)$$

so that, since $b_{ij} = b_{ji}$ (cf. Lemma 3)

$$\begin{gathered} A = A(\delta) = \big(A_{kn}(\delta)\big) \text{ is holomorphic for } |\delta - \delta_0| < \epsilon_4 \text{ and} \\ A(\delta) \text{ is a real symmetric matrix for } \delta \in \mathbb{R} . \end{gathered} \tag{2.33}$$

Hence, using (2.26)–(2.33), we find that solutions $\widetilde{W}$ of (2.26) must have the form

$$\widetilde{W}(r,\theta;\delta)=\sum_{j=1}^{N}\omega_j(\delta)\alpha_j(r,\theta;\delta) \tag{2.34}$$

where the unknown coefficients $\omega=[\omega_1,\ldots,\omega_N]^T$ satisfy

$$\omega(\delta)=\lambda_l(\delta)A(\delta)\omega(\delta)\,. \tag{2.35}$$

Since the reduction of (2.16) to (2.35) is reversible, we have that the operator $(I-\lambda_l(\delta)\mathcal{G})$ has a nontrivial nullspace if and only if so does the finite-dimensional operator $(I-\lambda_l(\delta)A(\delta))$. Thus, for $\delta\in\mathbb{R}$, $\lambda_l(\delta)^{-1}\in\mathbb{R}$ is an eigenvalue of $A(\delta)$ and, using (2.33), it follows from [34, Thm. 1–p. 33] that there exist a number $\epsilon_5=\epsilon_5(\epsilon_4)>0$, $\epsilon_5\le\epsilon_4$, and a function $\omega(\delta)=[\omega_1(\delta),\ldots,\omega_N(\delta)]^T\ne 0$, holomorphic for $|\delta-\delta_0|\le\epsilon_5$ and satisfying (2.35). Then, from (2.27) and (2.34) we see that the eigenfunction W_l that solves (2.16) with $\lambda(\delta)=\lambda_l(\delta)$ can be expressed as

$$W_l(r,\theta;\delta)=\sum_{j=1}^{N}\omega_j(\delta)\,(I-\lambda_l(\delta)\mathcal{R})^{-1/2}\,(\alpha_j)(r,\theta;\delta)\in\mathcal{H}_{\epsilon_5}(\delta_0)$$

and therefore, using (2.12),

$$w_l(r,\theta;\delta)\equiv\frac{W_l(r,\theta;\delta)}{s(r,\theta;\delta)}\in\mathcal{H}_{\epsilon_5}(\delta_0)\ . \tag{2.36}$$

In particular, from (2.9),

$$\frac{\partial w_l}{\partial r}(1,\theta+it;\delta)=\frac{\lambda_l(\delta)}{2\pi}\int_0^1\int_0^{2\pi}\left(\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\phi)}\right)$$
$$w_l(\rho,\phi+it;\delta)s(\rho,\phi+it;\delta)^2\,\rho\,d\phi\,d\rho$$

and therefore

$$\frac{\partial w_l}{\partial r}(1,\theta+it;\delta)\text{ is analytic for }|t|<\epsilon_5\,,\ |\delta-\delta_0|<\epsilon_5\,. \tag{2.37}$$

Finally, we may consider the (non-characteristic) Cauchy problem for a function $p=p(\vec{x};\delta)=p(r,\theta;\delta)$

$$\begin{cases}\dfrac{\partial^2 p}{\partial x_1^2}+\dfrac{\partial^2 p}{\partial x_2^2}-\lambda\left[\left(\dfrac{\partial U}{\partial x_1}\right)^2+\left(\dfrac{\partial V}{\partial x_1}\right)^2\right]p=0 & \text{in } B_1(0)\times\{|\delta-\delta_0|<\epsilon_5\}\\ p=0 & \text{on } \partial B_1(0)\times\{|\delta-\delta_0|<\epsilon_5\}\ ,\\ \dfrac{\partial p}{\partial r}(1,\theta;\delta)=\dfrac{\partial w_l}{\partial r}(1,\theta;\delta) & \text{on } \partial B_1(0)\times\{|\delta-\delta_0|<\epsilon_5\}\ ,\end{cases}$$

to conclude, from the Cauchy-Kowalevsky theorem, that there exists $\epsilon_0>0$ such that the function w_l is jointly analytic in $(\vec{x};\delta)$ for $(\vec{x};\delta)\in\overline{B_{1+\epsilon_0}(0)}\times\{|\delta-\delta_0|\le\epsilon_0\}$. Letting

$$u_l\,(\vec{x};\delta)=w_l\,(R\,(\vec{x};\delta)\,;\delta)\quad\text{[cf. (2.6) and (2.7)]}\,,$$

choosing ϵ_0 to also satisfy $\epsilon_0\le\epsilon_1$ and using Lemma 1, the theorem follows. □

3. The Algorithm

Theorem 2 asserts that it is possible to choose solutions to (2.1) that depend (jointly) analytically on $(\vec{x};\delta)$. In this section we shall describe an algorithm that produces such solutions; the results of a numerical implementation of these methods follows in Section 4.

To describe the process, let us consider the eigenvalue problem (2.1) and let us assume that

$$u(\vec{x};\delta)=\sum_{k\geq 0}u_k(\vec{x})\delta^k \quad\text{and}\quad \lambda(\delta)=\sum_{k\geq 0}q_k\delta^k\,.$$

Then, it follows from Theorem 2 that the coefficients $(u_n;q_n)$ satisfy the recursive relations

$$\begin{cases}\Delta u_n(r,\theta)+q_0u_n(r,\theta)=-\sum_{p=0}^{n-1}q_{n-p}u_p(r,\theta) & \text{for}\quad r\leq 1\,,\\ \text{and}\qquad u_n(1,\theta)=-\sum_{m=0}^{n-1}\frac{f(\theta)^{n-m}}{(n-m)!}\frac{\partial^{n-m}u_m}{\partial r^{n-m}}(1,\theta)\,. \end{cases}\tag{3.1}$$

The function u_0 is an eigenfunction of the Laplacian in the disk Ω_0, of course, and it is therefore given by

$$u_0(r,\theta)=\alpha_0 J_M\left(q_0^{1/2}r\right)e^{iM\theta}+\beta_0 J_M\left(q_0^{1/2}r\right)e^{-iM\theta},\quad \alpha_0,\beta_0\in\mathbb{R}\,,\tag{3.2}$$

where J_M denotes the Bessel function of order $M\geq 0$. The unperturbed eigenvalue q_0, in turn, satisfies

$$J_M\left(q_0^{1/2}\right)=0\,.$$

Note that the constants α_0, β_0 in (3.2) are arbitrary (any linear combination provides an eigenfunction in Ω_0) since q_0 has multiplicity two. However, upon boundary deformations this double eigenvalue will, in general, "split" into two simple ones, each having only a one-dimensional family of associated eigenfunctions. The requirement of analyticity (or even continuity) in δ of these eigenfunctions will thus force a very particular choice of the constants α_0, β_0. As we shall see below, however, this choice may not be apparent until several Taylor coefficients in the expansion for $(u(\cdot,\delta),\lambda(\delta))$ have been derived, as the aforementioned splitting may occur at any order in δ, depending on the perturbation function $f(\theta)$. For this reason, our algorithm to find the coefficients $(u_n(\cdot),q_n)$ in (3.1) proceeds in several steps, which we describe in what follows.

Step 1. Assume

$$u_0(r,\theta)=J_M\left(q_0^{1/2}r\right)e^{iM\theta}\,.\tag{3.3}$$

Note that, as we said, this assumption will generally be inconsistent with the desired analytic dependence of eigenfunctions on the perturbation parameter and will therefore have to be reconsidered at a latter stage of the algorithm (see Step 5). In any case, the question arises as to how to represent the successive derivatives u_n. A possible choice for basis functions are the actual eigenfunctions for the unperturbed geometry. However, such a choice would result in infinite series representations, and we therefore choose instead to define functions

$$\psi_{k,l}(r)=\frac{r^k}{(4q_0)^{k/2}\,k!}J_{l+k}\left(q_0^{1/2}r\right)\quad\text{and}\quad \phi_{k,l}(r,\theta)=\psi_{k,|l|}(r)e^{il\theta}\,.$$

These functions are characterized by

$$P_l\left(\psi_{k,l}\right)=\psi_{k-1,l}\quad\text{and}\quad \mathcal{L}\left(\phi_{k,l}\right)=\phi_{k-1,l}\tag{3.4}$$

where

$$P_l(\cdot) = \partial_r^2 + \frac{1}{r}\partial_r + \left(q_0 - \frac{l^2}{r^2}\right) \quad \text{and} \quad \mathcal{L}(\cdot) = \Delta + q_0 \,.$$

And, if $f(\theta)$ has a Fourier series expansion

$$f(\theta) = \sum_{l=-F}^{F} C_{1,l}e^{il\theta}$$

we will seek u_n of the form

$$u_n(r,\theta) = \sum_{\substack{0\le k\le n \\ M-(n-k)F\le l\le M+(n-k)F}} d_{k,l}^n\, \phi_{k,l}(r,\theta)\,. \tag{3.5}$$

Step 2. Find q_n. Multiplying the first equation in (3.1) by $\overline{u_0}$ and integrating on $\{r \le 1\}$ we find that

$$q_n = \frac{1}{A_{1,|M|}^0}\left[-\sum_{\substack{1\le p\le n-1 \\ 0\le k\le p}} q_{n-p} d_{k,M}^p A_{k+1,|M|}^0 + \sum_{\substack{0\le m\le n-1 \\ 0\le k\le m \\ M-\min(m-k,n-m)F\le s\le M+\min(m-k,n-m)F}} C_{n-m,M-s} d_{k,s}^m A_{k,|s|}^{n-m}\right]$$

where

$$A_{n,l}^m = \partial_r^m \psi_{n,l}\Big|_{r=a} \quad \text{and} \quad \frac{f(\theta)^n}{n!} = \sum_{l=-nF}^{nF} C_{n,l}e^{il\theta}\,.$$

Step 3. Find $d_{k,l}^n$ for $1 \le k \le n$ (all except $k = 0$). For this, we use the differential equation for u_n [cf. (3.1)] and the properties (3.4) which imply that

$$\mathcal{L}(u_n) = \sum_{k,l} d_{k,l}^n\, \phi_{k-1,l} = -\sum_p q_{n-p} \sum_k d_{k,l}^p\, \phi_{k,l}$$

and therefore

$$d_{k,l}^n = -\sum_{p=k-1}^{n-1} q_{n-p}\, d_{k-1,l}^p\,.$$

Step 4. Find $d_{0,l}^n$ and *check for eigenvalue "splitting."* For this, let $v_n = \sum_l d_{0,l}^n\, \phi_{0,l}$ and recall that $\phi_{0,l}(r,\theta) = J_{|l|}(q_0^{1/2}r)e^{il\theta}$ solves $(\Delta + q_0)\phi_{0,l} = 0$. Then, from (3.1), (3.5),

$$\begin{cases} \Delta v_n(r,\theta) + q_0 v_n(r,\theta) = 0 \quad \text{for} \quad r \le 1\,, \\ v_n(1,\theta) = -\sum_{m=0}^{n-1} \frac{f(\theta)^{n-m}}{(n-m)!}\, \partial_r^{n-m} u_m(1,\theta) - \sum_{\substack{1\le k\le n \\ l}} d_{k,l}^n\, \phi_{k,l}(1,\theta)\,. \end{cases}$$

Thus, if B_l^n denotes the l-th Fourier coefficient of the boundary values $v_n(1,\theta)$,

$$v_n(1,\theta) = \sum_l B_l^n e^{il\theta} \tag{3.6}$$

then, the solution takes the form

$$v_n(r,\theta) = \sum_l \frac{B_l^n}{J_{|l|}\left(q_0^{1/2}\right)} J_{|l|}\left(q_0^{1/2} r\right) e^{il\theta} . \tag{3.7}$$

Now, the choice of q_n (cf. Step 2) guarantees that the coefficient B_M^n of the (resonant) mode $l = M$ in (3.6) vanishes (recall $J_M(q_0^{1/2}) = 0$). However, the coefficient B_{-M}^n may or may not vanish. And, in fact, it can be shown that

$$\text{the eigenvalue "splits"} \iff B_{-M}^n \neq 0 \text{ for some n}$$

(see equation (3.8) below). Therefore, the procedure follows different paths depending on the value B_{-M}^n. If $B_{-M}^n = 0$, then we may indeed define v_n as in (3.7) and continue: replace $n \to n+1$ and go back to Step 2. Otherwise, if $B_{-M}^n \neq 0$ we proceed to the next step.

Step 5. (Only necessary under eigenvalue splitting as described in Step 4). "Recalculate" (the correct value of) q_n and choose appropriate eigenfunctions to order 0. As we said, once the splitting has been identified it needs to be accounted for by an appropriate choice of the constants α_0 and β_0 in (3.2) and the determination of coefficients q_n^+ and q_n^- corresponding to each of the two distinct eigenvalues, which have been found (cf. Step 4) to split at order n (i.e., $q_k^+ = q_k^- = q_k$ for $1 \leq k \leq n-1$). To this end, we first note that if we begin the procedure, in Step 1, with u_0 in (3.3) replaced by

$$u_0^1 = \alpha_0 u_0 + \beta_0 \overline{u_0}$$

we get that the corresponding higher order coefficients u_k will be given by

$$u_k^1 = \alpha_0 u_k + \beta_0 \overline{u_k} \qquad (1 \leq k \leq n-1)$$

which satisfy

$$\begin{cases} \Delta u_k^1(r,\theta) + q_0 u_k^1(r,\theta) = -\sum_{p=0}^{k-1} q_{k-p} u_k^1(r,\theta) & \text{for} \quad r \leq 1 , \\ \text{and} \qquad u_k^1(1,\theta) = -\sum_{m=0}^{k-1} \frac{f(\theta)^{k-m}}{(k-m)!} \frac{\partial^{k-m} u_m^1}{\partial r^{k-m}}(1,\theta) . \end{cases}$$

To find $q_n^{\pm}$, α_0, and β_0, we multiply the first equation above for $k = n$ by $\left(\frac{u_0}{\overline{u_0}}\right)$ and integrate in $\{r \leq 1\}$. After some straightforward calculations, and using the previously derived value of q_n, we find that

$$q_n^{\pm} = q_n \pm \frac{|B_{-M}^n|}{\psi_{1,|M|}(1)}, \qquad \alpha_0^{\pm} = \mp |B_{-M}^n| \qquad \text{and} \qquad \beta_0^{\pm} = B_{-M}^n \tag{3.8}$$

where, as we said, the superscripts $\pm$ differentiate the two (simple) eigenvalues. Thus, we define

$$u_k^{1,\pm} = \alpha_0^{\pm} u_k + \beta_0^{\pm} \overline{u_k} \text{ for } 0 \leq k \leq n-1$$

and now $u_n^{1,\pm}$ can be computed as in Steps 3 and 4, since the choices in (3.8) guarantee that the new $B_{\pm M}^n$ vanish.

Step 6. Calculate $q_{n+1+\nu}$ and choose appropriate eigenfunctions to order $1+\nu$ for each $\nu \geq 0$ (iterate in ν). For this, assume that for a fixed integer $\nu \geq 0$ we have calculated

$$\left\{u_k^{\nu+1,\pm}\right\}_{k=0}^{n+\nu} \quad \text{and} \quad \left\{q_k^{\pm}\right\}_{k=0}^{n+\nu} \qquad \left(q_k^{\pm} = q_k \text{ for } k = 0, \cdots, n-1\right) .$$

We want to define $q^{\pm}_{n+\nu+1}$ and new corrections $\{u_k^{\nu+2,\pm}\}_{k=1}^{n+\nu+1}$ where

$$u_k^{\nu+2,\pm} = u_k^{\nu+1,\pm} \quad \text{for} \quad k = 0, \cdots, \nu$$

For instance, if $\nu = 0$, we want to define $q^{\pm}_{n+1}$ and corrections $\{u_k^{2,\pm}\}_{k=0}^{n+1}$; however, $u_0^{2,\pm} = u_0^{1,\pm}$ has already been appropriately chosen in the previous step. We shall then look for solutions $u_k^{\nu+2,\pm}$ of the form

$$\begin{aligned} u_k^{\nu+2,\pm} &= u_k^{\nu+1,\pm} && \text{for } k = 0, \cdots, \nu \\ u_k^{\nu+2,\pm} &= u_k^{\nu+1,\pm} + \alpha^{\pm}_{\nu+1} u_{k-\nu-1} + \beta^{\pm}_{\nu+1} \bar{u}_{k-\nu-1} && \text{for } k = \nu + 1, \cdots, n + \nu \end{aligned}$$

where the u_m, $0 \le m \le n-1$, are those that were originally found (Steps 1–4). As before (Step 5), from the orthogonality conditions it is possible to find the values of $q^{\pm}_{n+1+\nu}$, $\alpha^{\pm}_{\nu+1}$ and $\beta^{\pm}_{\nu+1}$. And, finally, $u^{\nu+2,\pm}_{n+\nu+1}$ can be determined as in Steps 3 and 4.

4. Numerical Results

As it follows from Theorem 2, the procedure above produces the complete Taylor series of eigenvalues and eigenfunctions. Moreover, these series define analytic functions of δ throughout the domain $|\delta| < 1/\max|f|$. Of course, the series themselves may not be the appropriate representation of these functions as they may diverge for relatively small values of the deformation (see Figs. 1–4 below). On the other hand, since the functions are indeed analytic for $|\delta| < 1/\max|f|$, it should be

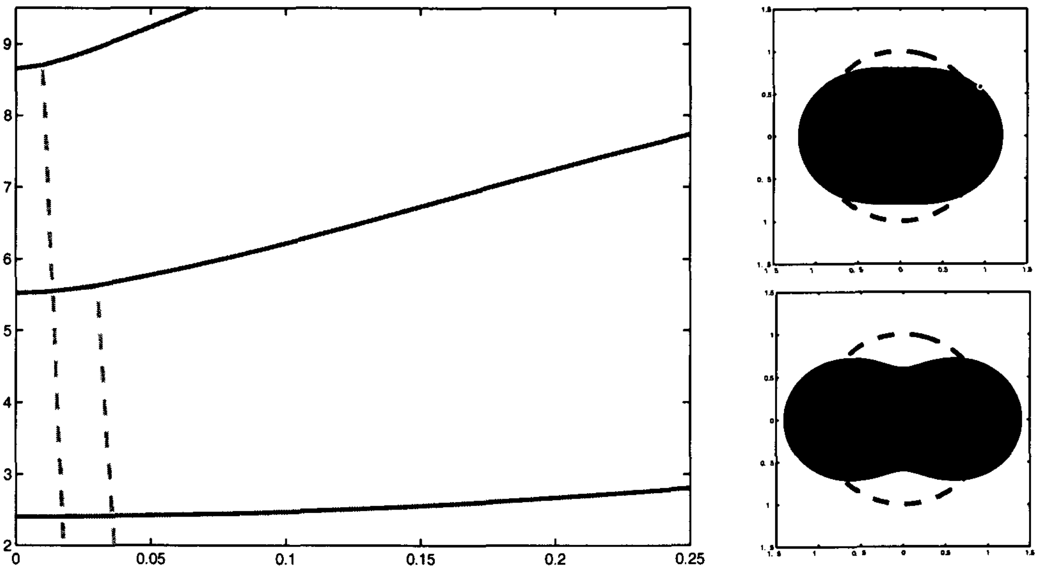

FIGURE 1 Continuation of three zeros of J_0 as eigenfrequencies for perturbations with $f(\theta) = 2\cos(2\theta)$. **No splitting.** Dashed: Taylor series (order 28); solid: Padé [14/14]. Inset: domains $r = 1 + \delta f(\theta)$ for $\delta = 0.10$ and $\delta = 0.20$.

possible to *analytically continue* the Taylor series to this domain. Several strategies are possible to numerically achieve this continuation (see [6, 7] and the references therein). Here we shall use Padé approximation (see e.g., [2]) as our continuation mechanism.

Below we present the results of our implementation of the algorithm described in Section 3 for a variety of perturbations $f(\theta)$. Figures 1–4 depict the results of our codes in cases where no

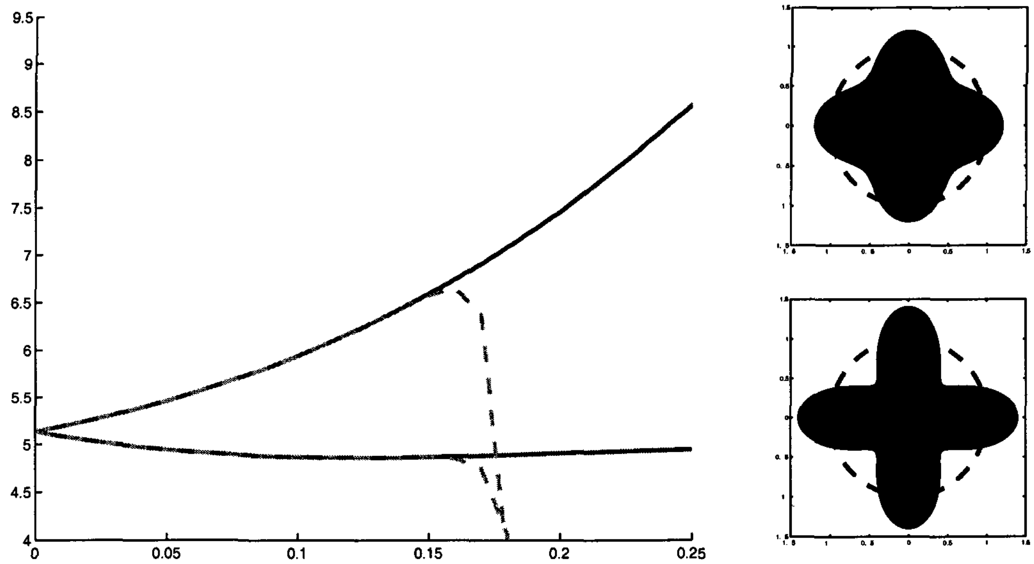

FIGURE 2 Continuation of a zero of J_2 as eigenfrequencies for perturbations with $f(\theta) = 2\cos(4\theta)$. **Splitting at order 1.** Dashed: Taylor series (order 28); solid: Padé [14/14]. Inset: domains $r = 1 + \delta f(\theta)$ for $\delta = 0.10$ and $\delta = 0.20$.

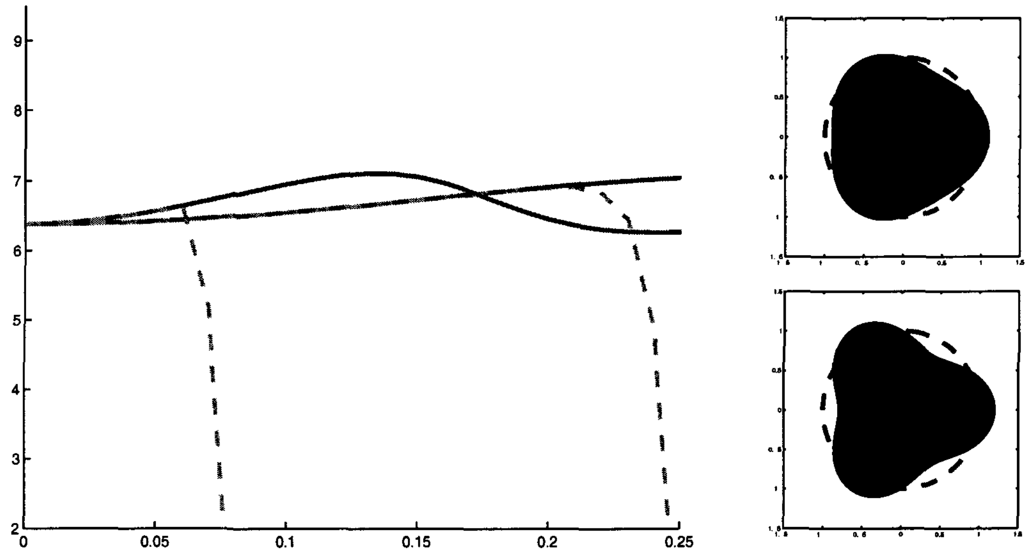

FIGURE 3 Continuation of a zero of J_3 as eigenfrequencies for perturbations with $f(\theta) = 2\cos(3\theta)$. **Splitting at order 2.** Dashed: Taylor series (order 28); solid: Padé [14/14]. Inset: domains $r = 1 + \delta f(\theta)$ for $\delta = 0.05$ and $\delta = 0.10$.

splitting occurs and where the eigenvalues separate at orders $n = 1, 2$ and 4, respectively. For the case of first order splitting (Fig. 2) we also include a table where we have recorded the outcome of convergence studies. These results show that, as claimed, very accurate approximations can be achieved for substantial deformations which may be well beyond the disk of convergence of the Taylor series.

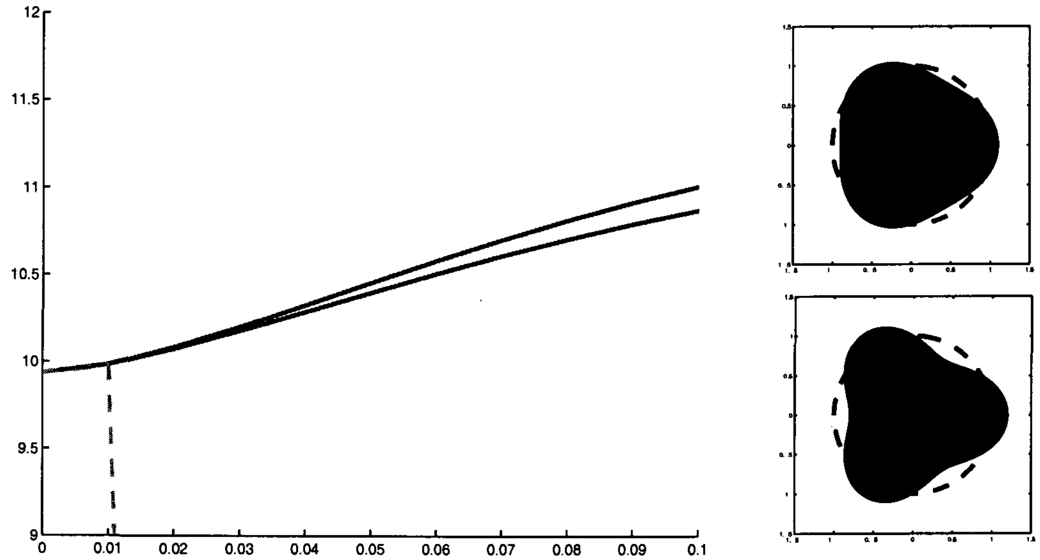

FIGURE 4 Continuation of a zero of J_6 as eigenfrequencies for perturbations with $f(\theta) = 2\cos(3\theta)$. **Splitting at order 4.** Dashed: Taylor series (order 28); solid: Padé [14/14]. Inset: domains $r = 1 + \delta f(\theta)$ for $\delta = 0.05$ and $\delta = 0.10$.

TABLE 4.1

Numerical values for the example of Figure 2: $f(\theta) = 2\cos(4\theta)$. Continuation of the zero $z = 5.1356223$ of $J_2(z)$.

N	δ	Padé ($[N/2, N/2]$)	Taylor (N)
4	0.10	5.932459814	5.933746721
8	0.10	5.939888161	5.939685615
12	0.10	5.940103147	5.940002737
16	0.10	5.940063852	5.940052067
20	0.10	5.940063913	5.940061333
24	0.10	5.940063912	5.940063302
28	0.10	5.940063912	5.940063758
N	δ	Padé ($[N/2, N/2]$)	Taylor (N)
4	0.20	7.250946252	7.290785243
8	0.20	7.430618958	7.338717672
12	0.20	7.476836436	7.147414978
16	0.20	7.459583127	6.463497406
20	0.20	7.462505263	2.837427216
24	0.20	7.462303523	Diverges

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