BIT
1998, Vol. 38, No. 3, pp. 597–611

0006-3835/98/3803-0597 $12.00

A NUMERICAL METHOD FOR THE BENJAMIN–ONO EQUATION *

V. THOMÉE[1] and A. S. VASUDEVA MURTHY[2]

[1] *Department of Mathematics, Chalmers University of Technology*
S-412 96 Göteborg, Sweden. email: thomee@math.chalmers.se
[1] *TIFR Centre, Indian Institute of Science, Bangalore 560 012, India*
email: vasu@math.tifrbng.res.in

Abstract.

This paper is concerned with the numerical solution of the Cauchy problem for the Benjamin–Ono equation $u_t + uu_x - Hu_{xx} = 0$, where H denotes the Hilbert transform. Our numerical method first approximates this Cauchy problem by an initial-value problem for a corresponding $2L$-periodic problem in the spatial variable, with L large. This periodic problem is then solved using the Crank–Nicolson approximation in time and finite difference approximations in space, treating the nonlinear term in a standard conservative fashion, and the Hilbert transform by a quadrature formula which may be computed efficiently using the Fast Fourier Transform.

AMS subject classification: 45K05, 65M10.

Key words: Benjamin–Ono equation, periodic, finite differences, quadrature.

1 Introduction.

This paper is concerned with the numerical solution of the Cauchy problem for the Benjamin–Ono equation

$$u_t + uu_x - Hu_{xx} = 0, \quad \text{for } x \in R, \ t > 0, \tag{1.1}$$

where H denotes the Hilbert transform. This integro-differential equation arises, e.g., in the study of long internal gravitation waves in deep stratified fluids, see Benjamin [3] and Ono [9], and models the propagation of nonlinear dispersive waves in a similar way as in the Korteweg–deVries equation. For mathematical analysis we refer to Abdelouhab et al. [1], Case [5], and Iório [6].

Because of the nonlocal character of the equation, the numerical method we propose replaces the pure initial-value problem for (1.1) by the periodic Cauchy problem with a large spatial period L. This may be justified by the decay of the solutions of the unrestricted problem as $|x| \to \infty$, but as this decay is only polynomial, in contrast to the exponential decay for the KdV equation, L has to be taken quite large in order to have a good approximation to the unrestricted problem. Our numerical method uses the Crank–Nicolson approximation in

*Received October 1997. Communicated by Åke Björck.

time and finite difference approximations in the spatial variable, treating the nonlinear term in a standard conservative fashion, and approximating the Hilbert transform by a quadrature formula, which may be computed efficiently by the Fast Fourier Transform (FFT). We show second order error estimates in both space and time for smooth solutions and also that the first two invariants of (1.1) are conserved by our numerical method.

Earlier published work on numerical methods for (1.1) include James and Weideman [7] and Miloh et al. [8]. In both these papers Fourier methods are used, and in [7] also a method based on rational approximating functions. Good computational results are reported but no error analyses are given. In recent work by Pelloni and Dougalis [10], L_2-norm error bounds have been shown for a spectral approximation in the spatial variables, and numerical computations have been carried out for associated explicit discretizations in time.

2 The Cauchy problem for the Benjamin–Ono equation.

In this section we collect some known background material for the Cauchy problem for the Benjamin–Ono equation,

$$u_t + uu_x - Hu_{xx} = 0, \quad \text{for } x \in R,\ t > 0, \quad \text{with } u(\cdot, 0) = u_0, \quad \text{in } R. \tag{2.1}$$

Here H is the Hilbert transform defined by the principal value integral

$$Hu(x) = \mathrm{PV}\frac{1}{\pi}\int_R \frac{u(x-y)}{y}\,dy.$$

We recall (see, e.g., Abdelouhab et al. [1]) that this equation has an infinite sequence of invariants, the first two of which are

$$\Phi_1(u) = \int_R u\,dx, \quad \Phi_2(u) = \tfrac{1}{2}\int_R u^2\,dx, \tag{2.2}$$

In the first case it follows at once formally by integration of (2.1) over R and integration by parts that $(d/dt)\Phi_1(u) = 0$, so that $\Phi_1(u)$ is constant in time. For $\Phi_2(u)$ we multiply (2.1) by u and integrate over R to obtain

$$\tfrac{1}{2}\frac{d}{dt}\|u\|^2 + (uu_x, u) - (Hu_{xx}, u) = 0, \quad \text{for } t > 0,$$

where

$$(u, v) = \int_{-\infty}^{\infty} u(x)\overline{v(x)}\,dx, \quad \text{and } \|u\| = (u, u)^{1/2}.$$

Here the second and third terms vanish because

$$(uu_x, u) = (u_x, u^2) = -(u, (u^2)_x) = -2(uu_x, u), \tag{2.3}$$

and, noting that since H is a convolution with an odd function it is skew-symmetric and commutes with differentiation,

$$(Hu_{xx}, u) = -(u_{xx}, Hu) = -(u, (Hu)_{xx}) = -(Hu_{xx}, u). \tag{2.4}$$

Thus $(d/dt)\Phi_2(u) = 0$, so that $\Phi_2(u) = $ constant.

Equation (2.1) has soliton solutions, such as, for c arbitrary,

$$u(x,t) = \frac{4c}{1+c^2(x-ct)^2}. \tag{2.5}$$

In fact, since $H((1+x^2)^{-1}) = x(1+x^2)^{-1}$, one finds easily for u defined in (2.5) that $(Hu)(x,t) = 4c^2(x-ct)/(1+c^2(x-ct)^2)$, from which (2.1) follows by differentiation. Another family of soliton solutions is given by

$$u(x,t) = \frac{4c_1c_2\big(c_1\lambda_1^2 + c_2\lambda_2^2 + (c_1+c_2)^3c_1^{-1}c_2^{-1}(c_1-c_2)^{-2}\big)}{\big(c_1c_2\lambda_1\lambda_2 - (c_1+c_2)^2(c_1-c_2)^{-2}\big)^2 + (c_1\lambda_1 + c_2\lambda_2)^2}, \tag{2.6}$$

where $\lambda_j = \lambda_j(x,t) = x - c_jt - d_j$, $j = 1,2$, and c_1, c_2, d_1, d_2 are arbitrary constants.

We have the following existence and uniqueness result from [1]. Here we denote by $\mathcal{H}^s = \mathcal{H}^s(R)$ the Sobolev space defined by

$$\|u\|_{\mathcal{H}^s} = \Big(\int_R (1+\xi^2)^{s/2}|\hat{u}(\xi)|^2 d\xi\Big)^{1/2}.$$

THEOREM 2.1. *Let $s \geq 3/2$ and assume $u_0 \in \mathcal{H}^s$. Then there exists a unique solution u of* (2.1) *such that $u \in C^k(R_+;\mathcal{H}^{s-2k})$ for integer $k \leq (s+1)/2$.*

We are also interested in the behavior of the solution for large $|x|$. For this we introduce the weighted Sobolev spaces $\mathcal{F}^s$ defined by the norms

$$\|u\|_{\mathcal{F}^s}^2 = \|u\|_{\mathcal{H}^s}^2 + \|(1+x^2)^{s/2}u\|_{L_2}^2.$$

We quote the following result from Iório [5]:

THEOREM 2.2. *Let $u_0 \in \mathcal{F}^2$. Then there is a unique solution u of* (2.1) *such that $u \in C(R_+;\mathcal{F}^2)$ and $u_t \in C(R_+;L_2)$.*

Let $u_0 \in \mathcal{F}^3$ with $\Phi_1(u_0) = 0$. Then there is a unique solution u of (2.1) *such that $u \in C(R_+;\mathcal{F}^3)$ and $\Phi_1(u(t)) = 0$. The condition $\Phi_1(u_0) = 0$ is necessary for the existence of a solution in $C(R_+;\mathcal{F}^3)$.*

If $u \in C([0,T];\mathcal{F}^4)$ for some $T > 0$, then $u(t) \equiv 0$ for $t \in [0,T]$.

For smooth initial functions u_0 which decay rapidly we thus have essentially $u(x,t) = O(|x|^{-2})$ as $|x| \to \infty$, but even if u_0 satisfies $\Phi_1(u_0) = 0$, faster decay than $u(x,t) = O(|x|^{-3})$ in the above sense is not possible.

For the Fourier transform of the Hilbert transform we have (see [4])

$$\widehat{Hu}(\xi) = -i\,\mathrm{sign}(\xi)\hat{u}(\xi), \quad \text{where } \hat{u}(\xi) = \frac{1}{\sqrt{2\pi}}\int_R e^{-ix\xi}u(x)\,dx, \tag{2.7}$$

with $\mathrm{sign}(\xi) = \xi/|\xi|$ for $\xi \neq 0$, $\mathrm{sign}(0) = 0$. The limit in the decay of the solution is related to the lack of regularity of $\widehat{Hu}$; note that the condition $\Phi_1(u_0) = \int_R u_0\,dx = 0$ means that $\hat{u}_0(0) = 0$.

For the purpose of indicating an argument in the subsequent analysis of our numerical method, we now show that the solutions of (2.1) are stable under perturbations of the initial data.

THEOREM 2.3. *Let u be a smooth solution of* (2.1) *with u_x uniformly bounded. Then, if v is any other smooth solution of* (2.1), *we have*

(2.8) $\|v(t) - u(t)\| \le e^{Mt}\|v(0) - u(0)\|,$ *for* $t \ge 0$, *where* $M = \frac{1}{4}\|u_x\|_{L_\infty}$.

PROOF. We have at once by subtraction

$$(v - u)_t + vv_x - uu_x - H(v - u)_{xx} = 0.$$

Setting $e = v - u$ and multiplying by e, we obtain

$$(e_t, e) + (vv_x - uu_x, e) - (He_{xx}, e) = 0. \tag{2.9}$$

Here the last term vanishes as in (2.4). For the second term we have

$$(vv_x - uu_x, e) = ((u+e)(u_x+e_x) - uu_x, e) = (u_x e, e) + (ue_x, e) + (ee_x, e). \tag{2.10}$$

The last term vanishes as in (2.3) and, using integration by parts,

$$|(u_x e, e) + (ue_x, e)| = |\tfrac{1}{2}(u_x e, e)| \le 2M\|e\|^2.$$

Thus (2.9) implies $(d/dt)\|e\|^2 \le 2M\|e\|^2$ which shows (2.8). □

We remark that since $\int_R u_x\, ds = 0$, cancellations are possible in $(u_x e, e)$, so that our choice of M could be pessimistic. For instance, if it is known that both $u(x,t)$ and $v(x,t)$ are symmetric in x around some point $\bar{x}(t)$, then so is $e(x,t)$, and $u_x(x,t)$ is antisymmetric around $\bar{x}(t)$, so that $(u_x e, e) = 0$. Thus in this case $\|e(t)\|$ is constant. We also note that by the invariance of the energies, $\|e(t)\| \le \|u(0)\| + \|v(0)\|$, and thus $\|e(t)\|$ is uniformly bounded for $t \ge 0$.

3 The periodic problem.

In order to be able to define a finite-dimensional approximation, we shall consider instead of (2.1) the corresponding $2L$-periodic problem and consider thus functions $u(x,t)$ with $u(x + 2L, t) = u(x,t)$ and such that

$$u_t + uu_x - \widetilde{H}u_{xx} = 0, \quad \text{for } x \in R,\ t > 0, \quad \text{with } u(\cdot, 0) = u_0, \quad \text{in } R, \tag{3.1}$$

where u_0 is $2L$-periodic. Here $\widetilde{H}$ is the periodic Hilbert transform (cf. [4])

$$\widetilde{H}u(x) = \mathrm{PV}\frac{1}{2L}\int_{-L}^{L} \cot\Big(\frac{\pi}{2L}y\Big)u(x-y)\,dy.$$

To see formally that this is what Hu reduces to for u periodic, we use the periodicity of u and transform $((2k-1)L, (2k+1)L)$ into $(-L, L)$ to obtain

$$\int_{\varepsilon \le |y| \le (2N+1)L} \frac{u(x-y)}{y}\,dy$$
$$= \int_{\varepsilon \le |y| \le L} \frac{u(x-y)}{y}\,dy + \sum_{k=-N, k\ne 0}^{N} \int_{-L}^{L} \Big(\frac{1}{y - 2kL} + \frac{1}{2kL}\Big)u(x-y)\,dy,$$

where the second term in the parentheses may be added since the positive and negative ones cancel. By the Mittag-Leffler representation of $\cot y$ we have

$$\frac{\pi}{2L}\cot\Big(\frac{\pi}{2L}y\Big)=\frac{1}{y}+\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\Big(\frac{1}{y-2kL}+\frac{1}{2kL}\Big),\quad \text{for } |y|\le L,$$

and taking limits, $\varepsilon\to 0$ and $N\to\infty$, we obtain the representation claimed.

Also in the periodic case one shows as earlier that

$$\Phi_1(u)=\int_{-L}^{L}u\,dx\quad\text{and}\quad \Phi_2(u)=\tfrac{1}{2}\int_{-L}^{L}u^2\,dx$$

are conserved for any solution of (3.1).

The main existence result for (2.1) carries over to the periodic case; with $\mathcal{H}^s$ now denoting $\mathcal{H}^s(R/[-L,L])$ we have the following (cf. [1]).

THEOREM 3.1. *Let $s\ge 3/2$ and assume $u_0\in\mathcal{H}^s$. Then there exists a unique solution u of* (3.1) *such that $u\in C^k(R_+;\mathcal{H}^{s-2k})$ for integer $k\le(s+1)/2$.*

We also note that the formula (2.7) for the Fourier transform of Hu has an analogue for the Fourier coefficients of $2L$-periodic functions $\widetilde{H}u$, namely (cf. [4])

$$\widehat{\widetilde{H}u}_n=-i\,\mathrm{sign}\,(n)\,\hat{u}_n,\quad\text{where } \hat{u}_n=\frac{1}{2L}\int_{-L}^{L}e^{-inx\pi/L}u(x)\,dx. \tag{3.2}$$

The periodic equation (3.1) also has soliton solutions, e.g., for c arbitrary,

$$u_L(x,t)=\frac{2c\delta^2}{1-\sqrt{1-\delta^2}\cos(c\delta(x-ct))},\quad\text{with } \delta=\frac{\pi}{cL}. \tag{3.3}$$

This follow easily as earlier from the fact that

$$\widetilde{H}\Big(\frac{1}{1-\mu\cos(c\delta x)}\Big)=\frac{\mu}{\sqrt{1-\mu^2}}\frac{\sin(c\delta x)}{1-\mu\cos(c\delta x)}.$$

We remark that it is easy to see that for $L\to\infty$, i.e., $\delta\to 0$, this $2L$-periodic soliton solution u_L tends to the one defined in (2.5).

4 The numerical method.

In order to define our numerical method for the periodic problem (3.1), we need some preliminaries. We introduce a spatial mesh with mesh-width $h=L/N$, so that the periodicity interval $[-L,L]$ is divided into $2N$ mesh-intervals by the nodal points $x_j=jh$, $j\in Z$. We consider the set $\mathcal{S}_h$ of discrete $2N$-periodic functions $V = \{V_j\}_{j=-\infty}^{\infty}$ defined on this mesh, thus satisfying $V_{j+2N} = V_j$ for all j, and introduce for this section the discrete inner product and norm

$$(V, W) = \frac{1}{2N} \sum_{j=0}^{2N-1} V_j \overline{W}_j = \frac{h}{2L} \sum_{j=0}^{2N-1} V_j \overline{W}_j, \quad \text{and} \quad \|V\| = \Big(\frac{h}{2L} \sum_{j=0}^{2N-1} |V_j|^2\Big)^{1/2}.$$

For the purpose of discretizing uu_x and $\widetilde{H} u_{xx}$ on the mesh, we set

$$\partial U_j = \frac{U_{j+1} - U_j}{h}, \ \bar{\partial} U_j = \frac{U_j - U_{j-1}}{h}, \quad \text{and} \quad \hat{\partial} U_j = \frac{U_{j+1} - U_{j-1}}{2h}.$$

Note that by summation by parts $(\partial U, V) = -(U, \bar{\partial} V)$ and $(\hat{\partial} U, V) = -(U, \hat{\partial} V)$. For our approximation of uu_x we choose, following Zabusky and Kruskal [12] (cf. also Richtmyer and Morton [11, Section 6.3]),

$$(4.1) \quad F_h(U) = \tfrac{1}{3}\hat{\partial}(U^2) + \tfrac{1}{3} U \hat{\partial} U = Q_h U \, \hat{\partial} U, \quad (Q_h U)_j = \tfrac{1}{3}(U_{j-1} + U_j + U_{j+1});$$

clearly, for u smooth, $F_h(u) - uu_x = O(h^2)$ as $h \to 0$. To approximate u_{xx} we use

$$\Delta_h U_j = h^{-1}(\partial - \bar{\partial}) U_j = h^{-2}(U_{j+1} - 2U_j + U_{j-1}),$$

which is also second order accurate.

To define a quadrature formula for the $2L$-periodic Hilbert transform $\widetilde{H}$ we assume N even, $N = 2M$, and divide the periodicity interval $(-L, L)$ into the $N = 2M$ intervals (x_{2k}, x_{2k+2}), $k = -M, \dots, M-1$ of length $2h$. We then apply the midpoint rule on each of these intervals, and set

$$(4.2) \qquad (\widetilde{H}_h V)_j = \frac{1}{2L} \sum_{k=-M}^{M-1} 2h \cot\Big(\frac{\pi}{2L} x_{2k+1}\Big) V_{j-2k-1}.$$

This may be thought of as a discrete convolution, writing

$$(4.3) \quad (\widetilde{H}_h V)_j = \sum_{k=-N}^{N-1} c_k V_{j-k}, \quad \text{where } c_k = \begin{cases} \frac{h}{L} \cot\big(\frac{\pi k h}{2L}\big), & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}. \end{cases}$$

Since c_j is odd, i.e., $c_{-j} = -c_j$, $\widetilde{H}_h$ is skew-symmetric. Note also that $\widetilde{H}_h$ commutes with translations, so that, in particular, $\widetilde{H}_h \Delta_h U = \Delta_h \widetilde{H}_h U$.

We wish to express the operator $\widetilde{H}_h$ in terms of a discrete Fourier transform. For $W = \{W_j\}_{j \in Z} \in \mathcal{S}_h$ we define the discrete Fourier transform by

$$(4.4) \qquad \widehat{W}_j = \sum_{k=-N}^{N-1} W_k e^{-2\pi i jk/N}.$$

which also belongs to $\mathcal{S}_h$. The inverse Fourier transform is then

$$(4.5) \qquad W_k = \frac{1}{2N} \sum_{j=-N}^{N-1} \widehat{W}_j e^{2\pi i jk/N}.$$

We note Parseval's formula

$$\|\widehat{W}\| = \sqrt{2N}\|W\|. \tag{4.6}$$

The following lemma shows that our discrete Hilbert transform defined in (4.2) has inherited the property (3.2) of the Fourier transform; this is, in fact, the reason for our choice of quadrature formula.

LEMMA 4.1. *We have for the operator* $\widetilde{H}_h$ *defined in* (4.2) *and* $\widetilde{sign} \in S_h$

$$\widehat{\widetilde{H}_h V}_j = -i\,\widetilde{\operatorname{sign}}(j)\widehat{V}_j, \quad \textit{where} \quad \widetilde{\operatorname{sign}}(j) = \begin{cases} 1, & \textit{if } 1 \le k \le N-1, \\ -1, & \textit{if } -N+1 \le k \le -1, \\ 0, & \textit{if } j = -N, 0. \end{cases}$$

PROOF. Since with c_j defined in (4.2),

$$\widehat{\widetilde{H}_h V}_j = \sum_{k=-N}^{N-1} \sum_{l=-N}^{N-1} c_l V_{k-l} e^{-\pi ijk/N} = \hat{c}_j \widehat{V}_j,$$

we need to show $\hat{c}_j = -i\,\widetilde{\operatorname{sign}}(j)$. This is equivalent to showing that if $\widehat{W}_j = -i\,\widetilde{\operatorname{sign}}(j)$, then $W_k = c_k$. But by (4.6)

$$\begin{aligned} W_k &= -\frac{i}{2N} \sum_{j=1}^{N-1} \left(e^{\pi ijk/N} - e^{-\pi ijk/N}\right) = \frac{1}{N} \sum_{j=1}^{N-1} \sin(\pi jk/N) \\ &= \begin{cases} \frac{1}{N} \cot\left(\pi k/N\right), & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}, \end{cases} \end{aligned}$$

which shows the lemma since $h = L/N$. □

Together with (4.6) the lemma immediately shows

$$\|\widetilde{H}_h V\| \le \|V\|. \tag{4.7}$$

Since obviously

$$\widehat{\Delta_h V}_j = \sum_{k=-N}^{N-1} (\Delta_h V)_k e^{-\pi ijk/N} = 2h^{-2}(\cos(\pi j/N) - 1)\widehat{V}_j,$$

we have thus $\widehat{\widetilde{H}_h \Delta_h V}_j = \mu_j \widehat{V}_j$ where $\mu_j = -2i\,\widetilde{\operatorname{sign}}(j)h^{-2}(\cos(\pi j/N) - 1)$. In our numerical work the evaluation of $\widetilde{H}_h \Delta_h V$, for V given, is therefore done by taking the Fourier transform of V, multiplying by μ_j, and then taking the inverse Fourier transform; in practise these operations are done using the FFT.

Although the midpoint rule is of second order accuracy when the integrand is smooth, it is now applied in (4.2) to a function with a singularity at the origin. In spite of this, $\widetilde{H}_h u$ is a second order approximation to $\widetilde{H} u$ when u is smooth.

LEMMA 4.2. *For u $2L$-periodic and smooth we have*

$$\|\widetilde{H}_h u - \widetilde{H} u\|_\infty \le Ch^2 \|u\|_{C^3}, \quad \textit{where } \|U\|_\infty = \max_j |U_j|.$$

PROOF. We may think of $(\widetilde{H}_h u)_j$ as the result of applying the midpoint rule to

$$\widetilde{H}u(x_j) = \frac{1}{2L}\int_0^L \cot\Big(\frac{\pi}{2L}y\Big)(u(x_j-y)-u(x_j+y))\,dy = \int_0^L \delta(x_j;y)\,dy,$$

where, with $\psi(y) = y\cot(\pi y)$ and $\omega(x;y) = (u(x-y)-u(x+y))/y$,

$$\delta(x;y) = \frac{1}{2L}\cot\Big(\frac{\pi}{2L}y\Big)(u(x-y)-u(x+y)) = \psi(\frac{y}{2L})\,\omega(x;y).$$

By the standard error estimate for the midpoint rule we therefore obtain, since ψ is smooth ($D_y = \partial/\partial y$),

$$|(\widetilde{H}_h u)_j - \widetilde{H}u(x_j)| \le Ch^2 \max_{x,y}|D_y^2\delta(x;y)| \le Ch^2\max_x \|\omega(x;\cdot)\|_{C^2} \le Ch^2\|u\|_{C^3},$$

which completes the proof. □

Let $k>0$ be a time step and $t_n = nk$. With U^n defined for $n \ge 0$ we set

$$\bar\partial_t U^n = (U^n - U^{n-1})/k \quad \text{and} \quad \bar U^{n-1/2} = \tfrac{1}{2}(U^n + U^{n-1}), \quad \text{for } n \ge 1.$$

Then the Crank–Nicolson scheme for our problem is

$$\begin{aligned}(4.8)\qquad \bar\partial_t U^n + F_h(\bar U^{n-1/2}) - \widetilde{H}_h\Delta_h \bar U^{n-1/2} &= 0, \quad \text{for } n\ge 1,\\ U_j^0 &= u_0(x_j), \quad \text{for } j \in Z.\end{aligned}$$

For U^{n-1} given this is a nonlinear equation for U^n. We shall return later to discuss the existence, uniqueness and computation of this solution, but start by showing that the discrete analogous of the invariants defined in (2.2) are conserved.

THEOREM 4.3. *The functionals $\Phi_{1,h}(U) = (U,1)$ and $\Phi_{2,h}(U) = \frac{1}{2}\|U\|^2$ are conserved for solutions of* (4.8).

PROOF. By summation by parts we have $(\widetilde{H}_h\Delta_h U, 1) = (\Delta_h \widetilde{H}_h U, 1) = 0$ and similarly $(F_h(U),1) = \frac{1}{3}(\hat\partial U, U) + \frac{1}{3}(\hat\partial(U^2),1) = 0$. Hence $\bar\partial_t \Phi_{1,h}(U^n) = 0$. Further, multiplying (4.8) by $\bar U^{n-1/2}$ we obtain

$$(4.9)\quad (\bar\partial_t U^n, \bar U^{n-1/2}) + (F_h(\bar U^{n-1/2}), \bar U^{n-1/2}) - (\widetilde{H}_h\Delta_h \bar U^{n-1/2}, \bar U^{n-1/2}) = 0.$$

For any U we have

$$(4.10)\quad (F_h(U),U) = \tfrac{1}{3}(\hat\partial U^2, U) + \tfrac{1}{3}(U\hat\partial U, U) = -\tfrac{1}{3}(U^2,\hat\partial U) + \tfrac{1}{3}(U\hat\partial U, U) = 0.$$

Further

$$(4.11)\qquad (\widetilde{H}_h\Delta_h U, U) = -(\Delta_h U, \widetilde{H}_h U) = -(U, \Delta_h\widetilde{H}_h U) = -(\widetilde{H}_h\Delta_h U, U).$$

Using (4.10) and (4.11) in (4.9) shows $\bar\partial_t\Phi_{2,h}(U^n) = 0$, completing the proof. □

LEMMA 4.4. *For U^{n-1} given there exists a solution U^n of* (4.8).

PROOF. In terms of $X = \bar{U}^n$ the equation may be written

$$\Psi(X) = 2(X - U^{n-1}) - k\widetilde{H}_h\Delta_h X + kF_h(X) = 0.$$

Using (4.10) and (4.11) we find

$$(\Psi(X), X) = 2(X - U^{n-1}, X) = 2\|X\|^2 - 2(U^{n-1}, X) \geq \|X\|^2 - \|U^{n-1}\|^2,$$

and hence $(\Psi(X), X) > 0$ for $\|X\| = q := (\|U^{n-1}\|^2 + 1)^{1/2}$, say. The equation $\Psi(X) = 0$ therefore has a solution $X \in B_q = \{Y; \|Y\| \leq q\}$. In fact, if we assume that $\Psi(Y) \neq 0$ for $Y \in B_q$, then the mapping $\Lambda(Y) = -q\Psi(Y)/\|\Psi(Y)\| : B_q \to B_q$, is continuous, and hence it has a fixed point $X \in B_q$ by Brouwer's fixed point theorem. For this fixed point we have $q^2 = \|\Lambda(X)\|^2 = (\Lambda(X), X) = -q(\Psi(X), X)/\|\Psi(X)\|$, which contradicts $(\Psi(X), X) > 0$. □

We postpone the discussion of uniqueness and turn to the error estimate. For this we shall need the following (cf. (2.10)).

LEMMA 4.5. *If u_x is uniformly bounded we have*

$$|(F_h(U) - F_h(u), U - u)| \leq M\|U - u\|^2, \quad \textit{where } M = \tfrac{1}{2}\|u_x\|_{L_\infty}.$$

PROOF. Set $e = U - u$. We have $U^2 - u^2 = (2u + e)e = 2ue + e^2$, and hence

$$(\hat{\partial}(U^2 - u^2), e) = 2(\hat{\partial}(ue), e) + (\hat{\partial}(e^2), e).$$

Similarly

$$(U\hat{\partial}U - u\hat{\partial}u, e) = (e\hat{\partial}u, e) + (u\hat{\partial}e, e) + (e\hat{\partial}e, e).$$

Thus

$$(F_h(U) - F_h(u), e) = -\tfrac{1}{3}(u\hat{\partial}e, e) + \tfrac{1}{3}(e\hat{\partial}u, e) + (F_h(e), e).$$

Setting $(\tau e)_j = e_{j+1}$ we have $e\,\hat{\partial}e = \frac{1}{2}\bar{\partial}(e\,\tau e)$ and hence

$$-(u\hat{\partial}e, e) = -\tfrac{1}{2}(u, \bar{\partial}(e\,\tau e)) = \tfrac{1}{2}(\bar{\partial}u, e\,\tau e) \leq M\|e\|^2.$$

Since $(e\hat{\partial}u, e) \leq 2M\|e\|^2$ and $(F_h(e), e) = 0$ by (4.10), the result follows. □

We now show the following error estimate:

THEOREM 4.6. *Let u be a smooth solution of* (3.1) *and U^n be a solution of* (4.8). *Then*

$$\|U^n - u^n\| \leq C_T(u)(h^2 + k^2), \quad \textit{for } t_n \leq T.$$

PROOF. Set $e^n = U^n - u^n$ where $u^n = u(t_n)$. Then, since u satisfies (3.1),

$$\bar{\partial}_t e^n - \widetilde{H}_h\Delta_h\bar{e}^{n-1/2} = -F_h(\bar{U}^{n-1/2}) + F_h(\bar{u}^{n-1/2}) - G^n, \tag{4.12}$$

where

$$G^n = \bar{\partial}_t u^n + F_h(\bar{u}^{n-1/2}) - \widetilde{H}_h \Delta_h \bar{u}^{n-1/2} - (u_t + uu_x - \widetilde{H} u_{xx})^{n-1/2},$$

with $u^{n-1/2} = u\big(t_n - \frac{1}{2}k\big)$. Here, since both approximations are of second order,

$$\|\bar{\partial}_t u^n - u_t^{n-1/2}\| + \|F_h(\bar{u}^{n-1/2}) - (uu_x)^{n-1/2}\| \le C(u)(h^2 + k^2),$$

and, using also (4.7),

$$\begin{aligned}\|\widetilde{H}_h \Delta_h \bar{u}^{n-1/2} - \widetilde{H} u_{xx}^{n-1/2}\| &\le \|\widetilde{H}_h(\Delta_h \bar{u}^{n-1/2} - \bar{u}_{xx}^{n-1/2})\| \\ &+ \|\widetilde{H}_h(\bar{u}_{xx}^{n-1/2} - u_{xx}^{n-1/2})\| + \|(\widetilde{H}_h - \widetilde{H})u_{xx}\| \le C(u)(h^2 + k^2),\end{aligned}$$

so that $\|G^n\| \le C(u)(h^2 + k^2)$.

Multiplication of (4.12) by $\bar{e}^{n-1/2}$ gives

$$\begin{aligned}&(\bar{\partial}_t e^n, \bar{e}^{n-1/2}) - (\widetilde{H}_h \Delta_h \bar{e}^{n-1/2}, \bar{e}^{n-1/2}) \\ &\quad = (F_h(\bar{u}^{n-1/2}) - F_h(\bar{U}^{n-1/2}), \bar{e}^{n-1/2}) - (G^n, \bar{e}^{n-1/2}).\end{aligned}$$

By (4.11) and Lemma 4.5 we thus obtain

$$\begin{aligned}\|e^n\|^2 - \|e^{n-1}\|^2 &\le Ck\|\bar{e}^{n-1/2}\|^2 + Ck\|G^n\|\,\|\bar{e}^{n-1/2}\| \\ &\le Ck(\|e^n\|^2 + \|e^{n-1}\|^2) + Ck\|G^n\|^2,\end{aligned}$$

and hence, for small k,

$$\|e^n\|^2 \le (1 + Ck)\|e^{n-1}\|^2 + Ck(h^2 + k^2)^2.$$

By repeated application we have, since $e^0 = 0$,

$$\|e^n\|^2 \le e^{CT}\|e^0\|^2 + Cnk(h^2 + k^2)^2 \le CT(h^2 + k^2)^2, \quad \text{for } t_n \le T,$$

which completes the proof. □

With U^{n-1} given, the nonlinear equation to be solved in (4.8) at time level n may be put in the form

$$\text{(4.13)} \quad W - \tfrac{1}{2}k\widetilde{H}_h \Delta_h W = g - k\bar{F}_h(W), \quad \text{where } \bar{F}_h(W) = F_h(\tfrac{1}{2}(W + U^{n-1})),$$

with $g = U^{n-1} + \frac{1}{2}k\widetilde{H}_h \Delta_h U^{n-1}$. For the solution of this equation we consider the iterative scheme

$$\text{(4.14)} \qquad (I - \tfrac{1}{2}k\widetilde{H}_h \Delta_h)W^{j+1} = g - k\bar{F}_h(W^j), \quad \text{for } j \ge 0,\ W^0 = U^{n-1}.$$

Since $\widetilde{H}_h \Delta_h$ is skew-symmetric, the matrix on the left is nonsingular, so that this linear problem has a unique solution for given right hand side. The following lemma shows the linear convergence of (4.14) for appropriately small k.

LEMMA 4.7. *For $R > 0$ given and $\gamma < 1$, let $kh^{-1} \leq \gamma R^{-1}$ and assume that the W^j defined in* (4.14) *are such that $\|W^j\|_\infty \leq R$. Then*

$$\|W^{j+1} - W^j\| \leq \gamma\|W^j - W^{j-1}\|, \quad \text{for } j \geq 1.$$

In particular, W^j converges to a solution of (4.13).

PROOF. We first note that since $F_h(U) - F_h(V) = Q_h U\hat{\partial}(U - V) + Q_h(U - V)\hat{\partial}V$,

$$\|F_h(U) - F_h(V)\| \leq 2Rh^{-1}\|U - V\|, \quad \text{if } \|U\|_\infty, \|V\|_\infty \leq R. \tag{4.15}$$

We have for $j \geq 1$

$$W^{j+1} - W^j - \tfrac{1}{2}k\widetilde{H}_h\Delta_h(W^{j+1} - W^j) = -k(\bar{F}_h(W^j) - \bar{F}_h(W^{j-1})).$$

Taking inner products with $W^{j+1} - W^j$ and using the skew-symmetry of $\widetilde{H}_h\Delta_h$, we get by (4.15), since $\|\frac{1}{2}(W^j + U^{n-1})\|_\infty \leq R$,

$$\begin{aligned} (4.15)\ \|W^{j+1} - W^j\| &\leq k\|\bar{F}_h(W^j) - \bar{F}_h(W^{j-1})\| \\ &= k\big\|F_h(\tfrac{1}{2}(W^j + U^{n-1})) - F_h(\tfrac{1}{2}(W^{j-1} + U^{n-1}))\big\| \\ &\leq Rkh^{-1}\|W^j - W^{j-1}\|, \end{aligned}$$

which implies the convergence stated since $Rkh^{-1} \leq \gamma$. □

We are now finally in a position to complete the proof that, for k appropriate, our numerical method has a unique solution, which may be obtained by the iterative scheme (4.14), and which approximates the exact solution as stated in Theorem 4.6.

THEOREM 4.8. *Let u be a smooth solution of* (2.1). *Given $T > 0$ and $\gamma < 1$, there is a number $R = R_T$ such that if h and k are small and $k \leq \gamma h R^{-1}$, then there exists a unique solution of* (4.8) *such that $\|U^n\|_\infty \leq R$ for $t_n \leq T$, and the conclusion of Theorem* 4.6 *holds. Further the W^j defined in* (4.14) *satisfy $\|W^j\|_\infty \leq R$.*

PROOF. Setting $B = B_T = \sup_{t \leq T}\|u(t)\|_\infty$ we shall show the theorem with $R = B + 2$. We first note that by Theorem 4.6, as long as U^n exists, we have since $k \leq \gamma h R^{-1}$, for h small,

$$\|U^n - u^n\|_\infty \leq Ch^{-1/2}\|U^n - u^n\| \leq C_T(u)h^{-1/2}(h^2 + k^2) \leq 1,$$

so that $\|U^n\|_\infty \leq B + 1$.

Assume now that U^{n-1} exists, and consider the iterative scheme (4.14) with $g = U^{n-1} + \frac{1}{2}k\widetilde{H}_h\Delta_h U^{n-1}$. We want to demonstrate that $\|W^j\|_\infty \leq R$ for $j \geq 1$ which then shows, by (4.15), that W^j converges to a solution of (4.13). This solution is then unique. In fact, if W and W' were two solutions we would have

$$W' - W - \tfrac{1}{2}k\widetilde{H}_h\Delta_h(W' - W) = -k(\bar{F}_h(W') - \bar{F}_h(W)).$$

After multiplication by $W' - W$ this shows (cf. (4.16)), $\|W' - W\| \leq \gamma \|W' - W\|$ so that $W' = W$.

It follows from (4.16) that $\|W^j - W^0\| \leq (1-\gamma)^{-1}\|W^1 - W^0\|$. In order to bound $W^1 - W^0$, with $W^0 = U^{n-1}$, we note that, using the equations for W^1 and U^n,

$$W^1 - U^n - \tfrac{1}{2}k\widetilde{H}_h\Delta_h(W^1 - U^n) = -k\big(F_h(U^{n-1}) - \bar{F}_h(U^n)\big).$$

Hence, after multiplication by $W^1 - U^n$ (cf. (4.15)),

$$\begin{aligned}\|W^1 - U^n\| &\leq C\|U^n - U^{n-1}\| \\ &\leq C(\|U^n - u^n\| + \|U^{n-1} - u^{n-1}\| + \|u^n - u^{n-1}\|) \leq C(k + h^2).\end{aligned}$$

This yields

$$\|W^j - U^{n-1}\| \leq C(\|W^1 - U^n\| + \|U^n - U^{n-1}\|) \leq C(k + h^2).$$

We conclude that $\|W^j - U^{n-1}\|_\infty \leq Ch^{-1/2}(k + h^2) \leq Ch^{1/2} \leq 1$ for h small, so that $\|W^j\|_\infty \leq \|U^{n-1}\|_\infty + 1 \leq B + 2 = R$. The proof is now complete. □

5 Numerical illustrations.

We have applied our method to simulate the periodic single soliton solution (3.3), with $L = 15$, $c = .25$. Sample results are shown in Table 5.1, where the l^2-norm $\|e\|$ and the maximum-norm $\|e\|_\infty$ of the error are listed at $t = 10$, 100 for certain combinations of N and k, and where the second order convergence can be observed. For instance, doubling N and halving k with $N = 512$, $k = .2$ reduces the error by a factor 3.9. Figure 5.1 shows the result for $N = 1024$, $k = .25$.

We have also simulated the double soliton solution (2.6) of the unrestricted Cauchy problem (2.1), using its given initial values on an interval $(-L, L)$ as initial values for a $2L$-periodic problem; numerical results at $t = 10, 90, 180$ ($t = 90, 180$ were employed in [7]) are shown in Table 5.2 for $L = 100$, $c_1 = 0.3$, $c_2 = 0.6$, $d_1 = -30$, $d_2 = -55$. We note that in this case the error stems from both the approximation of the unrestricted initial-value problem by a periodic one, and by the numerical approximation of the latter. In this case, in contrast to the single soliton case which is just a translation, there is interaction between the two solitons. We see that the error (especially in the maximum norm) is larger than the single soliton case; note the phase error which is evident in Figure 5.2 ($N = 2048$, $k = .1$).

We note that the spectral methods used in [7, 8], and [10] need fewer parameters in the spatial discretization, but use shorter time steps.

In our calculations we have used the iterative scheme (4.14) with a tolerance of 10^{-6}. This tolerance was chosen because it was found that decreasing the tolerance further did not change the error in the first four significant digits. The number of iterations required are listed in the tables under *It*. For faster convergence we have also considered the extrapolated initial approximation $W^0 = 2U^{n-1} - U^{n-2}$, which resulted in a certain reduction in the number of

Table 5.1: Periodic single soliton (3.3), with $L = 15$, $c = .25$.

t	N/k	$\|\|e\|\|$	$\|\|e\|\|_\infty$	It
10	256/0.5	0.713e−4	0.171e−3	5.0 (4.0)
	256/0.25	0.629e−4	0.155e−3	4.0 (3.0)
	512/0.5	0.279e−4	0.608e−4	5.0 (4.0)
	512/0.25	0.181e−4	0.436e−4	4.0 (3.0)
	1024/0.5	0.181e−4	0.358e−4	5.0 (4.0)
	1024/0.25	0.711e−5	0.157e−4	4.0 (3.0)
100	256/0.5	0.480e−3	0.909e−3	5.0 (4.0)
	256/0.25	0.375e−3	0.730e−3	4.0 (3.0)
	512/0.5	0.227e−3	0.409e−3	5.0 (4.0)
	512/0.25	0.121e−3	0.229e−3	4.0 (3.0)
	1024/0.5	0.164e−3	0.286e−3	5.0 (4.0)
	1024/0.25	0.574e−4	0.104e−3	4.0 (3.0)

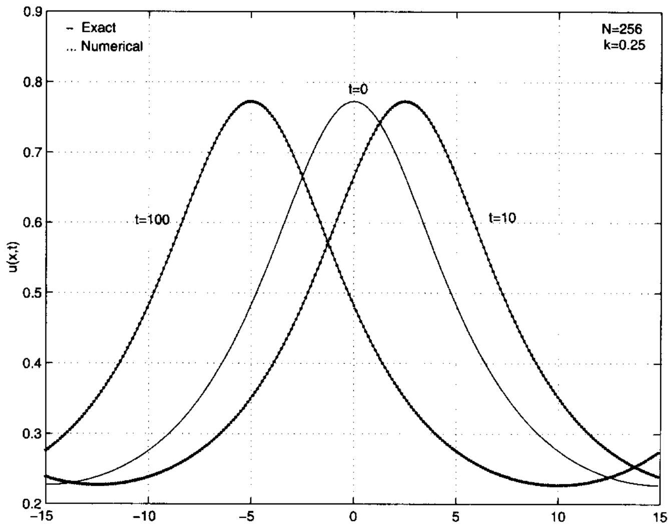

Figure 5.1: Simulation of periodic single soliton, with $N = 1024$, $k = .25$.

Table 5.2: Double soliton (2.6) with $L = 100$, $c_1 = 0.3$, $c_2 = 0.6$, $d_1 = -30$, $d_2 = -55$.

t	N/k	$\|\|e\|\|$	$\|\|e\|\|_\infty$	It
10	512/0.2	0.194e−1	0.158	9.0 (7.0)
	512/0.1	0.190e−1	0.155	6.0 (5.0)
	1024/0.2	0.574e−2	0.470e−1	9.6 (8.0)
	1024/0.1	0.528e−2	0.435e−1	7.0 (5.0)
	2048/0.2	0.203e−2	0.158e−1	10.0 (8.0)
	2048/0.1	0.156e−2	0.121e−1	7.0 (5.0)
90	512/0.2	0.624e−1	0.336	7.8 (6.3)
	512/0.1	0.613e−1	0.331	5.5 (4.2)
	1024/0.2	0.182e−1	0.979e−1	8.4 (7.0)
	1024/0.1	0.168e−1	0.912e−1	5.9 (4.5)
	2048/0.2	0.580e−2	0.305e−1	6.6 (5.1)
	2048/0.1	0.480e−2	0.248e−1	4.6 (3.2)
180	512/0.2	0.195	1.399	7.9 (6.4)
	512/0.1	0.192	1.377	5.6 (4.2)
	1024/0.2	0.678e−1	0.520	8.6 (7.2)
	1024/0.1	0.617e−1	0.475	6.0 (4.6)
	2048/0.2	0.242e−1	0.188	8.9 (7.3)
	2048/0.1	0.175e−1	0.136	6.2 (4.7)

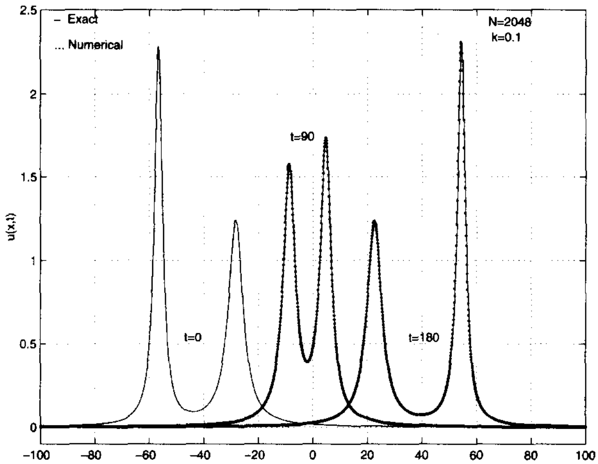

Figure 5.2: Simulation of double soliton (2.6), with $N = 2048$, $k = .1$.

iterations needed, listed within parentheses under *It*. For even faster convergence one could consider Newton's method. Since straightforward application of this method would require the solution of linear systems which are not in convolution form and therefore not immediately suitable for the FFT technique, it is then natural to use a modification proposed by Akrivis, Dougalis and Karakashian [2], in which an inner iteration with the matrix in (4.14) would be applied. We shall not pursue this here.

Acknowledgement.

The first author gratefully acknowledges useful discussions with Professor G. Akrivis on various aspects of this work.

REFERENCES

1. L. Abdelouhab, J. L. Bona, M. Felland, and J.–C. Saut, *Nonlocal models for nonlinear dispersive waves* , Physica D, 40 (1989), pp. 360–392.
2. G. Akrivis, V. A. Dougalis, and O. Karakashian, *Solving the systems of equations arising in the discretization of some nonlinear PDE's by implicit Runge–Kutta methods*, RAIRO Modél. Math. Anal. Numér , 31 (1979), pp. 251–288.
3. T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech., 29 (1967), pp. 559–592.
4. P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation, I. One-dimensional Theory*, Birkhäuser Verlag, Basel, 1971.
5. K. M. Case, *Properties of the Benjamin-Ono equation*, J. Math. Phys, 20 (1979), pp. 972–977.
6. R. J. Iório, Jr., *On the Cauchy problem for the Benjamin–Ono equation*, Comm. P.D.E., 11 (1986), pp. 1031–1081.
7. R. L. James and J. A. C. Weideman, *Pseudospectral methods for the Benjamin–Ono equation*, in Advances in Computer Methods for Partial Differential Equations VII, R. Vichnevetsky, D. Knight, and G.Richter (eds.), IMACS, Brunswick N.J., 1992, pp. 371–377.
8. T. Miloh, M. Prestin, L. Shtilman, and M. P. Tulin, *A note on the numerical and N-soliton solutions of the Benjamin–Ono evolution equation*, Wave Motion, 17 (1993), pp. 1–10.
9. H. Ono, *Algebraic solitary waves in stratified fluids*, J. Phys. Soc. Japan, 39 (1975), pp. 1082–1091.
10. B. Pelloni and V. A. Dougalis, *On spectral methods for the Benjamin–Ono equation*, in HERMIS '96, Proceedings of the Third Hellenic-European Conference on Mathematics and Informatics, E. A. Lipitakis, ed., LEA, Athens, 1996, pp. 229–237.
11. R. D. Richtmyer and K. W. Morton *Difference Methods for Initial-Value Problems*, Wiley, New York, 1967.
12. N. J. Zabusky and M. D. Kruskal, *Interaction of "solitons" in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett., 15 (1965), pp. 240–243.