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A Generalized Hecke Identity

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ABSTRACT. The classical Hecke identity gives the Fourier transform of the product of a homogeneous harmonic polynomial h times the Gaussian $e^{-\frac{1}{2}<.,.>}$. *A similar formula is valid when the Gaussian is replaced by the tempered distribution* $e^{\frac{i}{2}<.,.>}$. *It is shown that there is a similar identity when the inner product is replaced by an indefinite quadratic form* $\mathfrak{q}$ *and h is a* $\Box$*-harmonic distribution, where* $\Box$ *is the differential operator canonically associated to* $\mathfrak{q}$. *Another generalization is obtained in the context of representations of Jordan algebras, in the spirit of Herz's previous work on matrix spaces.*

1. The Hecke Identity (Positive Definite Case)

Let E, $<,>$ be a euclidean vector space of dimension N. Let $d\xi$ be the associated Lebesgue measure.

Theorem 1.
Let p be a harmonic polynomial homogeneous of degree k. Then

$$\int_E p(\xi)e^{-\frac{1}{2}<\xi,\xi>}e^{-i<\xi,\eta>}d\xi = (2\pi)^{\frac{N}{2}}(-i)^k p(\eta)e^{-\frac{1}{2}<\eta,\eta>} .$$

This is the classical Hecke identity. A variation of this identity can be easily obtained by analytic continuation. First, let a be a strictly positive real number. An obvious change of variable gives

$$\int_E p(\xi)e^{-\frac{1}{2}a<\xi,\xi>}e^{-i<\xi,\eta>}d\xi = (2\pi)^{\frac{N}{2}}a^{-\frac{N}{2}}(-i)^k a^{-k} p(\eta)e^{-\frac{1}{2}\frac{1}{a}<\eta,\eta>} .$$

Both sides make sense when a is replaced by a complex number z, $\Re z > 0$, and they define holomorphic functions in the right half-plane. As they already coincide on $]0,+\infty)$, they must coincide everywhere. Now, let z tend to $-i$. Then the expression $p(\xi)e^{-\frac{1}{2}z<\xi,\xi>}$ converges in $\mathcal{S}'(E)$ to the tempered distribution $p(\xi)e^{\frac{1}{2}i<\xi,\xi>}$. On the other hand, the term $a^{-\frac{N}{2}}(-i)^k a^{-k}$ tends to $e^{iN\frac{\pi}{4}}$, whereas $p(\eta)e^{-\frac{1}{2}\frac{1}{z}<\eta,\eta>}$ converges to the tempered distribution $p(\eta)e^{-\frac{1}{2}i<\eta,\eta>}$. With a little more care, one has the following result.

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Theorem 2.
Let p be a harmonic polynomial. The Fourier transform of the tempered distribution $p(\xi)e^{\frac{1}{2}i<\xi,\xi>}$ is given by

$$\int_E p(\xi)e^{\frac{1}{2}i<\xi,\xi>}e^{-i<\xi,\eta>}d\xi = (2\pi)^{\frac{N}{2}}e^{iN\frac{\pi}{4}}p(\eta)e^{-\frac{1}{2}i<\eta,\eta>} ,$$

where the equality is to be interpreted in the sense of Fourier transform of tempered distributions.

It is remarkable that in this version of the Hecke formula, the degree of homogeneity of the polynomial p (= the number k) disappears. In some sense, Theorem 2 seems more natural than Theorem 1. This remark suggests a possible generalization of Theorem 2. Notice first that if $\mathbf{q}$ is any non-degenerate quadratic form on E (not necessarily positive definite), then $e^{\frac{1}{2}i\mathbf{q}}$ is still a tempered distribution. Recalling moreover that in the positive-definite case, the harmonic polynomials are all the harmonic tempered distributions, this suggests there could be a version of the Hecke identity, where the inner product is replaced by the symmetric bilinear form associated to $\mathbf{q}$ and the harmonic polynomials are replaced by tempered distributions p satisfying $\Box p = 0$, where $\Box$ is the second order differential operator with constant coefficients canonically associated to $\mathbf{q}$. This is indeed the case, as we will show in the next section.

2. Hecke Formula for a General Non-Degenerate Quadratic Form

Let E be a real vector space of dimension N, and let β be a symmetric, non-degenerate bilinear form on E, of signature (p, q), so that $N = p + q$. Denote by $\mathbf{q}$ the associated quadratic form defined for $\xi \in E$ by $\mathbf{q}(\xi) = \beta(\xi, \xi)$. Let $d\xi$ be the Lebesgue measure on E, normalized so that if $\{\xi_1, \xi_2, \dots, \xi_N\}$ is an orthogonal basis of (E, β) with $\mathbf{q}(\xi_j) = \pm 1$ for any j, $1 \leq j \leq N$, then the N-tope constructed on the basis $\{\xi_1, \xi_2, \dots, \xi_N\}$ has measure 1.

The non-degenerate form β induces a canonical isomorphism between E and its dual E': to any vector $\xi \in E$ is associated the linear form $\eta \mapsto \beta(\xi, \eta)$. Denote the inverse map by $\widetilde{\beta}$. To any element ξ in E is associated a first order differential operator ∇_ξ defined by $\nabla_\xi f(\eta) = (\frac{d}{dt})_{t=0} f(\eta + t\xi)$. For φ any linear form on E, let $\partial(\varphi) = \nabla_{\widetilde{\beta}(\varphi)}$, and extend ∂ to an algebra isomorphism from the algebra $\mathcal{P}(E)$ of polynomials on E to the algebra $\mathcal{D}(E)$ of constant coefficients differential operators on E. A special case is obtained by considering the quadratic form $\mathbf{q}$, to get the "square" operator

$$\Box = \partial(\mathbf{q}) .$$

Functions f (or distributions) which satisfy $\Box f = 0$ are called $\Box$-*harmonic*.

For f any function in the Schwartz class $\mathcal{S}(E)$, define the (β-normalized) Fourier transform $\mathcal{F}_\beta\, f$ by

$$\mathcal{F}_\beta\, f(\xi) = \int e^{-i\beta(\xi,\eta)} f(\eta)d\eta .$$

By duality, one extends as usual the Fourier transform to tempered distributions.

Recall the Fourier transform of the distribution $e^{\frac{i}{2}\mathbf{q}}$:

$$\mathcal{F}_\beta \left(e^{\frac{i}{2}\mathbf{q}}\right) = (2\pi)^{\frac{N}{2}} e^{i\frac{\pi}{4}(p-q)} e^{-\frac{i}{2}\mathbf{q}} .$$

Next observe that any partial derivative of $e^{\frac{i}{2}\mathbf{q}}$ is a polynomial times the function $e^{\frac{i}{2}\mathbf{q}}$. Hence, if φ is a function in the Schwartz class $\mathcal{S}(E)$, then the function $e^{\frac{i}{2}\mathbf{q}} f$ also belongs to $\mathcal{S}(E)$, and the mapping $\varphi \mapsto \varphi e^{\frac{i}{2}\mathbf{q}}$ is a continuous map from $\mathcal{S}(E)$ into itself. So, using the pairing between $\mathcal{S}(E)$ and

$\mathcal{S}'(E)$, if T is any tempered distribution, the product $e^{\frac{i}{2}\mathfrak{q}}T$ is a well-defined tempered distribution. Transferring this result by the Fourier transform, it is clear that if $\varphi \in \mathcal{S}(E)$ (resp. $T \in \mathcal{S}'(E)$), then $\varphi \star e^{\frac{i}{2}\mathfrak{q}}$ is in $\mathcal{S}(E)$ (resp. $\mathcal{S}'(E)$).

For $t > 0$, define

$$k_t = (2\pi)^{-\frac{N}{2}} e^{-i\frac{\pi}{4}(p-q)} t^{-\frac{N}{2}} e^{i\frac{1}{2t}\mathfrak{q}}$$

As a tempered distribution, its Fourier transform is given by

$$\mathcal{F}_\beta\ (k_t) = e^{-\frac{i}{2}t\mathfrak{q}} .$$

Denote by K_t the operator of convolution with k_t. Equivalently,

$$K_t = \mathcal{F}_\beta^{-1} e^{-\frac{i}{2}t\mathfrak{q}} \mathcal{F}_\beta .$$

It is a continuous operator in $\mathcal{S}(E)$.

Lemma 1.

For $s, t > 0$, and $\varphi, \psi \in \mathcal{S}(E)$,

$$K_{s+t} = K_s \circ K_t \qquad (i)$$

$$\int_E (K_t\varphi)\ \psi = \int_E \varphi\ (K_t\psi) \qquad (ii)$$

$$\frac{d}{dt} K_t\varphi = \frac{i}{2} K_t \Box \varphi = \frac{i}{2} \Box K_t \varphi \qquad (iii)$$

as $t \to 0^+$, $K_t\,\varphi \longrightarrow \varphi$ in the topology of $\mathcal{S}(E)$.

The lemma is easily obtained by Fourier transform.

Lemma 2.

Let $\varphi \in \mathcal{S}(E)$. Then

$$\mathcal{F}_\beta \left(\varphi e^{\frac{i}{2}\mathfrak{q}}\right) = (2\pi)^{\frac{N}{2}} e^{i\frac{\pi}{4}(p-q)}\, e^{-\frac{i}{2}\mathfrak{q}}\ K_1\varphi .$$

In fact, by "completing the square,"

$$\int_E e^{\frac{i}{2}\beta(\xi,\xi)} e^{-i\beta(\xi,\eta)} \varphi(\xi) d\xi\, e^{-\frac{i}{2}\beta(\eta,\eta)} \int_E e^{\frac{i}{2}\beta(\xi-\eta,\xi-\eta)} \varphi(\xi) d\xi$$

and one recognizes the convolution (up to a constant) of φ with the kernel k_1. The lemma follows.

By the standard arguments using the pairing between $\mathcal{S}(E)$ and $\mathcal{S}'(E)$, the operators K_t can be extended to tempered distributions, and the preceding results are still valid.

Lemma 3.

Let $f \in \mathcal{S}'(E)$, such that $\Box f = 0$. Then

$$K_1 f = f .$$

Let $\varphi \in \mathcal{S}(E)$, and consider the function $\theta(t) = (K_t f, \varphi) = (f, K_t\varphi)$ defined for $t > 0$. It is a smooth function of t, and its derivative satisfies

$$\theta'(t) = \left(f, \frac{d}{dt} K_t\varphi\right) = \frac{i}{2}\,(f, \Box K_t\varphi) = \frac{i}{2}\,(\Box f, K_t\varphi) = 0 .$$

Hence, θ is a constant function on $]0, \infty)$. As $t \to 0$, $\theta(t)$ tends to (f, φ). Hence, $\theta(t) = (f, \varphi)$ for any $t > 0$. Making $t = 1$ gives the result.

Combining Lemma 2 (extended to tempered distributions) and Lemma 3 gives the next result.

Theorem 3 (Generalized Hecke Formula).
Let f be a tempered distribution satisfying $\Box f = 0$. Then

$$\mathcal{F}_\beta \left(f\, e^{\frac{i}{2}\mathbf{q}}\right) = (2\pi)^{(\frac{N}{2})} e^{i\frac{\pi}{4}(p-q)} f\, e^{-\frac{i}{2}\mathbf{q}} .$$

Remark. I wish to thank the referee who suggested the present proof. My original proof was more complicated, and less general. In [7] the classical formula of Bochner (= Fourier transform of a radial function times a spherical harmonic) is generalized to the context of indefinite quadratic forms. It does not seem obvious, however, how to deduce our Theorem 3 from this result, even for homogeneous distributions. □

3. Hecke Identity in the Context of Representations of Jordan Algebras

Most of the results (and definitions) needed in this section can be found in [1]. Let V be a simple Jordan algebra over $\mathbb{R}$, with identity element e. For any $x \in V$, denote by $L(x) : V \longrightarrow V$ the endomorphism $y \mapsto xy$ and recall that, by assumption $B(x, y) = \operatorname{tr} L(xy)$ is a non-degenerate symmetric bilinear form, for which the operators $L(x)$ and $P(x) = 2L(x)^2 - L(x^2)$ are symmetric. The norm function is denoted by det (to avoid confusion, we then use Det for the determinant of an endomorphism). The norm function is a polynomial, homogeneous of degree r equal to the rank of the Jordan algebra V.

Let E a real vector space of dimension N, with a symmetric bilinear non-degenerate form β. A *representation* of V on E is a map $\Phi : V \longrightarrow \operatorname{End}(E)$, satisfying the following assumptions:

$$\begin{aligned} \phi(xy) &= \tfrac{1}{2}(\phi(x)\phi(y) + \phi(y)\phi(x)) && (1)\\ \phi(e) &= \mathrm{Id} && (2)\\ \beta(\phi(x)\xi, \eta) &= \beta(\xi, \phi(x)\eta)\,, && (3) \end{aligned}$$

for all $x, y \in V, \xi, \eta \in E$.

To such a representation one associates a bilinear map $H : E \times E \longrightarrow V$, defined for $\xi, \eta \in E$ by

$$\forall x \in V, \qquad \beta(\phi(x)\xi, \eta) = B(x, H(\xi, \eta))\,.$$

As a consequence of property (3), H is symmetric. Let Q be the associated quadratic map, defined by

$$Q(\xi) = H(\xi, \xi)$$

and for $x \in V$, let β_x be the symmetric bilinear form on E defined by

$$\beta_x(\xi, \eta) = B(H(\xi, \eta), x) = \beta(\phi(x)\xi, \eta)\,,$$

and let $\mathbf{q}_x$ be the associated quadratic form, so that $\mathbf{q}_x(\xi) = \beta(\phi(x)\xi, \xi)$. Let Sym_β be the Jordan algebra of β-symmetric endomorphisms of E with the Jordan product $S.T = \frac{1}{2}(ST + TS)$. Then ϕ is a Jordan algebra homomorphism from V into Sym_β. This in particular implies the relation

$$\forall x, y \in V, \quad \phi(P(x)y) = \phi(x)\phi(y)\phi(x)\,,$$

and as a consequence

$$\forall x \in V,\ \forall \xi \in E, \qquad Q(\phi(x)\xi) = P(x)(Q(\xi))\,.$$

Note the formula

$$\forall x, y \in V, \qquad \mathbf{q}_y \circ \phi(x) = \mathbf{q}_{P(x)y}\,.$$

Denote by $V^\times$ the set of invertible elements. If x is invertible, then $\phi(x)$ is invertible, and as a consequence β_x is non-degenerate. Using notations from Section 2, we may introduce the mappings $\widetilde{\beta}_x : E' \longrightarrow E$ and $\partial_x : \mathcal{P}(E) \longrightarrow \mathcal{S}(E)$, and let $\Box_x = \partial_x(\mathbf{q}_x)$.

The following result will be needed later.

Lemma 4.

The dimension of E is a multiple of the rank of V and

$$\forall x \in V \qquad \operatorname{Det} \phi(x) = (\det x)^{\frac{N}{r}} .$$

For the proof, recall that in a simple Jordan algebra, the inversion $x \mapsto x^{-1}$ is a rational mapping. More precisely, there exists a V-valued polynomial R (of degree $r-1$) such that the identity $xR(x) = \det(x)e$ is satisfied on V. If $x \in V^\times$, $\phi(x)$ and $\phi(x^{-1})$ commute, hence the same is true for $\phi(x)$ and $\phi(R(x))$. Hence, $\phi(x)\phi(R(x)) = \det x \ \mathrm{Id}$. By continuity this is true for all $x \in V$. Take the determinant of each side, and notice that $\operatorname{Det}(\phi(R(x))$ is a polynomial function of x. This shows that $\operatorname{Det}(\phi(x))$ is a divisor of $(\det x)^N$. But the simplicity of V implies the absolute irreducibility of the "norm" function det (see [6]). Hence, $\operatorname{Det}(\phi(x))$ is (up to a constant) a power of $\det(x)$. Testing on elements $x = te$ for $t \in \mathbb{R}$ gives the lemma. In the case where V is euclidean, a different proof of this result is given in [1].

Let us recall some facts about distributions. For any element $D \in \mathcal{D}(E)$, and T any distribution, recall that DT is the distribution defined by $(DT, \varphi) = (T, D^*\varphi)$, where D^* is the adjoint of D. For $p \in \mathcal{P}(E)$, the adjoint of $\partial(p)$ is $\partial(\check{p})$, where $\check{p}$ is the polynomial defined by $\check{p}(\xi) = p(-\xi)$. If A is a linear isomorphism of E, then $T \circ A$ is the distribution given by the rule $(T \circ A, \varphi) = \det(A)^{-1}(T, \varphi \circ A^{-1})$ for $\varphi \in \mathcal{S}(E)$. Finally, the mapping $\varphi \longrightarrow D(\varphi \circ A^{-1}) \circ A$ is a constant coefficient differential operator, denoted by D^A. For A a β symmetric invertible operator, note the formulae

$$\left(D^A\right)^* = \left(D^*\right)^A \quad \text{and} \quad \partial(p)^A = \partial\left(p \circ A^{-1}\right) .$$

Lemma 5.

Let f be a distribution on E. Then the following properties are equivalent:

$$\begin{array}{lll} \forall x \in V^\times, & f \circ \phi(x) \ \textit{is} \ \Box \ \textit{harmonic} & (i) \\ \forall x \in V, & \partial(\mathbf{q}_x) f = 0 & (ii) \\ \forall x \in V^\times, & \Box_x f = 0 . & (iii) \end{array}$$

Let x be an invertible element of V. Then $\phi(x)$ is invertible, and hence we may apply the previous formula to $A = \phi(x)^{-1}$ and $p = \mathbf{q}_e$, to get

$$\partial\left(\mathbf{q}_{x^2}\right) = \Box^{\phi(x^{-1})} .$$

So, if f is a distribution on E, $f \circ \phi(x)$ is $\Box$-harmonic if and only if f is $\partial(\mathbf{q}_{x^2})$-harmonic. To conclude, we need to remark that any element in V is a limit of invertible elements, and that on the other hand, the linear combinations of squares generate V. This gives the equivalence between (i) and (ii).

For the other equivalence, let $x \in V^\times$. Then, for any $\varphi \in E'$, $\widetilde{\beta}_x(\varphi) = \phi(x^{-1})(\widetilde{\beta}(\phi))$, hence $\partial_x(\varphi) = \nabla_{\phi(x^{-1})\widetilde{\beta}(\varphi)}$. Now, for A a β-symmetric automorphism of E and $\xi \in E$, $\nabla_{A\xi} f = \nabla_\xi (f \circ A) \circ A^{-1}$ for all smooth functions f on E, and hence,

$$\partial_x(\varphi) f = \left(\partial(\varphi)\left(f \circ \phi\left(x^{-1}\right)\right)\right) \circ \phi(x) .$$

This formula can now be extended to any polynomial on E. In particular,

$$\partial_x(\mathbf{q}_x) f = \left(\partial(\mathbf{q}_x)\left(f \circ \phi\left(x^{-1}\right)\right)\right) \circ \phi(x) .$$

Assume $x = y^2$, with $y \in V^\times$; then $\mathbf{q}_x = \mathbf{q} \circ \phi(y)$, and hence

$$\partial_x(\mathbf{q}_x) f = \left(\left(\partial(\mathbf{q})\left(f \circ \phi\left(y^{-2}\right) \circ \phi(y)\right)\right) \circ \phi\left(y^{-1}\right)\right) \circ \phi\left(y^2\right) = \left(\Box\left(f \circ \phi\left(y^{-1}\right)\right)\right) \circ \phi(y)$$
$$= \partial\left(\mathbf{q} \circ \phi\left(y^{-1}\right)\right) f = \partial\left(\mathbf{q}_{x^{-1}}\right) f .$$

So, for any invertible square element x

$$\Box_x = \partial\left(\mathbf{q}_{x^{-1}}\right) .$$

As already seen in the proof of Lemma 4, the inversion is a rational map, so that the right-hand side depends rationally on x. For obvious reasons, this is also true for the left-hand side. As they coincide on invertible square elements (which form an open set in E), they must coincide on $V^\times$. This makes the equivalence between (ii) and (iii) clear.

A distribution f on E, which satisfies the equivalent properties of Lemma 5, is called *Q-pseudo-harmonic.*

Theorem 4.

Let f be a Q-pseudo-harmonic tempered distribution. Then, for any $x \in V^\times$,

$$\int_E e^{\frac{i}{2}\beta(\phi(x)\xi,\xi)}\, f(\xi)\, e^{-i\beta(\xi,\eta)}\, d\xi$$
$$= (2\pi)^{\frac{N}{2}} e^{i\frac{\pi}{4}\operatorname{sgn}(\mathbf{q}_x)}\, |\det(x)|^{-\frac{N}{2r}}\, f\left(\phi\left(x^{-1}\right)\eta\right)\, e^{-\frac{i}{2}\beta(\phi(x^{-1})\eta,\eta)} ,$$

where the equality should be interpreted in terms of distributions.

Let $x \in V^\times$ and consider the non-degenerate form β_x, as introduced before. Let $d_x\xi$ denote the normalization of the Lebesgue associated to β_x as explained in Section 2. Then clearly, $d\xi = |\operatorname{Det}\phi(x)|^{-\frac{1}{2}} d_x\xi$. Now, using in particular Lemma 4 the left-hand side can be written as

$$\int_E e^{\frac{i}{2}\mathbf{q}_x(\xi)}\, e^{-i\beta_x(\xi,\phi(x^{-1})\eta)}\, f(\xi)\, |\det(x)|^{-\frac{N}{2r}}\, d_x\xi .$$

By Lemma 5, the distribution f is $\Box_x$-harmonic. The result now follows from Theorem 3.

The result can be specialized to distributions which have some nice transformation law with respect to the action of the representation (cf. [3]). Let $\nu \in \mathbb{C}$ and $\epsilon = 0$ or 1. For t a real number define $|t|_\epsilon^\nu = (\operatorname{sgn}(t))^\epsilon |t|^\nu$. A distribution f is said to be *ϕ-homogeneous* of degree (ν, ϵ) if

$$\forall x \in V^\times, \quad f(\phi(x)\xi) = |\det x|_\epsilon^{\frac{\nu}{r}}\, f(\xi) .$$

Such a distribution is homogeneous of degree ν (choose $x = te, t > 0$), and if f is moreover $\Box$-harmonic, then f is automatically Q-pseudo-harmonic.

Theorem 5.

Let $\nu \in \mathbb{C}$, and $\epsilon = 0$ or 1. Let f be a $\Box$-harmonic distribution, ϕ-homogeneous of type (ν, ϵ). Then,

$$\mathcal{F}_\beta\left(f\, e^{\frac{i}{2}\mathbf{q}_x}\right) = (2\pi)^{\frac{N}{2}}\, e^{i\frac{\pi}{4}\operatorname{sgn}(\mathbf{q}_x)}\, |\det(x)|_\epsilon^{-\frac{\nu}{r}-\frac{N}{2r}}\, f\, e^{-\frac{i}{2}\mathbf{q}_{x^{-1}}} .$$

Except for the phase factor, this result was conjectured in [5]. The formula is the key step in constructing explicit intertwining operators in order to decompose the unitary action of the conformal group of a (real, simple) Jordan algebra V on $L^2(V)$ (cf. [4] where several examples are considered). Details will appear elsewhere.

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