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Reconstruction from Limited Data of Arc Means

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ABSTRACT. *Let f be a function in a Euclidean plane with compact support in a half disc H. The problem of reconstruction of the function from the data of its integrals over half circles $A \subset H$ with centers at the diameter of H is studied. An explicit formula and a microlocal analysis of stability of the reconstruction are given.*

1. Introduction

Let E_+ be an open halfplane in a Euclidean plane E and $\mathcal{A}_+$ be the family of all circular arcs $A \subset E_+$ that are orthogonal to the boundary ∂E_+. For a function f in E_+ we call *arc mean* the integral

$$Mf(A) \doteq \int_A f\, ds$$

over an arc $A \in \mathcal{A}_+$ against the Euclidean line density ds. The function Mf will be referred to as *arc mean* transform of f. Let $H \subset E_+$ be a half disc centered at a boundary point of E_+. We study the problem of reconstruction of the original f from its arc means transform known only for the subfamily $\mathcal{A} \subset \mathcal{A}_+$ of arcs $A \subset H$. We call it a *local* arc problem. The local arc problem is of practical importance in the following contexts:

• In seismic tomography the arc mean operator M is a linearization of the travel time mapping for the family of geodesics A of the Riemannian metric $d\sigma = c^{-1} ds$, where c stands for the velocity of elastic waves. Here f is a perturbation of the slowness c^{-1} and Mf is the residual of the travel time. See [18], [19], [7] for surveys and more bibliography. If the daylight surface is supposed to be a plane and a background velocity field v of a medium is a linear function of depth, the geodesics are arcs of circles orthogonal to the plane V parallel to daylight surface, where the extended field v vanishes. Hence, the three-dimensional case is reduced to the family of Euclidean planes E orthogonal to V and for each E the arc mean operator M is equal the linearization of the travel time mapping.

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The linearized inverse kinematic problem is reduced to a version of the arc problem. However, this linearization is reasonable only for some restricted regions of the earth surface. Therefore the local approach to the arc problem is more realistic comparing with the global version.

• A reconstruction of a ground reflectivity coefficient of electromagnetic waves by the method of synthetic aperture radar image processing can be also reduced to the local arc mean transform [8], [9].

A relation between a function and its spherical means in a Euclidean space E has been studied in [4] (unicity theorem). John [10] has given a reconstruction of a function from the family of spheres centered in a hyperplane $E' \subset E$. See [17], [18], [6], [9] for other approaches to the same problem. Note that this family of spheres is invariant with respect to a non-trivial translation group G of E, whereas the local sampling $\mathcal{A}$ as above admits no invariance group. Therefore, no straightforward harmonic analysis is applicable to the local arc problem. An original with compact support in H is uniquely determined from the local data of arc means [4], but the reconstruction is unstable. Our objective is to evaluate the stability; and hence, the reliability of reconstruction in the local arc problem. This is a very special case of the general problem of resolution and accuracy in the seismic tomography, which is considered as "the fundamental question" [20].

In Section 2 we reduce the local arc problem to the limited angle problem for the Euclidean Radon transform. In Sections 3 and 4 we give explicit formulae for a reconstruction and state in Section 5 a Plancherel-type identity that equalizes a weighted L_2-norm of the original f (we call it energy) with a Sobolev-type norm of its transform Mf. In Sections 6 through 8 we give a microlocal estimate of energy in terms of the arc mean transform.

If no *a priori* information is accessible, the energy of the original is assumed to be spread uniformly over the cotangent bundle $T^*(H)$. Take a curve $A \subset H$ and consider the conormal bundle $N^*(A) \subset T^*(H)$ of this curve. Denote by $N^*(\mathcal{A})$ the union of sets $N^*(A)$, $A \in \mathcal{A}$. This is a subset of $T^*(H)$, whose fibers $N_p^*(\mathcal{A})$, $p \in H$ are cones. We call this subset the *audible* zone. We show that the part of the energy of the original contained in the audible zone can be reasonably estimated by a norm of its arc mean transform. The complementary part of the energy in the *silent* zone $T^*(H) \setminus N^*(\mathcal{A})$ can be estimated with an appropriate weight. This weight is a function in the cotangent bundle that exponentially decreases, when the point moves away from the audible zone. It can be shown that an exponentially decreasing weight is in fact indispensable, hence our estimate in the silent zone cannot be essentially improved. In other words, the reconstruction of the function f is stable and reliable in the audible zone for all frequencies and in the silent zone for low frequencies; no method can give a stable reconstruction for high frequency in the silent zone.

Similar qualitative arguments for the limited data X-ray transform are due to Quinto [16]. He emphasized, in particular, the microlocal character of relations between smoothness of an original and its transform.

We do not discuss here the practical problem, how to improve stability of a reconstruction algorithm from the local data of arc means. This problem deserves another look.

2. Geometry of the Audible Zone

For an arbitrary point $p \in H$ the fiber $N_p^*(\mathcal{A})$ of $N^*(\mathcal{A})$ is the union of conormal lines to arcs $A \in \mathcal{A}$ through p. This set has a simple geometrical description:

Proposition 1.

For an arbitrary point $p \in H$ denote by S_p the circle arc through the point p and the ends of diameter D of H. The cone $N_p^*(\mathcal{A})$ is the union of alternate angles of magnitude $\pi - \mathrm{rad}(S_p)$, where rad *means the radian measure. The line through p and the center O_p of S_p is the bisectrix of these angles.*

The proof is elementary. The magnitude of $N_p^*(\mathcal{A})$ is close to 2π when the point p is close to a point of diameter D (except for its ends). For a point p at the arc $\partial H \setminus D$ the cone $N_p^*(\mathcal{A})$ shrinks to the line orthogonal to this arc, see Figure 1.

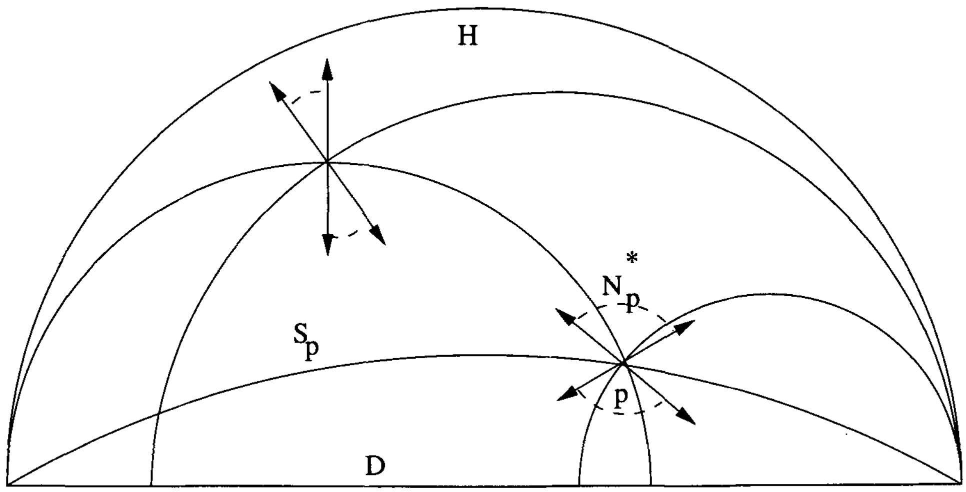

FIGURE 1

3. Reduction to the Radon Transform

We apply some classical mappings in the Euclidean plane to reduce the arc mean transform to the plane Radon transform. Similar methods were used earlier for the spherical Radon transform [3], for projective mappings [14] in Euclidean space and for the Radon transform in hyperbolic spaces [1], [2].

Let X_1, X_2 be Riemannian manifolds and $\mathcal{Y}$ be a family of k-dimensional submanifolds of X_1, $0 < k < \dim X_1$. A smooth mapping $F : X_1 \to X_2$ possesses the *factorization* property [15] for the family $\mathcal{Y}$ if the following equation holds for any $Y \in \mathcal{Y}$ and any $x \in Y$:

$$\frac{dV_2(F(x), F(Y))}{dV_1(x, Y)} = j_F(x) J_F(Y) \tag{3.1}$$

with some functions j_F in X_1 and J_F in $\mathcal{Y}$, where dV_i denotes k-dimensional volume form in X_i, $i = 1, 2$. These functions will be referred to as Jacobian factors. In particular, any conformal mapping F possesses the factorization property for the family of all submanifolds of arbitrary dimension k with the Jacobian factor $J_F = 1$.

Any projective transformation P of a Euclidean space E possesses the factorization property for the family $\mathcal{Y}$ of affine subspaces of arbitrary dimension [14].

If the factorization property holds for a family $\mathcal{Y}$ and a diffeomorphism F and a smooth mapping $G : X_2 \to X_3$ possesses this property for the family $F(\mathcal{Y}) = \{F(Y), Y \in \mathcal{Y}\}$, then the composition $GF : X_1 \to X_3$ has the factorization property for the family $\mathcal{Y}$ with the Jacobian factors $j_{GF}(x) = j_G(F(x)) j_F(x)$, $J_{GF}(Y) = J_G(F(Y)) J_F(Y)$.

Choose conformal coordinates x, y in E such that $E_+ = \{y > 0\}$ and half disc H is given in E_+ by $x^2 + y^2 < 1$. Let D_K, D_B be unit discs in the complex planes, $W = \{(u, v) : u^2 \geq v^2 + 1\}$ be the domain bounded by a hyperbola, and $U = W \cap \{u > 0\}$ be a convex component of W. We endow all these domains with the standard Euclidean metrics.

Proposition 2.
There is a sequence of diffeomorphisms

$$E_+ \xrightarrow{F} D_K \xrightarrow{G} D_B \xrightarrow{P} W$$

such that
(i) *the mapping F possesses the factorization property for the family $\mathcal{A}_+$;*
(ii) *the mapping G possesses the factorization property for the family $F(\mathcal{A}_+)$ of circles orthogonal to ∂D_K;*
(iii) *the mapping P possesses the factorization property for the family $G(F(\mathcal{A}_+))$ of all chords;*
(iv) *the image of H in W is equal to U and the image of the family $\mathcal{A}_+$ is the family of all chords in U. The image of $\mathcal{A}$ is the family of proper chords.*

Proof. The mappings F and G coincide with the classical isomorphisms between models of the Lobachewski plane, if the domains E_+, D_K, D_B are considered as the Poincaré, Klein, and Beltrami models, respectively (in spite of we do use no hyperbolic metric). Introduce complex coordinates $z = x + \imath y$, $z_1 = x_1 + \imath y_1$, $w_1 = u_1 + \imath v_1$ in H, D_K, D_B, respectively. Put

$$z_1 = F(z) \doteq \frac{\imath - z}{\imath + z}$$

This is a conformal mapping from H to D_K; hence, it satisfies Equation (3.1) with the factors

$$j_F(z) = \frac{2}{|\imath + z|^2}, \quad J_F \equiv 1$$

For any $A \in \mathcal{A}$ the image $F(A)$ is a circular arc orthogonal to the boundary ∂D_K and $F(H)$ is the right half of the disc D_K. The second mapping is that of Klein:

$$w_1 = G(z_1) \equiv \frac{2z_1}{1 + |z_1|^2}$$

This mapping possesses the factorization property for the family of arcs $A' = F(A)$ with the Jacobian factors

$$j_G(z_1) = \frac{2\left(1 - |z_1|^2\right)}{\left(1 + |z_1|^2\right)^2}, \quad J_G(A') = \frac{\sqrt{1 + r^2}}{r}, \tag{3.2}$$

where r is the radius of an arc A. The image $L = G(F(A))$ of this arc is a chord in D_B. To prove (3.2) we choose a unit tangent vector t to an arc A' at a point $z_1 \in D_K$. Its image $s = dw(t)$ is a tangent vector to the chord L with the length $|s| = 2|t - \bar{t}z_1^2|(1 + |z_1|^2)^{-2}$. It is easy to check that

$$\left|t - \bar{t}z_1^2\right| = \left|1 - (\bar{t}z_1)^2\right| = \left(1 - |z_1|^2\right) r^{-1}\sqrt{1 + r^2}$$

This implies (3.2). The image of the unit arc $\partial H \setminus D$ is the vertical diameter of the disc D_B and the set $G(F(H))$ is again the right half disc.

The last mapping P is the projective transformation:

$$(u, v) = P(u_1, v_1): \quad u = \frac{1}{u_1}, \ v = \frac{v_1}{u_1}$$

The vertical diameter is going to infinity and the unit circle is transformed to the hyperbola $v_1^2 + 1 = u_1^2$. The disc D_B maps onto the set $W = \{v_1^2 + 1 \leq u_1^2\}$ and the half disc $G(F(H))$ to the right component U of this set. The image of an arbitrary chord $L \subset D_B$ is a chord in U, since P is a projective transformation. If L is contained in the right half disc, the chord $P(L)$ is proper i.e., of finite length. This mapping possesses the factorization property with the factors

$$j_P(u_1, v_1) = |u_1|^{-2}, \quad J_P(L) = \sqrt{p(L)^2 + \sin\phi(L)^2}, \tag{3.3}$$

where $p(L)$ is the distance from the origin to the chord L and $(\cos\phi(L), \sin\phi(L))$ is its normal vector. □

Corollary 1.

The composition $S = PGF$ is a diffeomorphism from H to U that transforms the family $\mathcal{A}$ to the family $\mathcal{L}$ of proper chords in U and possesses the factorization property with the Jacobian factors

$$j_S(z) = j_P(G(F(z)))j_G(F(z))j_F(z), \quad J_S(A) = J_P(G(F(A)))J_G(F(A))J_F(A),$$

hence the following equation holds for an arbitrary arc A:

$$\int_{S(A)} f_W ds_W = J_S(A) \int_A f ds, \quad f_W \doteq f/j_S, \tag{3.4}$$

where ds_W is the Euclidean metric in W.

Now we calculate the Jacobian factors. By (3.2) we find

$$j_G(F(z)) = 2y|\imath + z|^2\left(1 + |z|^2\right)^{-2} \tag{3.5}$$

For an arbitrary arc $A \in \mathcal{A}$ we denote by $[a, b] \subset D$ its diameter; $-1 < a < b < 1$. Denote by α, β, γ the angles of the triangle $(a, b, \imath)$. We have $a = -\cot\alpha, b = \cot\beta$. The arc $A' := F(A)$ joins the points $F(a) = -\exp(2\imath\alpha)$ and $F(b) = -\exp(-2\imath\beta)$ and is orthogonal to the unit circle. The radius of the arc A' is equal $r = \tan\gamma$, hence $J_G(A') = \csc\gamma$ according to (3.2). The chord $L := G(A')$ has the same ends, hence $p(L) = -\cos(\alpha+\beta) = \cos\gamma$; $\phi(L) = \alpha - \beta$. By (3.3) we conclude that

$$j_P(G(F(z))) = \left(\frac{1 + |z_1|^2}{x_1}\right)^2 = \left(\frac{1 + |z|^2}{1 - |z|^2}\right)^2$$

and

$$J_P(L) = \sqrt{\cos^2\gamma + \sin^2(\alpha - \beta)} = \sqrt{1 - \sin 2\alpha \sin 2\beta}$$

Taking in account (3.5) and (3.3), we get

$$j_S(z) = \frac{4y}{(1 - |z|^2)^2}, \quad J_S(A) = \frac{\sqrt{1 - \sin 2\alpha \sin 2\beta}}{\sin\gamma}$$

We need to express the second factor in terms of the chord $L = S(A)$. Write the explicit formulae for the mapping S:

$$S(x, y) = \left(\frac{1 + x^2 + y^2}{1 - x^2 - y^2}, \frac{2x}{1 - x^2 - y^2}\right), \quad S^{-1}(u, v) = \left(\frac{v}{u+1}, \frac{\sqrt{u^2 - v^2 - 1}}{u+1}\right)$$

If the arc A is leaned on the diameter $[-\cot\alpha, \cot\beta]$, the chord L has the ends $S(a) = (-\sec 2\alpha, \tan 2\alpha)$, $S(b) = (-\sec 2\beta, -\tan 2\beta)$. The vector $(\cos\gamma, \sin(\beta - \alpha))$ is orthogonal to L. We have $|\sin(\beta - \alpha)| < \cos\gamma$, whence the angle ψ of this vector ranges in the interval $(-\pi/4, \pi/4)$. The angle ψ and the parameter $q \in \mathbb{R}$ (the distance to L from the origin) are coordinates on the variety $\mathcal{Y}_1$ of chords: $L = L(q, \psi)$. These coordinates relate to the parameters of arc as follows

$$\tan\psi = -\frac{a+b}{1+ab}, \quad q = \frac{\cos(\alpha-\beta)}{\sqrt{1-\sin 2\alpha \sin 2\beta}} = \frac{1-ab}{\sqrt{(1+ab)^2+(a+b)^2}} \tag{3.6}$$

The Jacobian factor J_S can be written as a function of ψ and q:

$$J_S(A) = \frac{\sqrt{1-\sin 2\alpha \sin 2\beta}}{\sin\gamma} = \frac{1}{\sqrt{q^2-\cos 2\psi}} \tag{3.7}$$

Note that the quantity $q^2 - \cos 2\psi$ vanishes simultaneously with the chord $L(q, \psi)$.

4. Interpolation and Reconstruction

From (3.4) and (3.7) we know the integral of the function $f_W(u, v) = (4y)^{-1}(1 - x^2 - y^2)^2 f(x, y)$ along an arbitrary proper chord L against the Euclidean line element ds_W. This is a continuous function with compact support in W. By the projection theorem we have for any $-\pi/4 < \psi < \pi/4$ and any $t \in \mathbb{R}$

$$\begin{aligned} \hat{f}_W(t\cos\psi, t\sin\psi) &= \int_{\mathbb{R}} \exp(-\iota qt) \int_{L(q,\psi)} f_W ds_W \\ &= \int \exp(-\iota qt) \frac{Mf(A(q,\psi))}{\sqrt{q^2-\cos 2\psi}} dq\,, \end{aligned} \tag{4.1}$$

Thus, the Fourier transform of the function f_W is known in the cone $D \doteq \{(\sigma, \tau) : \sigma^2 \geq \tau^2\}$.

Remark. The right side of (4.1) contains the integration along the family of arcs $A(q, \psi)$ with a constant angle ψ, which means that the quantity $x_A \doteq -\cot\psi = (1 + ab)/(a + b)$ is constant. Consider the complexification $E_{\mathbb{C}}$ of the plane E. An arbitrary circle A is the real part of a complex conic $A_{\mathbb{C}}$ that contains the points with the same abscissa and the ordinates $y_A = \sqrt{1 - x_A^2} = \sqrt{-\cos 2\psi}\csc\psi$. Consequently, the integral in (4.1) runs over the pencil of arcs A, whose complexifications $A_{\mathbb{C}}$ pass through the points (x_A, y_A). □

Now we use the interpolation method of [5] to reconstruct this function outside D:

$$\phi(\sigma) = \exp\left(\sqrt{\delta^2-\sigma^2}\right) \int_\Gamma \frac{\sin\left(\sqrt{\lambda^2-\delta^2}\right)}{\pi|\lambda-\sigma|} \phi(\lambda) d\lambda, \quad \Re\sqrt{\delta^2-\sigma^2} > 0 \tag{4.2}$$

where $\sigma \notin \Gamma := (-\infty, -\delta) \cup (\delta, \infty)$ and δ is an arbitrary positive number. The formula (4.2) is valid for an arbitrary function $\phi \in L_2(\mathbb{R})$ such that $\operatorname{supp}\hat{\phi} \subset [-1, 1]$. The support of the function f_W is compact and hence is contained in a strip $|u - a| \leq r$. Apply the interpolation method to the function $\phi_\tau(\sigma) := \hat{f}_1(\sigma, \tau)$ taking τ as a parameter and $f_1(u, v) \doteq f_W(ru + a, v)$. We have $\hat{f}_1(\sigma, \tau) = \exp(\iota a r^{-1}\sigma)\hat{f}_W(r^{-1}\sigma, \tau)$. The righthand side is known for $|\sigma| > r|\tau|$. We set $\delta = r|\tau|$ and get for an arbitrary τ the equation

$$\hat{f}_W(\sigma, \tau) = e^{r\sqrt{\tau^2-\sigma^2}} \int_{\lambda^2 \geq \tau^2} \frac{\sin\left(r\sqrt{\lambda^2-\tau^2}\right) e^{\iota a(\lambda-\sigma)}}{\pi|\lambda-\sigma|} \hat{f}_W(\lambda, \tau) d\lambda \tag{4.3}$$

for an arbitrary σ, τ. Now we apply the inverse Fourier transform and recover the function f.

We can use another interpolation formula [13] instead of (4.3)

$$\hat{f}_W(\sigma, \tau) = \cosh\left(r\sqrt{\tau^2-\sigma^2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\alpha_k}{\sigma_k}\left[\frac{e^{\imath(\sigma_k-\sigma)}}{\sigma_k-\sigma}\hat{f}(\sigma_k,\tau)+\frac{e^{-\imath(\sigma+\sigma_k)}}{\sigma_k+\sigma}\hat{f}(-\sigma_k,\tau)\right] \quad (4.4)$$

where

$$\alpha_k = \frac{\pi}{r}\left(k+\frac{1}{2}\right), \quad \sigma_k := \sqrt{\tau^2+\alpha_k^2}, \quad k = 0, 1, 2, \ldots$$

or this interpolation we need to measure the function $\hat{f}_W(\sigma, \tau)$ only in the hyperbolae $\sigma^2 = \tau^2 + a_k^2$, $k = 0, 1, \ldots$.

Remark. According to (4.3) as well as to (4.4) the main instability factor is equal exp $\left(r\sqrt{\tau^2-\sigma^2}\right)$. It can be shown that a factor of this form is inevitable for any method of reconstruction. □

Theorem 1.
For an arbitrary function $f \in L_2(E)$ with compact support supp $f \subset H$ *the formulae*

$$f(x, y) = \frac{y}{\pi^2\left(1-x^2-y^2\right)^2}\int_{\mathbb{R}^2}\exp(\imath(u(x, y)\sigma + v(x, y)\tau))\hat{f}_W(\sigma, \tau)d\sigma d\tau\,,$$

and (4.1), (4.3) [or (4.4)] give a reconstruction from the data $Mf(A)$, $A \in \mathcal{A}$.

5. Plancherel Theorem for Arc Means

For an arbitrary function $f \in L_2(H)$ we consider the global arc mean transform Mf defined on the family $\mathcal{A}_+$ of all arcs in the halfplane E_+ that are orthogonal to the boundary. The variety $\mathcal{A}_+$ is parameterized by the coordinates ψ, $-\pi/2 < \psi < \pi/2$ and $q \in \mathbb{R}$. We call the function

$$g(q, \psi) \doteq \frac{Mf(q, \psi)}{\sqrt{q^2-\cos 2\psi}}$$

the *normalized* arc mean transform.

Theorem 2.
For arbitrary square-integrable functions f_1, f_2 with compact supports in H and their normalized arc mean transforms g_1, g_2 the following identity holds:

$$2\pi^2\int\frac{1-x^2-y^2}{2y}f_1(x, y)\bar{f}_2(x, y)dxdy = -\int_{-\pi/2}^{\pi/2}\int_{\mathbb{R}^2}\frac{g_1(p, \psi)\bar{g}_2(q, \psi)}{(p-q)^2}dpdqd\psi\,,$$

where the principal value of the interior integral is taken.

For the Radon transform in plane, this result is due to Reshetnyak [11]. Write the right side in more explicit form:

$$\begin{aligned}
&\int_{-\pi/2}^{\pi/2}\int_{\mathbb{R}^2}\frac{g_1(p, \psi)\bar{g}_2(q, \psi)}{(p-q)^2}dpdqd\psi \\
&= -\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int\frac{g_1'(p, \psi)\bar{g}_2(q, \psi)-g_1(p, \psi)\bar{g}_2'(q, \psi)}{q-p}dpdqd\psi \\
&= \int_{-\pi/2}^{\pi/2}\int g_1'(p, \psi)\bar{g}_2'(q, \psi)\log|p-q|dpdqd\psi \qquad (5.1)
\end{aligned}$$

For a function $h = h(q)$ of one variable we introduce the convolution operator

$$D^{1/2}h \doteq d * h', \quad d \doteq \frac{dq}{2\pi|q|^{1/2}}, \quad h' = \frac{dh}{dq}$$

This is an operator of fractional derivative of order 1/2. Now we formulate the Plancherel-type theorem in a different form:

Theorem 3.

Under the same assumptions we have

$$(2\pi)^2 \int \frac{1 - x^2 - y^2}{2y} f_1(x, y)\bar{f}_2(x, y)dxdy = \int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}} D^{1/2}g_1(q, \psi)D^{1/2}\bar{g}_2(q, \psi)dqd\psi\ ,$$

Proof of Theorem 2. We calculate the lefthand side by means of Plancherel's theorem for the Fourier transform:

$$(2\pi)^2 \int \frac{1 - x^2 - y^2}{2y} f_1(x, y)\bar{f}_2(x, y)dxdy = (2\pi)^2 \int f_{1,w}\bar{f}_{2,w}dudv$$
$$= \int \hat{f}_{1,w}\bar{\hat{f}}_{2,w}d\xi d\eta = \int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}} \hat{g}_1(t, \psi)\bar{\hat{g}}_2(t, \psi)|t|dtd\psi\ ,$$

where $\det \partial(u, v)/\partial(x, y) = -8y(1 - x^2 - y^2)^{-3}$. Consider the interior integral in the right side and again apply Plancherel's identity

$$\int \hat{g}_1(t, \psi)\bar{\hat{g}}_2(t, \psi)|t|dt = 2\pi \int F^*\left(\hat{g}_1(t, \psi)|t|\right) \bar{g}_2(q, \psi)dq \tag{5.2}$$

where

$$F^*(h) \doteq \int \exp(\imath qt)h(t)dt$$

The image of the product $\hat{g}_1(t)|t|$ is equal the convolution of g_1 and of $F^*(|t|)$. Since of the equation $F^*(|t|) = -1/\pi|q|^2$, the righthand side of (5.2) is equal to the principal value of the integral

$$-2\int_{\mathbb{R}^2} \frac{g_1(p, \psi)\bar{g}_2(q, \psi)dpdq}{(p - q)^2}$$

Integrating this quantity against the density $d\psi$ we complete the proof of Theorem 2. By partial integration in the right side we get the formula (5.1). □

Proof of Theorem 3. Applying Plancherel's theorem once again we find

$$2\pi \int \hat{g}_1(t)\bar{\hat{g}}_2(t)|t|dt = \int G_1(q)\bar{G}_2(q)dq\ , \tag{5.3}$$

where

$$G_i \doteq F^*\left(\hat{g}_i(t)|t|^{1/2}\right) = g_i * F^*\left(|t|^{1/2}\right),\ i = 1, 2$$

We have

$$2\pi F^*\left(|t|^{1/2}\right) = -\frac{1}{2|q|^{3/2}} = \frac{d}{dq}\frac{1}{|q|^{1/2}}$$

Therefore, the right side is equal

$$(2\pi)^{-1} \int D^{1/2}g_1(q, \psi)D^{1/2}\bar{g}_2(q, \psi)dq$$

and Theorem 3 follows. □

Definition. We call the integral

$$\|f\|_H^2 \doteq \int_H \frac{1-x^2-y^2}{2y} |f(x,y)|^2 dxdy$$

the *energy* of a function f in H. By Theorem 3 we have

$$\|f\|_H^2 = \int_{-\pi/2}^{\pi/2} \int \left| D^{1/2} g(q,\psi) \right| dq d\psi \,, \tag{5.4}$$

where g is the normalized arc mean transform of f. It follows that the arc mean transform can be extended to the space of all originals f with finite energy. The normalized image g of the extended arc mean transform belongs to the non-isotropic Sobolev space $W^{1/2,0}(\mathbb{R} \times [-\pi/2, \pi/2])$. In fact, this transform is defined in terms of two Fourier–Plancherel transforms. Theorems 2 and 3 still hold for the extended transform. □

For the arc mean transform restricted to the family of arcs $\mathcal{A}$ we get the following identity:

Corollary 2.

For any function $f \in L_2(H)$ we have

$$\int_D \left| \hat{f}_W(\sigma,\tau) \right|^2 d\sigma d\tau = \int_{-\pi/4}^{\pi/4} \int_{\sqrt{\cos 2\psi}}^{\infty} \left| D^{1/2} \frac{Mf(q,\psi)}{\sqrt{q^2 - \cos 2\psi}} \right|^2 dq d\psi \,, \tag{5.5}$$

where $D = \{(\sigma,\tau) : \sigma^2 \geq \tau^2\}$ is the set of conormals to proper chords in U.

This follows from (4.1) by integrating against $d\psi$ in the interval $[-\pi/4, \pi/4]$.

Corollary 3.

For any function f with finite energy, the equation $Mf = 0$ implies $f = 0$ a.e.

Indeed Equation (5.5) implies that $\hat{f}_W = 0$ in D. The function $\hat{f}_W$ is holomorphic, hence it vanishes everywhere.

6. Microlocal Evaluations of Energy

We show that for an arbitrary original f with sufficiently small compact support in H the part of its energy which is held in the audible zone $N^*(\mathcal{A})$ is estimated in terms of the arc mean transform. We parametrize the variety $\mathcal{A}$ by the coordinates q, ψ [see (3.6)], where q runs over $\mathbb{R}$ and ψ over the interval $(-\pi/4, \pi/4)$. For a function $h = h(q,\psi)$ in the variety $\mathcal{A}$ we define the norm

$$\|h\|_{\mathcal{A},1/2}^2 \doteq \int_{-\pi/4}^{\pi/4} \int_{\sqrt{\cos 2\psi}}^{\infty} \left| D^{1/2} \frac{h(q,\psi)}{\sqrt{q^2 - \cos 2\psi}} \right|^2 dq d\psi \,.$$

Now we state an estimate of energy for originals with small support. Introduce Euclidean coordinates θ in E^* and denote by S^* the unit sphere.

Theorem 4.

Let $p \in H$ and V be a cone in E^* such that $\{p\} \times V \cap S^* \Subset N^*(\mathcal{A})$. There exist a compact neighborhood K of the point p and a positive constant C such that an arbitrary function f with finite energy such that supp $f \subset K$ *satisfies the inequality:*

$$\int_V \left| \hat{f}(\theta) \right|^2 d\theta \leq C \|Mf\|_{\mathcal{A},1/2} \cdot \|f\|_H \tag{6.1}$$

For originals with arbitrary support the following statement is true:

Theorem 5.
For arbitrary compact sets K, L and an arbitrary cone $V \subset E^*$ such that

$$K \Subset L \Subset H, \quad L \times V \cap S^* \Subset N^*(\mathcal{A}) \tag{6.2}$$

there exist positive constants a, c such that for any function f with finite energy the following inequality holds

$$c \int_V |F(e_K f)|^2 d\theta \leq \|Mf\|^2_{\mathcal{A},1/2} + \|\exp(-a|\theta|)\hat{f}\|_{L_2} \cdot \|f\|_H ,$$

where e_K is a smooth function such that $e_K = 1$ in K and $e_K = 0$ in $H \setminus L$.

In other words, the part of energy of f in the audible zone $N^*(\mathcal{A})$ is estimated in terms of its arc mean data Mf plus a term, which is small for high frequencies of the original f. We shall deduce both theorems from lemmata of Section 7 and complete the proof in Section 8.

For the part of energy in the silent zone, the reconstruction is exponentially unstable. For a nonempty compact set $K \subset E$ we denote its supporting function in E^* as follows:

$$\lceil \theta \rceil_K \doteq \max\{\theta(p),\ p \in K\}, \quad \theta \in E^*$$

Take a quadratic form Q of signature $(1, 1)$ in E^* and consider the cone $V(Q) \doteq \{\theta : Q(\theta) \leq 0\}$. Denote by $\delta(K, Q)$ the number such that

$$\min_{V(Q)} \left(\lceil \theta \rceil_K + \lceil -\theta \rceil_K - 2\delta(K, Q)\sqrt{-Q(\theta)} \right) = 0 , \tag{6.3}$$

It is well defined and positive.

Theorem 6.
Let K be a compact set in E and Q be a quadratic form in E^* of signature $(1, 1)$. *For an arbitrary function $h \in L_2(E)$ such that* $\operatorname{supp} h \subset K$ *the following inequality holds:*

$$\int_{V(-Q)} \left|\exp\left(-\delta(K, Q)\sqrt{Q(\theta)}\right) \hat{h}(\theta)\right|^2 d\theta \leq \int_{V(Q)} \left|\hat{h}(\theta)\right|^2 d\theta \tag{6.4}$$

Combine this estimate with Theorem 4 by taking a quadratic form Q such that $V(Q) = V$:

$$\int \left|\exp\left(-\delta(K, Q)\sqrt{Q(\theta)}\right) \hat{f}(\theta)\right|^2 d\theta \leq C\|Mf\|_{\mathcal{A},1/2} \cdot \|f\|_H$$

In the audible zone this inequality follows immediately from (6.1). The local silent zone $E^* \setminus N^*_p(\mathcal{A})$ is contained in the cone $V(-Q)$, consequently the integral of the density $|f|^2 d\theta$ with the fast decreasing weight $\exp\left(-\delta\sqrt{Q}\right)$ is estimated in terms of the arc means transform Mf.

7. Estimates in the Audible Zone

First we find a bound for the Fourier transform of f in terms of the function

$$\hat{f}_W(\rho) \doteq \int \exp(-\imath(\sigma u + \tau v)) f_W(x, y) du dv, \quad \rho \doteq (\sigma, \tau)$$

Consider the complexification $E_{\mathbb{C}}$ of the space E with the complex coordinates $\tilde{x} = x + \imath\check{x}$, $\tilde{y} = y + \imath\check{y}$. We abbreviate these notations to $p = (x, y)$, $\check{p} = (\check{x}, \check{y})$ and $\tilde{p} = (\tilde{x}, \tilde{y}) = p + \imath\check{p}$. First construct a family of quasianalytic cutting functions.

Lemma 1.

For arbitrary sets $K \Subset L \Subset E$ there exist positive constants ϵ, C such that for arbitrary positive $s > 0$ there exists a smooth function $e_K = e_K(\tilde{p}, s)$ in $E_{\mathbb{C}}$ such that

$$
\begin{aligned}
0 \le e_K(p,s) &\le 1 && \text{if } p \in E\,; && (7.1)\\
e_K(p,s) &= 1, && \text{if } p \in K\,; && (7.2)\\
e_K(p,s) &= 0, && \text{if } p \in E \setminus L\,; && (7.3)\\
|e_K(\tilde{p},s)| &\le C\exp(s), && \text{if } |\check{p}| \le \epsilon\,; && (7.4)\\
\left|\bar{\partial} e_K(\tilde{p},s)\right| &\le C\left|\check{p}\right|\exp(-s), && \text{if } \left|\check{p}\right| \le \epsilon\,. && (7.5)
\end{aligned}
$$

Proof. Take a smooth function $a : \mathbb{R} \to \mathbb{R}$ such that $a = 1$ for $t \le 2$, $a = 0$ for $t > 9/4$ and $0 \le a \le 1$ otherwise. Set

$$\tilde{a}(\tilde{p}) \doteq a\left(p^2\right) + 2\imath a'\left(p^2\right) p\check{p}, \quad p^2 = x^2 + y^2, \quad p\check{p} = x\check{x} + y\check{y}$$

The function $\tilde{a}$ is smooth in $E_{\mathbb{C}}$ and $\bar{\partial}\tilde{a}(\tilde{p}) = 2\imath a''(p^2)p\check{p}\cdot p d\bar{\tilde{p}}$. Define the function

$$e(\tilde{p},s) \doteq n(s)^{-1}\exp\left(-s\tilde{p}^2\right)\tilde{a}(\tilde{p}), \quad \tilde{p}^2 \doteq \tilde{x}^2 + \tilde{y}^2, \quad n(s) \doteq \int \exp\left(-sp^2\right) a\left(p^2\right) dp$$

that depends on a parameter $s \ge 0$. We have $n(s) > 0$ for $s \ge 0$, $sn(s) \to \pi$ as $s \to \infty$ and

$$\int |e(\tilde{p},s)|\,dp \le n(s)^{-1}\exp\left(s\check{p}^2\right)\int \exp\left(-sp^2\right)|\tilde{a}(\tilde{p})|\,dp \le C\exp(s) \qquad (7.6)$$

for $|\check{p}^2| \le 1$, since $|\exp(-s\tilde{p}^2)| = \exp(s\check{p}^2 - sp^2)$ and $\int \exp(-sp^2)|\tilde{a}|dp \sim \pi/s$ as $s \to \infty$. We have

$$\bar{\partial} e(\tilde{p},s) = n(s)^{-1}\exp\left(-s\tilde{p}^2\right) 2\imath a''\left(p^2\right) p\check{p}\cdot p d\bar{\tilde{p}}\,,$$

since $\exp(-s\tilde{p}^2)$ is an holomorphic function. Note that $a''(p^2)$ vanishes for $p^2 \le 2$. This implies the estimate

$$\int \left|\bar{\partial} e(\tilde{p},s)\right| dp \le M\left|\check{p}\right| n(s)^{-1}\exp\left(s\check{p}^2\right)\int_{p^2>2}\exp\left(-sp^2\right)dp^2\,,$$

where $M = \max|ta''(t^2)|$. The integral in the right-hand side is equal $s^{-1}\exp(-2s)$. The product $sn(s)$ is bounded from below by a positive constant c. Therefore, the previous inequality implies for $|\check{p}| \le 1$

$$\int \left|\bar{\partial} e(\tilde{p},s)\right| dp \le C\left|\check{p}\right|\exp\left(s\check{p}^2 - 2s\right) \le C\left|\check{p}\right|\exp(-s), \quad C \doteq M/c \qquad (7.7)$$

We have $\operatorname{supp} e(\cdot,s) \subset 3/2B$ for any s, where B means the unit ball in E. Choose $\epsilon > 0$ so small that $K + 3\epsilon B \subset L$ and denote by χ the indicator function of the compact $K + 3\epsilon/2B$. Set $e_\epsilon(\tilde{p},s) \doteq \epsilon^{-2}e(\tilde{p}/\epsilon,s)$ and

$$e_K(\tilde{p},s) \doteq \int e_\epsilon\left(p' + \imath\check{p},s\right)\chi\left(p - p'\right)dx'dy', \quad p' \doteq \left(x',y'\right)$$

The function $e_K(p,s)$ is nonnegative since the functions a, e, χ are. The inequality $e_K(p,s) \le 1$ follows from $\chi \le 1$ and $\int e_\epsilon(p,s)dp = 1$. This proves (7.1). The properties (7.2) and (7.3) follow from the inclusion $\operatorname{supp} e_\epsilon \subset 3\epsilon/2B$. By (7.6) we have for $|\check{p}| \le \epsilon$

$$|e_K(\tilde{p},s)| \le \int \left|e_\epsilon\left(p' + \imath\check{p},s\right)\right| dp' \le C\exp(s)\,,$$

which implies (7.4). The inequality (7.5) follows from (7.7). □

Take the family of functions e_K constructed by means of Lemma 1 and write

$$(2\pi)^2 F\,(e_K f)\,(\theta) = \int_E \int_{W^*} \exp(\imath\Phi(\theta, p, \rho)) e_K(p) j(p) \hat{f}_W(\rho) d\rho \wedge dx \wedge dy\,, \qquad (7.8)$$

where $\theta = (\xi, \eta),\ d\rho = d\sigma \wedge d\tau$ and

$$\Phi(\theta, p, \rho) \doteq -\xi x - \eta y + \sigma u(x, y) + \tau v(x, y), \quad j(p) \doteq 4y\left(1 - x^2 - y^2\right)^{-2}$$

Consider the 4-form in $E \times W^*$:

$$\omega \doteq \exp(\imath\Phi(\theta, p, \rho)) j(p) \hat{f}_W(\rho) d\rho \wedge dx \wedge dy$$

This form has a meromorphic continuation to the complex space $E_{\mathbb{C}} \times W^*_{\mathbb{C}}$ with a pole at the variety $\tilde{x}^2 + \tilde{y}^2 - 1 = 0$.

Lemma 2.

There exists a continuous field $q = q(\theta, p, \rho)$ in $V \setminus 0 \times H \times W^* \setminus D$ that is homogeneous of degree 0 with respect to $V \times W^*$ and which satisfies the inequalities:

$$|q(\theta, p, \rho)| \leq \epsilon\,, \qquad (7.9)$$
$$t\,(a|\theta| + b|\rho|) \leq \Im\Phi(\theta, p + \imath t q(\theta, p, \rho), \rho) \qquad (7.10)$$

for $\theta \in V,\ p \in L,\ \rho \in W^* \setminus D,\ 0 \leq t \leq 1$ and some positive a, b.

A proof will be given in the next section. Take an arbitrary point $\theta \in V$, choose a big parameter r, and consider the 5-chain $M(r)$ in $E_{\mathbb{C}} \times W^*$ given by the mapping

$$H \times \left(B^*(r) \setminus D\right) \times [0, 1] \to E_{\mathbb{C}} \times W^*, \ (p, \rho, t) \mapsto (p + \imath t q(\theta, p, \rho), \rho) \qquad (7.11)$$

where $B^*(r)$ is the ball in W^* of radius r. Define an orientation by the form $dx \wedge dy \wedge d\sigma \wedge d\tau \wedge dt$. By Stokes Theorem we get

$$\int_{\partial M(r)} e_K \omega = \int_{M(r)} d\,(e_K \omega) = \int_{M(r)} \bar{\partial} e_K \wedge \omega \qquad (7.12)$$

The boundary $\partial M(r)$ consists of the four pieces:

$$\partial M(r) = M_0(r) \cup M_1(r) \cup H(r) \cup W(r)\,, \qquad (7.13)$$

where $M_0(r), M_1(r)$ are the intersections of $M(r)$ with the hyperplanes $t = 0$ and $t = 1$ oriented by the forms $\mp d\rho \wedge dx \wedge dy$, respectively. The pieces $H(r), W(r)$ are the intersections of $M(r)$ with the hypersurfaces $p \in \partial H,\ \rho \in \partial(B^*(r) \setminus D)$, respectively. We have

$$e_K|H(r) = 0, \quad \omega|W(r) = 0\,. \qquad (7.14)$$

Indeed, the first equation follows from (7.3). The images of the vectors $\partial/\partial x,\ \partial/\partial y,\ \partial/\partial t,\ \gamma$ under the mapping (7.11) generate the tangent space of the chain $W(r)$, where γ is an arbitrary tangent vector to the boundary of $B^*(r) \setminus D$. The form $d\rho = d\sigma \wedge d\tau$ vanishes on any pair of these vectors, which implies the second Equation (7.14). Combining (7.12), (7.13), and (7.14), we get the relation

$$\int_{M_0(r)} e_K \omega = \int_{M_1(r)} e_K \omega - \int_{M(r)} \bar{\partial} e_K \wedge \omega\,. \qquad (7.15)$$

On the other hand, we have

$$\int_{M_+} e_K \omega \to \int_E e_K j \int_{W^*} \exp(\imath\Phi)\, \hat{f}_W \, d\rho dx dy = (2\pi)^2 F\,(e_K f)\,(\theta)$$

as $r \to \infty$, where $M_+ = M_0(r) \cup L \times D$.
For an arbitrary $\beta > 0$ we introduce the notation $\hat{f}_{W,\beta}(\rho) \doteq \exp(-\beta|\rho|)\hat{f}_W(\rho)$.

Lemma 3.

For any compact sets K, L, and cone V that satisfy (6.2) and any constant $\beta < b$ there exists a constant $c > 0$ such that for any $s \geq 0$ and an arbitrary function f in H with finite energy the following inequality holds:

$$c\int_V |F\,(e_K f)|^2\, d\theta \leq \exp(2s) \int_{W^* \setminus D} \left|\hat{f}_{W,\beta}\right|^2 d\rho + \int_D \left|\hat{f}_W\right|^2 d\rho + \exp(-2s)\|f\|_H^2 \qquad (7.16)$$

Proof. Write (7.15) as follows

$$\int_{M_+} e_K \omega = I_1 + I_2 + I_3\,,$$

where

$$\begin{aligned}
I_1(\theta) &\doteq \int_{L\times D} e_K \omega\,,\\
I_2(\theta) &\doteq \exp(-a|\theta|) \int\int_{W^*\setminus D} e_K\,(\tilde{p}, s)\, j\,(\tilde{p}) \exp(\imath\Psi\,(\theta, \tilde{p}, \rho))\, \hat{f}_{W,b}\, d\rho \wedge d\tilde{x} \wedge d\tilde{y}\,,\\
I_3(\theta) &\doteq -\int_{M(r)} j\,(\tilde{p}) \exp(\imath\Phi\,(\theta, \tilde{p}, \rho))\, \hat{f}_W(\rho) \bar{\partial} e_K\,(\tilde{p}, s) \wedge d\rho \wedge d\tilde{x} \wedge d\tilde{y}\\
\Psi\,(\theta, \tilde{p}, \rho) &\doteq \Phi(\theta, p + \imath tq, \rho) - \imath t(a|\theta| + b|\rho|)
\end{aligned}$$

Write the first integral in the form $I_1(\theta) = F\left(e_K j F^*(\chi \hat{f}_W)\right)$, where $F = F_{x,y\to\theta}$, $F^* = F_{\rho\to-(u,v)}$ are Fourier transforms and χ is the indicator function of the cone D. By Plancherel's identity for the Fourier transform we have

$$\begin{aligned}
\int_V |I_1|^2\, d\theta &= (2\pi)^2 \int \left|e_K j F^*\left(\chi \hat{f}_W\right)\right|^2 dx dy\\
&= (2\pi)^2 \int_L \frac{2y}{1 - x^2 - y^2} \left|e_K F^*\left(\chi \hat{f}_W\right)\right|^2 du dv\\
&\leq (2\pi)^2 C_L \int \left|F^*\left(\chi \hat{f}_W\right)\right|^2 du dv = (2\pi)^4 C_L \int_D \left|\hat{f}_W(\rho)\right|^2 d\rho\,, \qquad (7.17)
\end{aligned}$$

where $C_L \doteq \max_L 2y(1 - x^2 - y^2)^{-1}$. The second integral is estimated pointwise:

$$|I_2(\theta)| \leq C'_L \exp(-a|\theta| + s) \int_{W^*\setminus D} \left|\hat{f}_{W,b}(\rho)\right| d\rho \qquad (7.18)$$

with the constant $C'_L \doteq \max\{|j(p + \imath q)|,\ p \in L, |q| \leq \epsilon\}$. Here we use (7.4) and the inequality $|\exp(\imath\Psi)| \leq 1$. The kernels $\exp(-a|\theta|)$ and $\exp(-b'|\rho|)$, $b' > 0$ are square-integrable. Therefore, (7.18) implies the following inequality:

$$\int_V |I_2|^2\, d\theta \leq C' \exp(2s) \int_{W^*\setminus D} \left|\hat{f}_{W,\beta}\right|^2 d\rho \qquad (7.19)$$

for an arbitrary $\beta < b$ (and the constant $C' = C(b-\beta)^{-2}$).
Write the third integral as a three-fold one

$$I_3(\theta) = \int_H \left(\int_{B^*(r)\setminus D} \left(\int_0^1 j(\tilde{p}) \exp(\imath\Phi(\theta, \tilde{p}, \rho)) \hat{f}_W \bar{\partial} e_K(\tilde{p}, s) \right) d\rho \right) dx \wedge dy ,$$

where $\tilde{p} = p + \imath tq$. From (7.5) follows the estimate $|\bar{\partial} e_K(\tilde{p}, s)| \leq Ct \exp(-s)$, where C does not depend on s, hence in virtue of (7.10) we have

$$\begin{aligned} |I_3(\theta)| &\leq C \exp(-s) \int_{W^*\setminus D} \left(\int_0^1 \exp(-t(a|\theta| + b|\rho|)) t dt \right) \left| \hat{f}_W \right| d\rho \\ &\leq C \exp(-s) \int_{W^*\setminus D} (1 + |\theta| + |\rho|)^{-2} \left| \hat{f}_W(\rho) \right| d\rho \end{aligned}$$

The kernel in the right side is square-integrable and, moreover,

$$\int_{W^*} (1 + |\theta| + |\rho|)^{-4} d\rho \leq C(1 + |\theta|)^{-2} ;$$

hence, the right side is estimated by the quantity

$$\frac{C \exp(-s)}{(1 + |\theta|)^2} \left(\int_{W^*\setminus D} \left| \hat{f}_W \right|^2 d\rho \right)^{1/2} .$$

We extend the integration to W and apply Plancherel's Theorem:

$$\int_{W^*\setminus D} \left| \hat{f}_W \right|^2 d\rho \leq \int |f_W|^2 du dv = \|f\|_H^2$$

This gives the estimate

$$|I_3(\theta)| \leq \frac{C \exp(-s)}{(1 + |\theta|)^2} \|f\|_H ,$$

and

$$\int_V |I_3|^2 d\rho \leq C \exp(-2s) \|f\|_H^2$$

This together with (7.17) and (7.19) imply the lemma. □

Lemma 4.

For arbitrary sets K, L, V as above there exists positive c such that

$$c \int_V |F(e_K f)|^2 d\theta \leq \|Mf\|_{A,1/2}^2 + \left(\int_{W^*\setminus D} \left| \hat{f}_{W,\beta}(\rho) \right|^2 d\rho \right)^{1/2} \|f\|_H \qquad (7.20)$$

Proof. Take the number s that validates the equation

$$\exp(4s) \int_{W^*\setminus D} \left| \hat{f}_{W,\beta} \right|^2 d\rho = \|f\|_H^2 .$$

We have $s \geq 0$ in virtue of the inequality

$$\int \left| \hat{f}_{W,\beta} \right|^2 d\rho \leq \int \left| \hat{f}_W \right|^2 d\rho = (2\pi)^2 \int_E |f_W|^2 du dv = (2\pi)^2 \|f\|_H^2$$

Substitute this parameter in (7.16) and get (7.20). □

Now we assume that the compact K is so small that $\delta(S(K), R)\sqrt{R(\rho)} \leq \beta|\rho|$, where $R(\rho) \doteq \tau^2 - \sigma^2$. Apply (6.4) to $h \doteq \hat{f}_W$:

$$\int_{W^*\setminus D} \left|\hat{f}_{W,\beta}\right|^2 d\rho \leq \int_{W^*} \left|\exp\left(-\delta(S(K), R)\sqrt{R}\right) \hat{f}(\theta)\right|^2 d\theta \leq \int_D \left|\hat{f}_W\right|^2 d\rho$$

By Corollary 2 the right side is equal $\|Mf\|^2_{\mathcal{A},1/2}$. Therefore, the second term in (7.20) is estimated by the product $\|Mf\|_{\mathcal{A},1/2} \|f\|_H$. The first term is also bounded by this product, since (5.4). Consequently, Lemma 4 implies Theorem 4. Theorem 5 follows from this lemma and:

Lemma 5.

There exist positive numbers c, C such that

$$\int_{W^*\setminus D} \left|\hat{f}_{W,\beta}(\rho)\right|^2 d\rho \leq C \int \left|\exp(-c|\theta|)\hat{f}(\theta)\right|^2 d\theta$$

for any function f with finite energy.

We shall prove this lemma in the next section.

8. The Estimate in the Silent Zone and Lemmata

Proof of Theorem 6. Choose Euclidean coordinates ξ, η in E^* such that the minimum in (6.3) is reached at the vector $\theta_0 = (1, 0)$, i.e.,

$$\lceil \theta_0 \rceil_K + \lceil \theta_0 \rceil_K = 2\delta(K, Q)\sqrt{-Q(\theta_0)}$$

Choose a point $q \in E$ such that $\lceil \theta_0 \rceil_K + \theta_0(q) = \lceil -\theta_0 \rceil_K - \theta_0(q)$ and set $K' \doteq K + q$. We have

$$\lceil \theta_0 \rceil_{K'} = \lceil -\theta_0 \rceil_{K'} = \delta(K, Q)\sqrt{-Q(\theta_0)}$$

This implies that there exists a constant b such that the inequality

$$\lceil (\xi, \eta) \rceil_{K'} \leq \delta(K, Q)\sqrt{-Q(\xi, \eta)} + b|\eta|$$

holds in E^*. The support of the function $h_q(p) \doteq h(p - q)$ is contained in K'. By the above inequality and the Paley–Wiener theorem we have

$$\left|\hat{h}_q\left(\tilde{\xi}, \tilde{\eta}\right)\right| \leq C \exp\left(\lceil \left(\Im\tilde{\xi}, \Im\tilde{\eta}\right) \rceil_{K'}\right) \leq C(\eta) \exp(\delta(K, Q)\sqrt{-Q(\xi, \eta)})$$

in the complexified space $E^*_{\mathbb{C}}$. Therefore, we can apply the interpolation formula (4.2) to $\hat{h}_q(\theta) = \exp(-\imath\theta(q))\hat{h}$ with respect to the variable ξ:

$$\hat{h}_q(\xi, \eta) = \exp\left(\delta(K, Q)\sqrt{Q(\theta)}\right) \int_{Q(\lambda,\eta)\leq 0} \frac{\sin\left(\delta(K, Q)\sqrt{-Q(\lambda, \eta)}\right)}{\pi|\lambda - \xi|} \hat{h}(\lambda, \eta) d\lambda ,$$

It can written in the form of convolution in variable ξ:

$$(1 - \chi) \exp\left(-\delta(Q, K)\sqrt{Q}\right) \hat{h} = e * \tilde{h} ,$$

where

$$e(\xi) = \frac{d\xi}{\pi|\xi|}, \quad \tilde{h} \doteq \chi \sin\left(\delta(L, Q)\sqrt{-Q}\right)\hat{h}$$

and χ is the indicator function of the cone $V(Q)$. The convolution with the kernel e is a unitary operator in $L_2(\mathbb{R})$. Therefore, for each η we have

$$\int (1-\chi)\left|\exp\left(-\delta(L, Q)\sqrt{Q}\right)\hat{h}\right|^2 d\xi = \int \left|\tilde{h}\right|^2 d\xi \le \int \left|\chi\hat{h}\right|^2 d\xi \, .$$

Integrating this inequality with respect to η, we obtain the following inequality for L_2-norms:

$$\left\|(1-\chi)\exp\left(-\delta(L, Q)\sqrt{Q}\right)\hat{h}\right\|^2 \le \left\|\chi\hat{h}\right\|^2 \qquad \square$$

Proof of Lemma 2. Assume that $\theta \in V$ and $\rho \in W^* \setminus D$. The gradient of the phase function

$$\nabla\Phi \doteq \left(\Phi'_x, \Phi'_y\right) = -\theta + (\sigma\nabla u + \tau\nabla v) \tag{8.1}$$

is the sum of the vector $-\theta \in V$ and of the vector $\rho^* \doteq \sigma\nabla u + \tau\nabla v$ which is contained in the cone $V_p \doteq dS^*_p(W^* \setminus D)$. The closures of the cones V and V^*_p have no common non-zero element because of (6.2). The angle between these cones is a positive continuous function of $p \in L$. For arbitrary non-zero vectors $\theta, \theta' \in E^*$ we denote by $\phi(\theta, \theta')$ one half of the angle between these vectors such that $0 \le \phi(\theta, \theta') \le \pi/4$. Similarly, we define an angle $\phi(\theta, U)$ for a vector and a cone U. For arbitrary elements $\theta \in V, \theta' \in V'$ we have

$$\left|\theta + \theta'\right|^2 \ge \left(1 - \cos 2\phi\left(\theta, \theta'\right)\right)\left(|\theta|^2 + \left|\theta'\right|^2\right) \ge \sin^2\phi\left(\theta, V'\right)|\theta|^2 + \sin^2\phi\left(\theta', V\right)\left|\theta'\right|^2$$

It follows from (8.1) that

$$\sin^2\phi\left(\theta, V_p\right)|\theta|^2 + \sin^2\phi\left(\rho^*, V\right)\left|\rho^*\right|^2 \le |\nabla\Phi|^2 \le \left(|\theta| + \left|\rho^*\right|\right)^2 \, ,$$

Set

$$S(\theta) \doteq \min_{p\in L}\sin^2\phi\left(\theta, V_p\right)|\theta|^2, \quad T(\rho) \doteq \min_{p\in L}\sin^2\phi\left(\rho^*, V\right)\left|\rho^*\right|^2 \, ,$$

and

$$q \doteq \delta\,\frac{S(\theta) + T(\rho)}{|\nabla\Phi(\theta, p, \rho)|^2}\nabla\Phi(\theta, p, \rho)$$

in $V \times L \times W^* \setminus D$, where δ is a positive parameter to be specified. The functions S and T are continuous and homogeneous of degree 1, hence the field q is continuous and homogeneous of order 0 with respect to the variables θ, ρ. It has the following upper bound:

$$|q(\theta, p, \rho)| \le \delta \max_L \frac{\left(\sin\phi\left(\theta, V_p\right) + \sin\phi\left(\rho^*, V\right)\right)^2}{2\sin\phi\left(\theta, V_p\right)\sin\phi\left(\rho^*, V\right)}$$

Choose δ small enough to validate (7.9). Show that the field q satisfies (7.10). By Lagrange theorem we have for some t', $0 \le t' \le t$

$$\begin{aligned}
&\Im\Phi(\theta, p + \imath tq, \rho) \\
&= t\nabla\Phi(\theta, p, \rho)q(\theta, p, \rho) - \frac{t^2}{2}\Im\nabla^2\Phi\left(\theta, p + \imath t'q, \rho\right)(q(\theta, p, \rho)) \\
&= t(S(\theta) + T(\rho)) - \frac{t^2}{2}\frac{(S(\theta) + T(\rho))^2}{|\nabla\Phi(\theta, p, \rho)|^4}\nabla^2\Im\Phi\left(\theta, p + \imath t'q, \rho\right)(\nabla\Phi) \\
&= \left(t + t^2B\right)(S(\theta) + T(\rho)) \, , \\
B &\doteq \frac{S(\theta) + T(\rho)}{2|\nabla\Phi|^4}\nabla^2\Im\Phi\left(\theta, p + \imath t'q, \rho\right)(\nabla\Phi)
\end{aligned}$$

The quantity B is uniformly bounded with respect to all variables, since (7.9) and the relation $|\nabla^2\Phi(\theta, p + \imath q, \rho)| = O(|\theta| + |\rho|)$. Therefore, for sufficiently small t the right side is estimated from below by, f.e. $2t/3(S + T)$. This proves (7.10), if we take into account that S and T are continuous and positive. □

Proof of Lemma 5. Write $\hat{f}_{W,\beta}(\rho) = \int_{W\times E^*} \upsilon$, where

$$\upsilon \doteq \exp\left(-\imath\Phi\left(\theta, \tilde{p}, \rho\right) - \beta|\rho|\right) \hat{f}(\theta) d\theta \wedge du \wedge dv$$

Choose a number $\beta' < \beta$. By the method of the previous proof we find a field $q = q(\theta, p, \rho)$ such that

$$-\Im\Phi(\theta, p + \imath q, \rho) + \beta'|\rho| \geq c'|\theta| \tag{8.2}$$

for a positive c'. We construct this field in the form

$$q = -\sigma \frac{\nabla\Phi(\theta, p, \rho)}{|\nabla\Phi(\theta, p, \rho)| + \beta'|\rho|},$$

where σ is a positive parameter. The denominator does not vanish in $V \times W^* \setminus D$. This field is well-defined continuous and homogeneous of degree 0. We have

$$-\Im\Phi(\theta, p + \imath q, \rho) + \beta'|\rho| = \sigma \frac{|\nabla\Phi|^2}{|\nabla\Phi| + \beta'|\rho|} + \beta'|\rho| + O\left(\sigma^2\right)$$

If σ is sufficiently small, the right side is estimated from below by $c'|\theta|$ for some positive c'. Fix $\theta \in E^*$, a number r and consider the 5-chain $N(r)$ in $E^* \times E_{\mathbb{C}}$ given by the parameterization:

$$(\theta \in B(r), p \in E, t \in [0, 1]) \mapsto (\theta, \tilde{p} = p + \imath t q(\theta, p, \rho))$$

The form υ is holomorphic in θ and $\tilde{p}$; hence,

$$\int_{\partial N(r)} \upsilon = \int_{N(r)} d\upsilon = 0$$

Inequality (8.2) implies that integral of υ over the piece of $\partial N(r)$, where $|\theta| = r$ tends to zero as $r \to \infty$. Comparing integrals over two pieces of the boundary of $N(r)$, where $t = 0$ and $t = 1$ correspondingly, we get the equation

$$\int_{W\times E^*} \upsilon = \int_{W\times E^*} \exp\left(-\imath\Phi(\theta, p + \imath q, \rho) - \beta|\rho|\right) \hat{f}(\theta) d\xi d\eta du dv$$

By (8.2) we can estimate the right side by the integral over E^* of the density $\exp(-c'|\theta|)\hat{f}(\theta)d\theta$. This integral can be estimated by L_2-norm of the function $\exp(-c|\theta|)\hat{f}$ for any $c < c'$, whence $|\hat{f}_{W,\beta'}| \leq C\|\exp(-c|\theta|)\hat{f}\|$ in $W^* \setminus D$. The L_2-norm of $\hat{f}_{W,\beta}$ in this cone is estimated by the supremum of the left side, since $\beta' < \beta$. Consequently, the last inequality implies Lemma 5. □

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