SCIENTIFIC AND TECHNICAL SECTION

ON DETERMINATION OF VIBRATION CHARACTERISTICS OF A BEAM WITH A CLOSING CRACK IN BENDING VIBRATIONS

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We consider free vibrations of an elastic cantilever beam with an edge crack, which is simulated as a beam portion with a decreased moment of inertia. The weight of this portion is assumed to be constant, while the dimensions of the portion are determined by the energy criterion of equivalence. We propose an analytical approach to determination of natural frequencies and vibration modes of a beam with an open or closing crack and to investigation of nonlinear distortions of the displacement wave and acceleration and deformation of various sections of a beam with a closing crack. The solution allows for the possibility that more than one vibration mode of a beam can be generated at the moment of the crack opening and includes the effect of the crack on the strain distribution in the beam volume. It is demonstrated that the approach we propose gives reliable relationships between various vibration characteristics of a beam and the crack parameters and makes it possible to solve an inverse problem of damage diagnostics.

Notation

L — beam length

L_c — coordinate of the cracked section

h — height of the cross section

b — width of the cross section

a — crack depth

γ — relative crack depth

d — parameter to be determined

F — cross-sectional area

I — moment of inertia of the section

I_o — moment of inertia of the cracked section

I_m — moment of inertia of mass

m_L — weight of a mass at the beam end

m — intensity of the beam weight

E — elastic modulus

ρ — density

$w(x)$ — natural mode of vibration of a beam

$\theta(x)$ — distribution of the slopes of the elastic curve of a beam along its length

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$M(x)$	— bending moment distribution along the beam length
$Q(x)$	— lateral force distribution along the beam length
ω	— natural angular frequency of an intact beam
ω_o and ω_c	— natural angular frequencies of a beam with an open crack and a beam with a closing one, respectively
$f_{a=0}$ and f_a	— resonance frequencies of intact and cracked specimens, respectively
K_{I}	— stress intensity factor

Introduction. As a rule, an intact body and a body with a discontinuity differ substantially in vibration characteristics. This difference, which is due to a change in stiffness, can be used for the diagnostics of a damage (e.g., a fatigue crack) and for the determination of the damage size (dimensions and location).

The majority of the published analytical investigations into this problem have been attempts to use the natural frequency of vibration as a damage characteristic [1–9]. Some works were dedicated to the study of distortions of natural modes of vibrations or a combination of modes and frequencies of vibration of a cracked body [1, 2, 9]. These scientists considered open cracks; this case, however, does not adequately represent the behavior of fatigue cracks, which behave as closing ones (the stiffness of a beam with an open crack is assumed to be equal in both half-cycles of vibration, while that of a beam with a closing crack is taken to be different).

The results of numerous investigations have demonstrated the great potential of using the distortion of the vibration waveform of a system simulating a cracked body for the diagnostics of closing cracks which cause such a distortion [10–15]. The waveform distortion was evaluated by the presence of higher harmonics in the Fourier expansion of the respective time functions.

The accuracy of the analytical solution to a vibration problem for a cracked body depends greatly on the choice of the crack model. In published works one can find a wide range of such models: a crack is simulated by a spring [16, 17], an elastic hinge [7], a notch [1–3], a zone with a decreased elastic modulus [17]. Shen and Pierre [9] take account of the crack influence by semiempirical functions that describe the stress and strain distribution in the whole volume of a beam. In all cases, except for [16], the vibrations of a beam with an open crack were simulated.

The objective of the present work is to develop an analytical approach that would make it possible to simulate vibrations of a beam with a closing edge crack in order to solve both the direct problem (the determination of characteristics of a beam with a preset crack size and location) and the inverse problem (the evaluation of crack parameters from the known values of the respective vibration characteristics) in the damage diagnostics. The following characteristics were chosen for the investigation: the natural frequencies and vibration modes of a beam, the displacement waveform distortion, and strains (taking into account the close-to-actual distribution of strains near a crack) and accelerations of arbitrary section of a beam.

Model of a Cracked Beam. The principle of the analytical approach we propose can be illustrated by the example of a prismatic cantilever beam of constant cross section with an end mass because specimens of similar type were used during the testing of this approach. It is known that free lateral vibrations of this beam without regard for damping are given by the differential equation

$$\frac{\partial^4 y(x,t)}{\partial x^4}+\frac{\rho F}{EI}\frac{\partial^2 y(x,t)}{\partial t^2}=0, \tag{1}$$

where E and ρ are the elastic modulus and density of the beam material, respectively; $I = bh^3/12$ and $F = bh$ are the moment of inertia and the cross-sectional area of the beam, respectively; b and h are the width and height of the cross section, respectively.

A general solution to Eq. (1) can be represented as

$$y(x,t)=\sum_{i=1}^{\infty} w_i(x)(P_i \sin\omega_i t + R_i \cos\omega_i t), \tag{2}$$

where $w_i(x)$ and ω_i are the natural modes and the angular frequencies of vibration of the beam and i is the modal number. The natural modes of vibration of the beam are given by

$$w_i(x) = A_i S(k_i x) + B_i T(k_i x) + C_i U(k_i x) + D_i V(k_i x), \tag{3}$$

where $S(k_i x)$, $T(k_i x)$, $U(k_i x)$, and $V(k_i x)$ are the Krylov functions;

$$S(k_i x) = \frac{1}{2}(\cosh\ k_i x + \cos k_i x), \qquad T(k_i x) = \frac{1}{2}(\sinh\ k_i x + \sin k_i x),$$

$$U(k_i x) = \frac{1}{2}(\cosh\ k_i x - \cos k_i x), \qquad V(k_i x) = \frac{1}{2}(\sinh\ k_i x - \sin k_i x),$$

$$k_i^4 = \omega_i^2 \frac{\rho F}{EI}.$$

The factors A_i, B_i, C_i, and D_i in expression (3) are determined from the boundary conditions

$$\begin{gathered} w_i(0) = 0, \quad \theta_i(0) = \frac{\partial w_i(0)}{\partial x} = 0, \\ M_i(L) = EI\frac{\partial^2 w_i(L)}{\partial x^2} = I_m \omega_i^2 \frac{\partial w_i(L)}{\partial x}, \quad Q_i(L) = EI\frac{\partial^3 w_i(L)}{\partial x^3} = -m_L \omega_i^2 w_i(L), \end{gathered} \tag{4}$$

where θ_i is the slope of the elastic curve, M_i is the bending moment, Q_i is the lateral force, L is the beam length, m_L is the weight of the end mass, and I_m is the moment of inertia of mass.

In this case, the frequency equation is of the form

$$[S(k_i L) - qT(k_i L)][S(k_i L) + gV(k_i L)] - [T(k_i L) - qU(k_i L)][V(k_i L) + gU(k_i L)] = 0, \tag{5}$$

where

$$q = \frac{I_m k_i^3}{\rho F}, \quad g = \frac{m_L k_i}{\rho F}.$$

We determine the C_i factor from the boundary condition (4) for the bending moment, $C_i = M(0)/(EIk_i^2)$. Then,

$$w_i(x) = \frac{M(0)}{EIk_i^2}\left[U(k_i x) - \frac{V(k_i L) + gU(k_i L)}{S(k_i L) + gV(k_i L)} V(k_i x)\right]. \tag{6}$$

The P_i and R_i factors in expression (2) are found [18] from the initial conditions at the moment of time t_s for the displacement $y(x, t_s) = y_s(x)$, velocity $\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=t_s}$, and the slopes $\theta(x, t_s) = \theta_s(x)$ of the elastic curve of the beam, using the formulas

$$P_i = \frac{\omega_i G_1 \sin\omega_i t_s + G_2 \cos\omega_i t_s}{\omega_i \left[\int_0^L m w_i^2(x)dx + m_L w_i^2(L) + I_m \theta_i^2(L)\right]}, \tag{7}$$

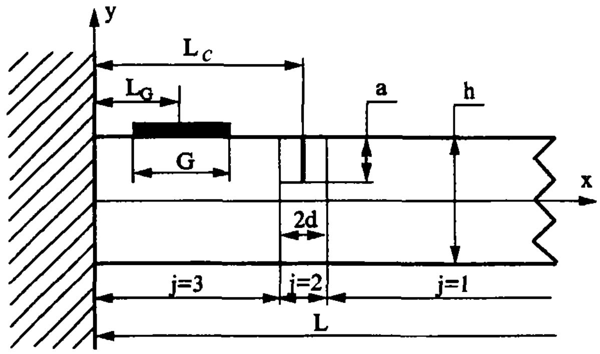

Fig. 1. Schematic representation of a cantilever beam.

$$R_i = \frac{\omega_i G_1 \cos\omega_i t_s - G_2 \sin\omega_i t_s}{\omega_i \left[\int_0^L m w_i^2(x)dx + m_L w_i^2(L) + I_m \theta_i^2(L) \right]}, \tag{8}$$

where $m = \rho F$ is the intensity of the beam weight;

$$G_1 = \int_0^L m y_s(x) w_i(x) dx + m_L y_s(L) w_i(L) + I_m \theta_s(L) \theta_i(L),$$

$$G_2 = \int_0^L m v_s(x) w_i(x) dx + m_L v_s(L) w_i(L) + I_m \left[\frac{\partial \theta(L,t)}{\partial t} \right]_{t=t_s} \theta_i(L).$$

Let us now consider a beam with an open edge crack (this crack is assumed to be open in both half-cycles of vibration) which is located at a distance L_c from the base (Fig. 1). We replace this beam with an equivalent one in which the crack (as well as the stress relaxation at the crack faces) is simulated as a portion with a decreased moment of inertia of the cross section (portion *2* in Fig. 1). The size of this portion $2d$ is varied according to the energy-based criterion of equivalence. The essence of this criterion is described below.

The free lateral vibrations of each portion of this beam, the damping being neglected, are represented by a differential equation of the form (1) [19], where $I = I_j$ is the moment of inertia of the cross section of the jth portion of the beam, $j = 1, 2, 3$; $I_1 = I_3 = I$; $I_2 = I_o = b(h-a)^3/12$; a is the crack depth (the subscript "o" stands for the open crack). In this case, a general solution to Eq. (1) for the jth portion of the beam is taken as

$$y_{oj}(x,t) = \sum_{i=1}^{\infty} w_{ij}(x)(P_{oi} \sin\omega_{oi} t + R_{oi} \cos\omega_{oi} t). \tag{9}$$

In the strict sense, from the above formula for the moment of inertia of the second portion of the equivalent beam it follows that its cross-sectional area is given by $F_2 = b(h-a)$. This, in turn, results formally in a decrease of the beam weight as the crack grows. However, it is known that the crack initiation and growth in a body does not cause any decrease of the weight of the body. This drawback of the model can be overcome either by taking the density of the second portion as $\rho_2 = \rho h/(h-a)$ or the cross-sectional area of the second portion as $F_2 = bh$. The sought-for dynamic characteristics in bending vibrations are invariant with respect to the above approaches. Therefore, taking into consideration that the length of the portion with an actual crack is negligible in comparison with the beam length and that the second approach is much simpler, we assume that $F_2 = F$. Thus, the weight of the equivalent beam remains unchanged with any crack length.

The equations that represent the natural modes of vibration of an equivalent beam are written for each beam portion j:

$$w_{ij}(x) = A_{ij}S(k_{ij}x) + B_{ij}T(k_{ij}x) + C_{ij}U(k_{ij}x) + D_{ij}V(k_{ij}x), \tag{10}$$

where

$$k_{ij}^4 = \omega_{oi}^2 \frac{\rho F}{EI_j}.$$

The boundary conditions and the conditions of conjugation of portions of an equivalent beam with an end mass are of the form

$$M_{i1}(L) = I_m\omega_{oi}^2 k_{i1}[A_{i1}V(k_{i1}L) + B_{i1}S(k_{i1}L) + C_{i1}T(k_{i1}L) + D_{i1}U(k_{i1}L)],$$

$$Q_{i1}(L) = -m_L\omega_{oi}^2[A_{i1}S(k_{i1}L) + B_{i1}T(k_{i1}L) + C_{i1}U(k_{i1}L) + D_{i1}V(k_{i1}L)],$$

$$w_{i1}(x_2) = w_{i2}(x_2), \quad \theta_{i1}(x_2) = \theta_{i2}(x_2), \quad M_{i1}(x_2) = M_{i2}(x_2),$$

$$Q_{i1}(x_2) = Q_{i2}(x_2), \quad w_{i2}(x_1) = w_{i3}(x_1), \quad \theta_{i2}(x_1) = \theta_{i3}(x_1),$$

$$M_{i2}(x_1) = M_{i3}(x_1), \quad Q_{i2}(x_1) = Q_{i3}(x_1), \quad w_{i3}(0) = 0, \quad \theta_{i3}(0) = 0,$$

where

$$x_1 = L_c - d, \quad x_2 = L_c + d.$$

Taking into account the properties of the Krylov function [$S(0) = 1$, $T(0) = U(0) = V(0) = 0$] and the last two boundary conditions, we find $A_{i3} = B_{i3} = 0$. Taking into account the remaining boundary conditions and the conjugation conditions, we obtain a set of ten equations whose determinant can be represented as

$$\begin{vmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\
-S(\delta) & -T(\delta) & -U(\delta) & -V(\delta) & S(\chi) & T(\chi) & U(\chi) & V(\chi) & 0 & 0 \\
-V(\delta) & -S(\delta) & -T(\delta) & -U(\delta) & r_1V(\chi) & r_1S(\chi) & r_1T(\chi) & r_1U(\chi) & 0 & 0 \\
-U(\delta) & -V(\delta) & -S(\delta) & -T(\delta) & r_2U(\chi) & r_2V(\chi) & r_2S(\chi) & r_2T(\chi) & 0 & 0 \\
-T(\delta) & -U(\delta) & -V(\delta) & -S(\delta) & r_1^{-1}T(\chi) & r_1^{-1}U(\chi) & r_1^{-1}V(\chi) & r_1^{-1}S(\chi) & 0 & 0 \\
0 & 0 & 0 & 0 & -S(\xi) & -T(\xi) & -U(\xi) & -V(\xi) & V(\eta) & U(\eta) \\
0 & 0 & 0 & 0 & -V(\xi) & -S(\xi) & -T(\xi) & -U(\xi) & r_1^{-1}U(\eta) & r_1^{-1}T(\eta) \\
0 & 0 & 0 & 0 & -U(\xi) & -V(\xi) & -S(\xi) & -T(\xi) & r_2^{-1}T(\eta) & r_2^{-1}S(\eta) \\
0 & 0 & 0 & 0 & -T(\xi) & -U(\xi) & -V(\xi) & -S(\xi) & r_1S(\eta) & r_1V(\eta)
\end{vmatrix} = 0, \tag{11}$$

where

$$a_{11} = U(\varphi_i) - q_0V(\varphi_i), \quad a_{12} = V(\varphi_i) - q_0S(\varphi_i), \quad a_{13} = S(\varphi_i) - q_0T(\varphi_i), \quad a_{14} = T(\varphi_i) - q_0U(\varphi_i),$$

$$a_{21} = T(\varphi_i) + g_0S(\varphi_i), \quad a_{22} = U(\varphi_i) + g_0T(\varphi_i), \quad a_{23} = V(\varphi_i) + g_0U(\varphi_i), \quad a_{24} = S(\varphi_i) + g_0V(\varphi_i),$$

$$\varphi_i = k_{i1}L, \quad \delta_i = k_{i1}x_2 = \frac{x_2}{L}\varphi_i, \quad \chi_i = k_{i2}x_2 = \frac{x_2}{L}r_1\varphi_i, \quad \xi_i = k_{i2}x_1 = \frac{x_1}{L}r_1\varphi_i, \quad \eta_i = k_{i3}x_1 = \frac{x_1}{L}\varphi_i,$$

$$r_1 = r^{-0.25}, \quad r_2 = r^{0.5}, \quad r = \frac{I_o}{I}, \quad q_0 = \frac{I_m\varphi_i^3}{\rho FL^3}, \quad g_0 = \frac{m_L\varphi_i}{\rho FL}.$$

In solving the above set of equations by the Gauss method, we preset the C_{i3} factor as for an intact beam. In doing so, we take $M_{i3}(0) = M(0)$ and assume also that the moment of inertia at the boundaries of the portions is equal to I.

The solution to the frequency equation (11) with respect to φ_i allows us to determine the natural frequencies of a cantilever beam with an open crack:

$$\omega_{oi} = \frac{\varphi_i^2}{L^2}\sqrt{\frac{EI}{\rho F}}. \tag{12}$$

If the variation of the bending moment $M(x)$ with the length of an equivalent beam is neglected, then the change of strain energy in the second portion of the equivalent beam is given by the equation

$$\Delta U_2 = \frac{12M^2 d}{Ebh^3}\left[1 - \frac{1}{(1-\gamma)^3}\right], \tag{13}$$

where $\gamma = a / h$.

In a linearly elastic body the strain-energy change due to the presence of a mode-I crack (under the assumption of a plane stress state) is determined by the following expression [20]:

$$\Delta U = \frac{b}{E}\int_0^a K_{\mathrm{I}}^2 da. \tag{14}$$

In this work, we used the stress intensity factor obtained by Cherepanov for a strip with an edge notch [21]:

$$K_{\mathrm{I}} = \frac{4.2M}{bh^{3/2}}\sqrt{(1-\gamma)^{-3} - (1-\gamma)^3}. \tag{15}$$

Since the strain wavelength at the vibration frequencies realized in this work is several orders of magnitude larger than the crack size, the elastic field near the crack can be considered as a quasistatic one [21]. This allows us to neglect the influence of dynamic effects on the stress intensity factor. Substitution of Eq. (15) into (14) gives

$$\Delta U = \frac{4.41M^2}{bh^2 E}(1-\gamma)^{-2}[(1-\gamma)^6 - 3(1-\gamma)^2 + 2]. \tag{16}$$

From the condition $\Delta U_2 = \Delta U$ we determine the parameter d:

$$d = \frac{0.3675h(1-\gamma)}{1-(1-\gamma)^3}[(1-\gamma)^6 - 3(1-\gamma)^2 + 2]. \tag{17}$$

For example, for a beam with a height of the section of $h = 13.8$ mm, the parameter d ranges from 1.7 to 3.8 mm.

Now let us consider the vibration of a beam with a so-called closing edge crack: this crack is open during one half-cycle of vibration and is closed during the other. We also assume that the properties of a beam with a closed crack do not differ from those of an intact beam. Let us preset the following initial conditions for the displacement and velocity of the beam section at a moment of time $t_s = -\pi / 2\omega_{st}$:

$$y_s(x) = w_{st}(x), \quad v_s(x) = 0 \tag{18}$$

(the subscript "*st*" indicates a starting preset natural mode of vibration). Then, in view of the orthogonality condition of the beam vibration

$$\int_0^L m w_{st}(x) w_i(x) dx + m_L w_{st}(L) w_i(L) + I_m \theta_{st}(L) \theta_i(L) = 0, \qquad i \neq st, \tag{19}$$

we find $P_{st} = 1$, $P_{i \neq st} = R_i = 0$ from Eqs. (7) and (8). In this case, expression (2) is simplified to the form

$$y(x,t) = w_{st}(x) \sin \omega_{st} t. \tag{20}$$

During the other vibration half-cycle, when the crack opens, the equation of vibrations takes a form similar to (9) with $y_{oj} = y_{cj}$, $P_{oj} = P_{cj}$, $R_{oj} = R_{cj}$ (the subscript "c" stands for the "closing crack"). Assume that the crack starts opening at a moment of time $t_{s2} = 0$, i.e., when the beam passes through a neutral position. Then, from Eq. (20) we obtain initial conditions for a beam with a closing crack at the moment of crack closure:

$$y_{s2}(x) = 0, \quad v_{s2}(x) = \omega_{st} w_{st}(x). \tag{21}$$

Using expressions (7) and (8), taking into account (21), we obtain $R_{cj} = 0$ and

$$P_{ci} = \frac{\omega_{st}}{\omega_{oi}} \frac{\int_0^{L_c-d} m w_{st}(x) w_{i3}(x) dx + \int_{L_c-d}^{L_c+d} m w_{st}(x) w_{i2}(x) dx + \int_{L_c+d}^{L} m w_{st}(x) w_{i1}(x) dx + G_3}{\int_0^{L_c-d} m w_{i3}^2(x) dx + \int_{L_c-d}^{L_c+d} m w_{i2}^2(x) dx + \int_{L_c+d}^{L} m w_{i1}^2(x) dx + G_4}, \tag{22}$$

Where $G_3 = m_L w_{st}(L) w_{i1}(L) + I_m \theta_{st}(L) \theta_{i1}(L)$ and $G_4 = m_L w_{i1}^2(L) + I_m \theta_{i1}^2(L)$. Then, the solution to the equation of vibrations of the beam during the crack-opening half-cycle takes the form

$$y_{cj}(x,t) = P_{cst} w_{stj}(x) \sin \omega_{ost} t + \sum_{i \neq st}^{\infty} P_{ci} w_{ij}(x) \sin \omega_{oi} t. \tag{23}$$

Analysis of Eqs. (22) and (23) shows that other (satellite) modes which differ from the initial preset mode can arise during the crack opening.

Equations (20) and (23) determine a solution for the first cycle of vibration of the beam. In a similar way, we can construct a solution for the second and subsequent cycles. In this case, the initial conditions for the beam at the moment of crack closure are found from Eq. (23), while those at the moment of crack opening are found from Eq. (2). In the general case, each vibration cycle is described by the appropriate pair of equations (2) and (23).

However, the estimation of the value of the P_{ci} factors for the specimens used in this work (see the last section of the paper) has demonstrated that the amplitudes of the satellite second and third modes did not exceed 3% of the amplitude of the first mode ($st = 1$). This means that for the first vibration mode of the specimens under consideration we can restrict ourselves to finding a solution for the first vibration cycle only.

The natural frequency of the stth vibration mode of a beam with a closing crack is given by the formula [22]

$$\omega_{cst} = \frac{2\omega_{st}\omega_{ost}}{\omega_{st} + \omega_{ost}}. \tag{24}$$

Expressions (20) and (23) represent the waveform of displacement of sections of a beam with a closing crack. They lead to expressions for the determination of the waveform of acceleration of various sections of the beam during different vibration half-cycles:

$$\frac{\partial y^2(x,t)}{\partial t^2} = -\omega_{st}^2 w_{st}(x) \sin \omega_{st} t, \tag{25}$$

$$\frac{\partial y_{cj}^2(x,t)}{\partial t^2} = -\omega_{ost}^2 P_{cst} w_{stj}(x)\sin\omega_{ost}t - \sum_{i\neq st}^{\infty}\omega_{oi}^2 P_{ci} w_{ij}(x)\sin\omega_{oi}t. \tag{26}$$

During the vibration half-cycles, when the crack is open and closed, respectively, the normalized function of the strain distribution along the beam length can represented as

$$\bar{\varepsilon}(x,t) = \overline{M}_{st}(x)\sin\omega_{st}t, \tag{27}$$

$$\bar{\varepsilon}_c(x,t) = f_\varepsilon(x,\gamma)\left(P_{cst}\overline{M}_{stj}(x)\sin\omega_{ost}t + \sum_{i\neq st}^{\infty}P_{ci}\overline{M}_{ij}(x)\sin\omega_{oi}t\right), \tag{28}$$

where the function $f_\varepsilon(x, y)$ allows for the influence of the crack on the strain distribution (the types of this function are shown below). The functions $\overline{M}_{st}(x)$ and $\overline{M}_{ij}(x)$ are the normalized functions of distribution of the bending moment along the beam length for an intact specimen and for a specimen with an open crack, respectively ($\overline{M}_{st}(0) = \overline{M}_{ij}(0) = 1$).

In [9], the function $f_\varepsilon(x, y)$ on the surface of a real beam on the crack side was represented as

$$f_\varepsilon(x,\gamma) = 1 - \exp\left(-\frac{2\alpha|x - L_c|}{h}\right), \tag{29}$$

where it was assumed that $\alpha = 1.276$. The finite-element analysis of the strain distribution near a crack* has revealed a significant dependence of the α factor on the crack depth and has made it possible to represent this dependence by a function of the form $\alpha(\gamma) = 0.683 + 0.152/\gamma$ (for $\gamma = 0$ it is assumed that $f_\varepsilon(x, y) = 1$). Applying this analysis, we obtained also the strain distribution function for the beam surface opposite to the crack:

$$f_\varepsilon(x,\gamma) = 1 + [\beta(\gamma) - 1]\exp\left[-\left(\frac{L_c - x}{\upsilon(\gamma)h}\right)^2 \ln\beta(\gamma)\right], \tag{30}$$

where $\beta(\gamma) = 0.123 + 0.813\exp(\gamma) + 0.064\exp(7\gamma)$, $\upsilon(\gamma) = 0.063 + 0.45\gamma$.

It is seen from Fig. 2 that the intensity of the function $f_\varepsilon(x, y)$ is quite significant near the crack. This means that in measuring the strains in the immediate vicinity of the cracked section one should take into account the influence of both the strain-gauge location L_G and the gauge nominal length G on the strain wave.

Thus, the proposed model of a cracked beam yields relationships between the natural frequencies and modes of vibrations of a cantilever beam and the crack depth and location for both open and closing edge cracks. Moreover, the derived equations make it possible to study the emergence of higher harmonics in the Fourier expansion of the waves of displacement, acceleration, and strain of various beam sections with a closing crack in the *st*th vibration mode:

$$F_c(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\omega_{cst}t + b_n\sin n\omega_{cst}t), \tag{31}$$

where

$$a_n = \frac{\omega_{cst}}{\pi}\left[\int_{-\frac{\pi}{\omega_{st}}}^{0} f(x,t)\cos n\omega_{cst}t\,dt + \int_{0}^{\frac{\pi}{\omega_{ost}}} f_c(x,t)\cos n\omega_{cst}t\,dt\right], \quad n = 1, 2, 3, \ldots, \tag{32}$$

* The stress distributions near a crack were obtained by A. Yu. Chirkov by the finite-element method.

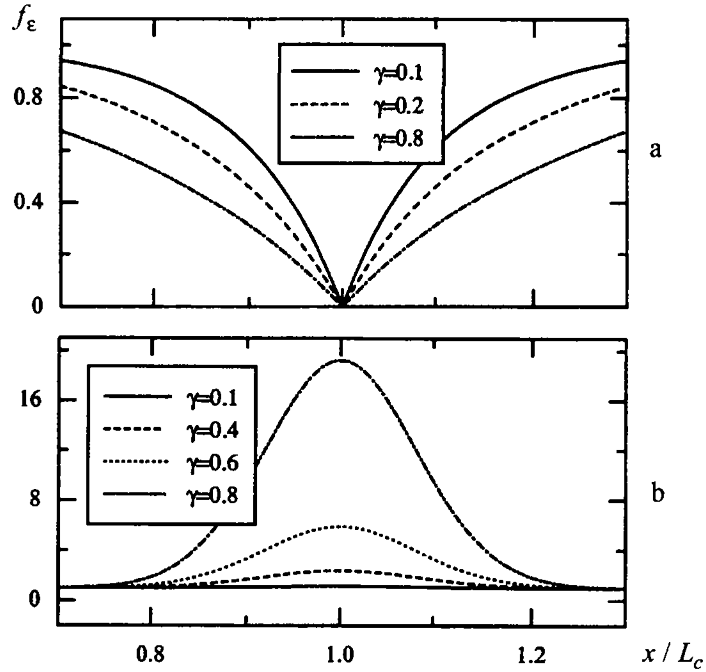

Fig. 2. Strain distribution over the specimen surface ($h = 13.8$ mm) near the crack: a – surface on the cracked side; b – surface opposite to the cracked side.

$$b_n = \frac{\omega_{cst}}{\pi}\left[\int_{-\frac{\pi}{\omega_{st}}}^{0} f(x,t)\sin n\omega_{cst}t\,dt + \int_{0}^{\frac{\pi}{\omega_{ost}}} f_c(x,t)\sin n\omega_{cst}t\,dt\right], \quad n = 1, 2, 3, \ldots . \tag{33}$$

The functions $f(x, t)$ and $f_c(x, t)$ are determined, respectively, by Eqs. (20) and (23) for the displacement wave, by Eqs. (25) and (26) for acceleration wave, and by Eqs. (27) and (28) for the strain wave in the x section of the beam. For the strain wave as measured by a strain gauge these functions are determined through the values of the strain (which is given by Eqs. (27) and (28)) averaged over the gauge nominal length.

Assessment of the Adequacy of the Cracked-Beam Model. The adequacy of determination of the relative change of natural vibration frequency of a cantilever beam with an edge crack was assessed by comparing the calculated data obtained by the proposed analytical approach and the results of testing specimens with an end mass (the geometrical and mechanical characteristics of these specimens are summarized in Table 1 [23, 24]) and the results of calculations of the first-mode natural frequency of a specimen without any end mass ($st = 1$) by the Rayleigh–Ritz method [9].

Figure 3 and Table 2 give several examples of such a comparison for some of the specimens with closing and open cracks. It is seen that there is a good agreement between the calculated and experimental results.

However, analysis of the results of theoretical and experimental determination of the relative change of natural frequency for 15Kh2NMFA steel specimens with a length of $L = 220$ mm has demonstrated that in the case of $\gamma \le 0.5$ the discrepancy between calculated and experimental data is in the range $-6.4 \le \Delta_f \le 2.1\%$. When $\gamma > 0.5$ this discrepancy is as large as -37.8%. The most probable reason for this is that during the tests when the crack growth is quite large, one cannot avoid the generation of considerable plastic strains in the weakened section, and therefore the crack becomes partially or fully open. This is indirectly confirmed by the calculation results for specimens without end mass, which were obtained under the assumption that the crack is open: in this case, the maximum difference between the calculated and experimental results for the $0.5 < \gamma \le 0.87$ did not exceed 8.3%. For specimens of VT-8 titanium alloy with an open crack, the discrepancy between the results of theoretical and experimental determination of the relative change of natural frequency was found to be in the range $-0.6 \le \Delta_f \le 16.8\%$ with $0.05 \le \gamma \le 0.8$.

TABLE 1. Test-Portion Size and Mechanical Properties of Metals

Specimen material	Crack type	L, mm	L_c/L	h, mm	b, mm	m, kg	E, GPa	ρ, kg/m^3
15Kh2NMFA	Closing crack	184	0.011	13.8	4	0.234	200	7800
		220	0.036	13.8	4	0.154	200	7800
		220	0.077	13.8	4	0.259	200	7800
		220	0.136	13.8	4	–	200	7800
		220	0.155	13.8	4	0.154	200	7800
		220	0.277	13.8	4	0.259	200	7800
08Kh18N10	Closing crack	150	0.200	20.0	4	3.520	200	7900
VN-8	Open crack	220	0.091	20.0	4	0.150	110	4480
		220	0.182	20.0	4	–	110	4480
		220	0.518	20.0	4	0.255	110	4480

TABLE 2. Experimental and Calculated Values of the Crack Depth and Relative Change in Natural Frequency

Specimen material	L_c/L	a, mm		Δ_a, %	$f_a/f_{a=0}$		Δ_f, %
		experimental	calculated		experimental	calculated	
15Kh2NMFA (L = 184 mm)	0.011	2.50	2.6	−4.0	0.984	0.985	−0.1
		4.90	5.0	−2.0	0.942	0.944	−0.2
		6.30	6.0	4.8	0.911	0.901	1.1
		7.00	6.7	4.3	0.884	0.872	1.4
		8.30	8.3	0	0.797	0.797	0
		9.90	9.9	0	0.655	0.654	0.2
08Kh18N10	0.200	1.90	3.7	−94.7	0.979	0.995	−1.6
		4.00	5.1	−27.5	0.960	0.976	−1.7
		6.10	6.1	0	0.943	0.943	0
		7.80	7.4	5.1	0.913	0.903	1.1
		10.10	9.7	4.0	0.845	0.830	1.8
		12.00	11.7	2.5	0.765	0.749	2.1
VT-8	0.091	1.00	1.1	−10.0	0.996	0.997	−0.1
		2.00	2.3	−15.0	0.987	0.990	−0.3
		4.00	4.2	−5.0	0.953	0.959	−0.6
		8.00	8.1	−1.3	0.830	0.833	−0.4
		12.00	11.9	0.8	0.629	0.624	0.8
		16.00	15.9	0.6	0.343	0.338	1.5
Steel [24]	0.200	1.33	1.5	−12.8	0.991	0.993	−0.2
		2.53	2.5	0	0.974	0.974	0
		4.00	3.9	2.5	0.933	0.928	0.5
		6.00	5.8	3.3	0.812	0.797	1.8
		8.00	7.9	1.3	0.522	0.510	2.3

The difference between the results of calculation of the first-mode natural frequency of a cantilever beam with an open crack, which was carried out by Shen and Pierre [9] by the Rayleigh–Ritz method, and the experimental results obtained by Wendtland [24] was found to be in the range $-1.2 \le \Delta_f \le 11.9\%$ for $0.133 \le \gamma \le 0.8$. However, Table 2 shows that the approach we proposed provides a more accurate description of the experimental data obtained by Wendtland ($-0.2 \le \Delta_f \le 2.3\%$).

The proposed analytical approach makes it possible also to solve an inverse problem of damage diagnostics. Using the experimental data on the relative change of first-mode resonance frequency of specimens, we calculated the crack-depth values that correspond to these experimental data. It is seen from Table 2 that the relative difference Δ_a of the crack-depth values measured during the tests and those calculated from the relative change of the resonance frequency of specimens is inversely proportional to the crack size. Note that the calculation did not include the

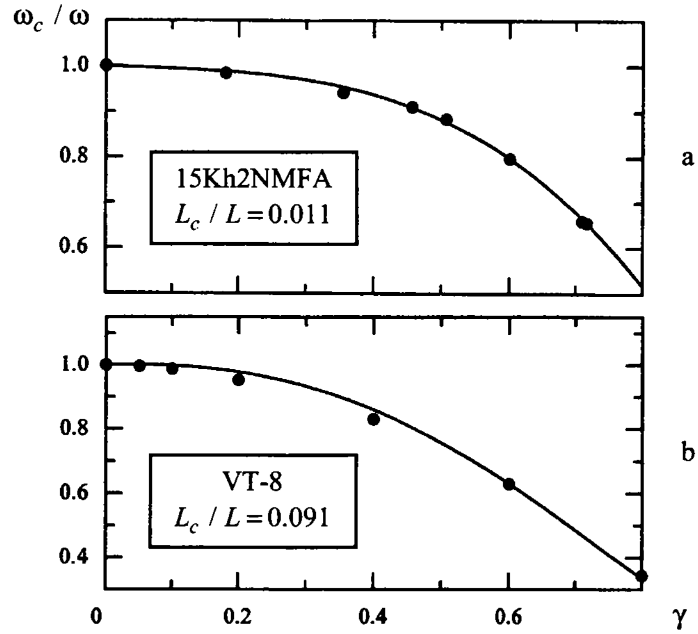

Fig. 3. Influence of the depth of a closing (a) and open (b) crack on the change of natural frequency of specimens (lines and dots indicate the calculated and experimental values, respectively).

crack-front ellipticity, which might be the main factor affecting the discrepancy between the calculation and experimental results in the case of small cracks (i.e., the real crack was somewhat larger than that fixed during the tests). To a certain extent, this explains why in the case of small cracks the calculations give an overestimated crack depth.

Thus, in the determination of absolute values of the crack depth there is a general tendency that the shorter the crack the larger the discrepancy between the calculated and experimental results. However, it is with comparatively short cracks (with $\gamma \leq 0.5$) that the difference between the calculated and measured values of the relative change of natural frequency is the smallest. This becomes evident when we compare the respective values of Δ_a and Δ_f in Table 2. Hence, the high accuracy of the analytical determination of the frequency ratio is not always evidence of the same accuracy of the crack-size determination. This finding supports the conclusions made in [23].

The experimental method of harmonic analysis of the strain wave, as described in [25], was used for revealing higher harmonics in the Fourier expansion of time dependence of strain and acceleration during the first-mode vibration testing of the specimens given in Table 1. To measure strains we used strain gauges with a nominal length $G = 5$ mm, whose location L_G with respect to the beam fixed end and the crack on the test specimens was varied. The acceleration was measured using an accelerometer that was attached to the specimen end.

In analytical determination of higher harmonics of the strain wave, we took into consideration the possibility of generation of the second and third vibration modes of a beam with a closing crack. Note that the influence of these modes on the higher harmonics turned out to be insignificant for the specimens tested.

Figures 4 and 5 demonstrate a good agreement between the experimental and calculated values of the strain wave second-harmonic amplitudes (in Figs. 4–7, the amplitudes of higher harmonics are shown relative to the first harmonic b_1 because in our case the fundamental harmonic is a sinusoid). Note that in the case of 08Kh18N10 steel specimens the strain gauge was placed on the surface on the cracked side, while in the case of 15Kh2NMFA steel specimens it was attached to the side opposite to the cracked one (the L_G / L values are given in the figures). In some cases, the quantitative discrepancy between the experimental and calculated results, e.g., for the zero harmonic during the tests of 08Kh18N10 (Fig. 4), was substantial. However, it should be mentioned that all the calculated functions describe the experimental results qualitatively and that the error of measurement of the zero-harmonic depends on the zero drift of amplifiers and can be significant. The dashed lines in Fig. 5 illustrate how the second-harmonic amplitude responds to a change in the strain-gauge location L_G near the crack by ± 1 mm (in this way we simulated the error of determination of L_G). It is evident that the influence of the L_G measurement error on the higher harmonics is comparatively insignificant. Analytical investigation has demonstrated that this influence decreases with distance from the cracked section and can be neglected.

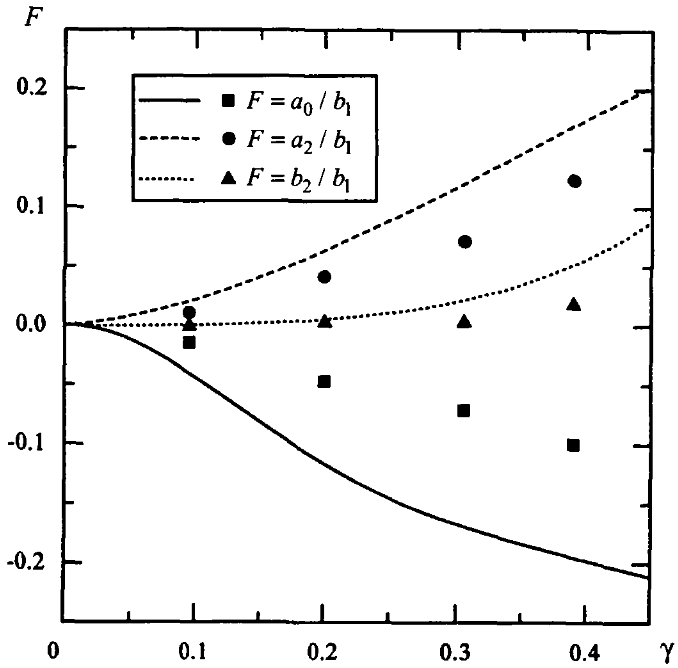

Fig. 4

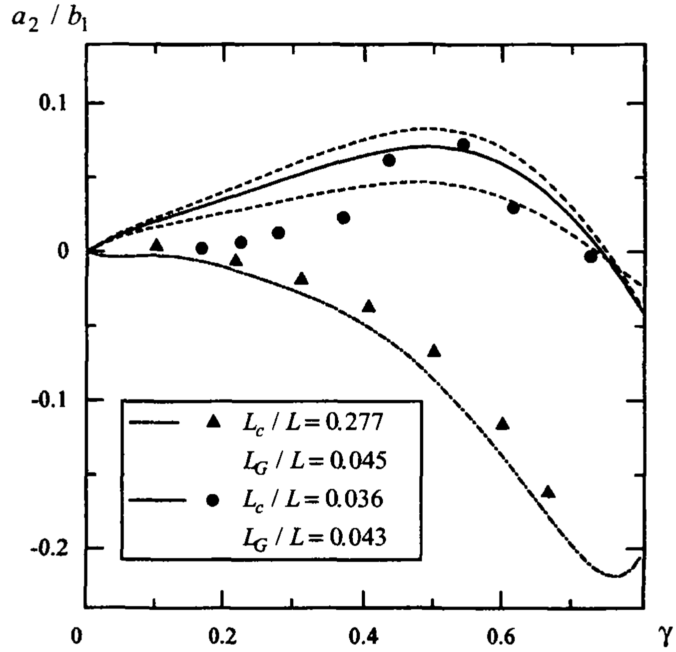

Fig. 5

Fig. 4. Influence of the crack depth on the strain-wave higher-harmonic amplitudes for 08Kh18N10 steel specimen, $L_G / L = 0.283$. (Lines and dots indicate the calculated and experimental values, respectively.)
Fig. 5. Effect of the crack depth on the strain-wave second-harmonic amplitude for 15Kh2NMFA steel specimens. (Solid lines – calculation, dots – experiment, and dashed lines – $L_G / L = 0.041$–0.050.)

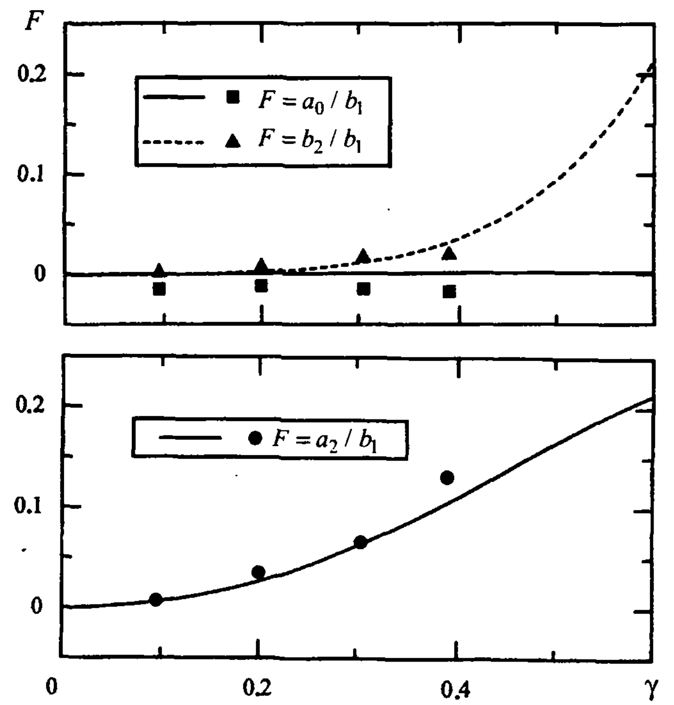

Fig. 6

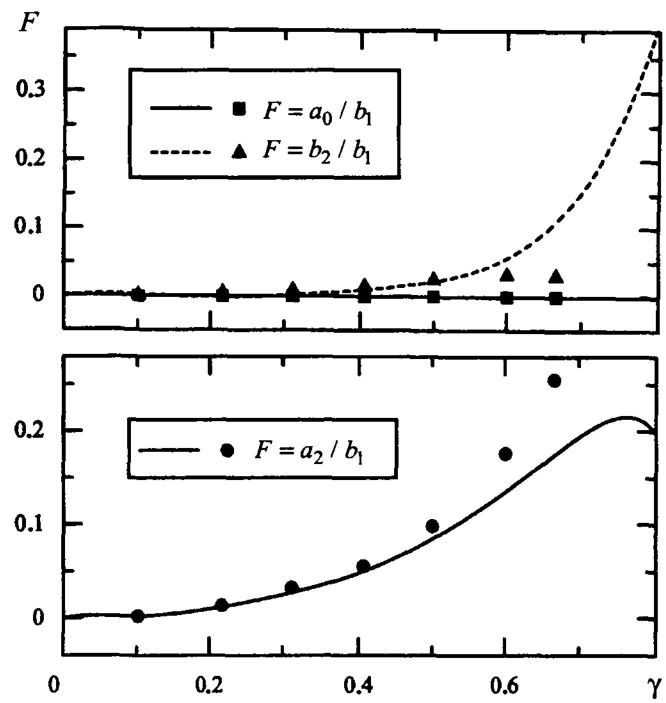

Fig. 7

Fig. 6. Influence of the crack depth on the acceleration-wave higher-harmonic amplitudes for 08Kh18N10 steel specimens. (Lines – calculation; dots – experiment.)
Fig. 7. Effect of the crack depth on the strain wave higher-harmonic amplitude for 15Kh2NMFA steel specimens, $L_c / L = 0.277$ (Lines – calculation; dots – experiment.)

During the analytical determination of higher harmonics of the acceleration wave we took into account also the possibility of generation of the second and third vibration modes of a beam. As is evidenced by Figs. 6 and 7, we have obtained a good agreement between the calculation and experimental results for the second harmonics and have confirmed the small values of zero harmonics in these cases. Similar results were also obtained for other specimens with a closing crack.

For specimens with an open crack (VT-8 alloy) the calculation results are close to experimental ones because the calculation does not reveal either zero or higher harmonics of the strain and acceleration wave. The experimental data have shown also that the amplitudes of these harmonics are close to zero.

The quantitative discrepancy between the results of calculation of the higher harmonics of the strain and acceleration wave and the respective experimental data may be due to the fact that the analytical approach does not include the possible influence of damping, collision of crack faces, crack-front ellipticity, etc., on the higher harmonics, on the one hand, and may be caused by errors of experimental determination of the dynamic characteristics under study, on the other hand.

Conclusions. We have proposed a model of a cantilever beam with an edge crack, which simulates the crack as a portion with a decreased moment of inertia.

We have derived expressions to determine the natural frequencies and modes of vibration of a beam with an open or a closing edge crack. In solving a vibration problem for a beam with a closing edge crack, we obtained equations which make it possible to study the crack-induced distortion of the wave of displacement, acceleration, and strain of various beam sections. This approach allows for the possibility that vibration modes different from the starting preset mode can be generated at the moment of crack opening and includes the special features of strain distribution near the crack.

The testing of the proposed analytical approach on a considerable body of experimental data and finite-element calculation results has demonstrated that this approach provided reliable relationships between various vibration characteristics of a beam and the crack parameters.

REFERENCES

1. P. G. Kirshmer, "The effect of discontinuities on the natural frequency of beams," *Proc. ASTM*, **44**, 897–904 (1944).
2. W. T. Thomson, "Vibration of slender bars with discontinuities in stiffness," *J. Appl. Mech.*, **16**, 203–207 (1949).
3. H. J. Petroski, "Simple static and dynamic models for the cracked elastic beams," *Int. J. Fract.*, **17**, No. 4, R71–R76 (1981).
4. P. Gudmundson, "Eigenfrequency changes of structures due to cracks, notches or other geometrical changes," *J. Mech. Phys. Solids*, **30**, No. 5, 339–353 (1982).
5. V. M. Chushko, "The dependence of natural vibration frequency of a beam on the crack size," in: *Strength of Materials and Structural Elements under Loading at Sonic and Ultrasonic Frequencies* [in Russian], Naukova Dumka, Kiev (1983), pp. 249–253.
6. A. Joshi and B. S. Madhusudhan, "A unified approach to free vibration of locally damaged beams having various homogeneous boundary conditions," *J. Sound Vibration*, **147**, No. 3, 475–488 (1991).
7. W. M. Ostachowich and M. Krawczuk, "Analysis of the effects of cracks on the natural frequencies of a cantilever beam," *J. Sound Vibration*, **150**, No. 2, 191–201 (1991).
8. M. L. Kikidis and C. A. Papadopoulos, "Slenderness ratio effect on cracked beams," *J. Sound Vibration*, **155**, No. 1, 1–11 (1992).
9. M.-H. Shen and C. Pierre, "Free vibrations of beams with a single-edge crack," *J. Sound Vibration*, **170**, No. 2, 237–259 (1994).
10. V. A. Karasev and A. B. Roitman, *Operation Development of Machines. Vibrodiagnostic Methods* [in Russian], Mashinostroenie, Moscow (1986).
11. V. P. Golub, V. P. Butseroga, and A. D. Pogrebnyak, "Investigation of fatigue crack kinetics by the differential compliance method," *Prikl. Mekh.*, **31**, No. 12, 66–74 (1995).

12. N. P. Plakhtienko and S. A. Yasinskii, "On second-order resonance in vibration of a beam with a cross crack," *Probl. Prochn.*, No. 3, 56–63 (1995).
13. V. V. Matveev, "On the efficiency of the method of spectral vibrodiagnostics of the fatigue damage of structural elements. Part 1. Longitudinal vibrations. Analytical solution," *Probl. Prochn.*, No. 6, 5–20 (1997).
14. V. V. Matveev and A. P. Bovsunovskii, "On the efficiency of the method of spectral vibrodiagnostics of the fatigue damage of structural elements. Part 2. Bending vibrations. Analytical solution," *Probl. Prochn..*, No. 6, 9–22 (1998).
15. A. B. Roitman, A. A. Pylov, and N. B. Aleksandrova, "Longitudinal vibrations of a crosscracked cantilever beam. Part 1. Small vibrations," *Probl. Prochn.*, No. 2, 23–34 (1999).
16. A. Ibrahim, F. Ismail, and H. K. Martin, "Modeling of the dynamics of a continuous beam including nonlinear fatigue crack," *Int. J. Anal. Exp. Modal Analysis*, **2**, No. 2, 76–82 (1987).
17. R. D. Adams, P. Cawley, C. J. Pye, and B. J. Stone, "A vibration technique for nondestructively assessing the integrity of structures," *J. Mech. Eng. Sci.*, **20**, No. 2, 93–100 (1978).
18. Ya. G. Panovko, *Fundamentals of Applied Theory of Elastic Vibrations* [in Russian], Mashgiz, Moscow (1957).
19. G. S. Pisarenko, *Vibration of Mechanical Systems Taking into Account the Imperfect Elasticity of the Material* [in Russian], Naukova Dumka, Kiev (1970).
20. J. Rise in: G. Libovitz (Ed.), *Mathematical Techniques in Fracture Mechanics*. Vol. 2. *Fracture* [Russian translation], Mir, Moscow (1975), pp. 204–335.
21. G. P. Cherepanov, *Brittle Fracture Mechanics* [in Russian], Nauka, Moscow (1974).
22. I. I. Blekhman (Ed.), *Vibration in Engineering. Handbook*. Vol. 2. *Vibration of Nonlinear Mechanical Systems* [in Russian], Mashinostroenie, Moscow (1979).
23. A. V. Bovsunovskii, "On determination of the natural frequency of transverse and longitudinal vibration of a cracked beam. Part 2. Experimental and calculation results," *Probl. Prochn.*, No. 3, 45–53 (1999).
24. D. Wendtland, *Änderung der Biegeeigenfrequenzen einer Idealisierten Schaufel durch Risse*. Ph.D. Thesis, University of Karlsruhe (1972).
25. A. P. Bovsunovskii, "Mechanical hysteresis loop shape for metallic material under harmonic stresses below the fatigue limit. Part 2. Experimental technique and results," *Probl. Prochn.*, No. 3, 119–130 (1997).