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Extensions of the Virasoro Group and the Virasoro Algebra by Modules of Tensor Densities on S^1

V. Yu. Ovsienko and C. Roger UDC 513.836

1. The Lie group $\mathrm{Diff}^+(S^1)$ (of all orientation preserving diffeomorphisms of the circle) has a unique (up to isomorphism) nontrivial central extension defined by the Thurston–Bott cocycle [1] $B(\Phi,\Psi) = \int_{S^1}\log((\Phi\circ\Psi)')\,d\log(\Psi')$, where $\Phi,\Psi\in\mathrm{Diff}^+(S^1)$, $\Psi' = d\Psi/dx$, and x is the parameter on S^1. The resulting group is called the *Bott–Virasoro group.*

The corresponding Lie algebra is called the *Virasoro algebra.* It is defined by a unique (up to isomorphism) nontrivial central extension of the Lie algebra $\mathrm{Vect}(S^1)$ (of all smooth vector fields on the circle) by means of the Gelfand–Fuks cocycle [3] $\omega(f,g) = \int_{S^1} f'g''\,dx$, where $f = f(x)\,d/dx$, $g = g(x)\,d/dx$.

2. Denote by $\mathscr{F}_\lambda$ the $\mathrm{Diff}^+(S^1)$- and $\mathrm{Vect}(S^1)$-module of tensor densities of degree λ of the form $a = a(x)(dx)^\lambda$, $a(x)\in C^\infty(S^1)$.

In this paper we study extensions of the group $\mathrm{Diff}^+(S^1)$ and of the Lie algebra $\mathrm{Vect}(S^1)$ by modules of tensor densities on S^1. The obtained Lie groups and Lie algebras are analogs of the Virasoro group and the Virasoro algebra.

We calculate the group $H^2_c(\mathrm{Diff}^+(S^1);\mathscr{F}_\lambda)$ of differentiable cohomology in the sense of Van-Est. This means a classification of the extensions given by differentiable operators. We find four new infinite-dimensional Lie groups and give explicit formulas for nontrivial cocycles on $\mathrm{Diff}^+(S^1)$.

Theorem 1. $H^2_c(\mathrm{Diff}^+(S^1);\mathscr{F}_\lambda) = \begin{cases}\mathbb{R}, & \lambda = 0,1,2,5,7,\\ 0, & \lambda\neq 0,1,2,5,7.\end{cases}$

3. Let us describe explicitly 2-cocycles on $\mathrm{Diff}^+(S^1)$ that define nontrivial cohomology classes. Recall that the mappings

$$l(\Phi) = \log(\Phi'),\qquad dl(\Phi) = \frac{\Phi''}{\Phi'}\,dx,\qquad S(\Phi) = \left[\frac{\Phi'''}{\Phi'} - \frac{3}{2}\left(\frac{\Phi''}{\Phi'}\right)^2\right](dx)^2$$

define nontrivial 1-cocycles on $\mathrm{Diff}^+(S^1)$ with values in $\mathscr{F}_0$, $\mathscr{F}_1$, and $\mathscr{F}_2$, respectively. The cocycle S is the so-called Schwarzian derivative, and dl is the logarithmic derivative. Define an operator $D\colon \mathscr{F}_\lambda\to\mathscr{F}_{\lambda+1}$ by the formula $D(a(x)(dx)^\lambda) = a'(x)(dx)^{\lambda+1}$.

Theorem 2. *The nontrivial* 2*-cocycles*

$$B_1(\Phi,\Psi) = (l(\Phi)\circ\Psi)\cdot dl(\Psi),\qquad B_2(\Phi,\Psi) = (l(\Phi)\circ\Psi)\cdot S(\Psi),$$
$$B_5(\Phi,\Psi) = (S(\Phi)\circ\Psi)\cdot DS(\Psi) - S(\Psi)\cdot D(S(\Phi)\circ\Psi),$$
$$B_7(\Phi,\Psi) = 2\begin{vmatrix} S(\Phi)\circ\Psi & S(\Psi)\\ D^3(S(\Phi)\circ\Psi) & D^3(S(\Psi))\end{vmatrix} - 9\begin{vmatrix} D(S(\Phi)\circ\Psi) & D(S(\Psi))\\ D^2(S(\Phi)\circ\Psi) & D^2(S(\Psi))\end{vmatrix}$$
$$-\frac{32}{3}(S(\Psi) + S(\Phi\circ\Psi))\,B_5(\Phi,\Psi),$$

together with $B_0(\Phi,\Psi) = B(\Phi,\Psi)$ (*the Thurston–Bott cocycle*), *define a basis of the cohomology group* $H^2_c(\mathrm{Diff}^+(S^1);\mathscr{F}_\lambda)$, *where* $\lambda = 0,1,2,5,7$.

Here the last of these cocycles (the Thurston–Bott cocycle) means a constant function on S^1.

C. N. R. S., Centre de Physique Théorique, Marseille; Université Claude Bernard—Lion I. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 30, No. 4, pp. 86–88, October–December, 1996. Original article submitted August 20, 1995; in revised form January 24, 1996.

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4. A classification of nontrivial extensions of the Lie algebra $\mathrm{Vect}(S^1)$ is given by the following Tsujishita assertion [5] (cf. [2, p. 147 of the Russian edition]):

$$H^2(\mathrm{Vect}(S^1);\mathscr{F}_\lambda)=\begin{cases}\mathbb{R}^2, & \lambda=0,1,2,\\ \mathbb{R}, & \lambda=5,7,\\ 0, & \lambda\neq 0,1,2,5,7.\end{cases}$$

Let us describe 2-cocycles on $\mathrm{Vect}(S^1)$ with values in $\mathscr{F}_\lambda$ that represent nontrivial cohomology classes.

Theorem 3. *The following* 8 *nontrivial* 2*-cocycles: the Gelfand–Fuks cocycle* $c_0(f,g)=\omega(f,g)$ *and the cocycles*

$$\bar{c}_0(f,g)=fg'-f'g,\qquad c_1(f,g)=(f'g''-f''g')\,dx,$$
$$\bar{c}_1(f,g)=(fg''-f''g)\,dx,\qquad c_2(f,g)=(f'g'''-f'''g')(dx)^2,$$
$$\bar{c}_2(f,g)=(fg'''-f'''g)(dx)^2,\qquad c_5(f,g)=(f'''g^{(IV)}-f^{(IV)}g''')(dx)^5,$$
$$c_7(f,g)=(2(f'''g^{(VI)}-f^{(VI)}g''')-9(f^{(IV)}g^{(V)}-f^{(V)}g^{(IV)}))(dx)^7$$

form a basis of the cohomology group $H^2(\mathrm{Vect}(S^1);\mathscr{F}_\lambda)$, *where* $\lambda=0,1,2,5,7$.

Remark. 1. The cocycles c_0, c_1, c_2, c_5, c_7 on the Lie algebra $\mathrm{Vect}(S^1)$ correspond to the group cocycles B_0, B_1, B_2, B_5, B_7. The algebra cocycles $\bar{c}_0$, $\bar{c}_1$, $\bar{c}_2$ cannot be integrated to cocycles on the group $\mathrm{Diff}^+(S^1)$.

2. The cocycle c_5 was found in [4].

Denote by $\mathfrak{g}_i$ the Lie algebra given by a nontrivial cocycle c_i and by $\bar{\mathfrak{g}}_i$ the Lie algebra given by the nontrivial cocycle $\bar{c}_i$.

5. We describe the central extensions of the Lie algebras $\bar{\mathfrak{g}}_i$ and $\mathfrak{g}_i$. Each of the Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_5$, and $\mathfrak{g}_7$ and $\bar{\mathfrak{g}}_0$, $\bar{\mathfrak{g}}_1$, and $\bar{\mathfrak{g}}_2$ has a nontrivial central extension given by the cocycle $c((f,a),(g,b))=\omega(f,g)$.

Proposition 1. *There are exactly two other nontrivial central extensions*:

1) *an extension, of the Lie algebra* $\bar{\mathfrak{g}}_1$, *given by the cocycle* $c((f,a),(g,b))=\int_{S^1}(fb-ga)\,dx$;

2) *an extension, of the Lie algebra* $\mathfrak{g}_1$, *given by the cocycle* $c((f,a),(g,b))=\int_{S^1}(f'b-g'a)\,dx$.

Now we consider central extensions of the semi-direct product.

Proposition 2. $H^2(\mathrm{Vect}(S^1)\triangleright\mathscr{F}_\lambda;\mathbb{R})=\begin{cases}\mathbb{R}^3, & \lambda=0,1,\\ \mathbb{R}, & \lambda\neq 0,1.\end{cases}$

For $\lambda=0$ and $\lambda=1$, there exist two nontrivial cocycles that are not equivalent to the extension of the Gelfand–Fuks cocycle. These can be given in the form $c((f,a),(g,b))=\int_{S^1}(f''b-g''a)\,dx$ and $c((f,a),(g,b))=\int_{S^1}(a\,db-b\,da)$, respectively.

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