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Semigroups of Difference Operators in Spectral Analysis of Linear Differential Operators*

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Let X be a complex Banach space, and let $\operatorname{End} X$ be the Banach algebra of bounded linear operators in X. We denote by $\mathscr{F}(\mathbb{R}, X)$ (or simply by $\mathscr{F}$) one of the following four Banach spaces: the space $L_p = L_p(\mathbb{R}, X)$ ($p \in [1,\infty]$) of pth-power integrable (for $p=\infty$, essentially bounded) Bochner measurable functions on $\mathbb{R} = (-\infty, \infty)$ ranging in X ($\|\cdot\|_p$ is the norm in $L_p(\mathbb{R}, X)$), the Banach space $S_p = S_p(\mathbb{R}, X)$ ($p \in [1,\infty)$) of locally pth-power integrable measurable functions on $\mathbb{R}$ ranging in X for which the norm $\|x\|_{S_p} = \sup_{t\in\mathbb{R}}(\int_0^1 \|x(s+t)\|^p ds)^{1/p}$ ($x \in S_p(\mathbb{R}, X)$) is finite, the subspace $C = C(\mathbb{R}, X)$ of continuous functions in $L_\infty(\mathbb{R}, X)$, and the subspace $C_0 = C_0(\mathbb{R}, X) \subset C(\mathbb{R}, X)$ of functions such that $\lim_{|t|\to\infty} \|x(t)\| = 0$ for $x \in C_0(\mathbb{R}, X)$.

We consider a family of evolution operators (a propagator) $\mathscr{U} = \{\mathscr{U}(t,s);\ -\infty < s \le t < \infty\} \subset \operatorname{End} X$; in other words, it is assumed that the following conditions hold:

1) the family $\mathscr{U}$ is strongly continuous on $\Delta = \{(t,s) \in \mathbb{R}^2 : s \le t\}$;
2) $\mathscr{U}(t,s)\mathscr{U}(s,\tau) = \mathscr{U}(t,\tau)$, $-\infty < \tau \le s \le t < \infty$;
3) $\mathscr{U}(t,t) = I$ for any $t \in \mathbb{R}$;
4) $\sup_{0\le t-s\le 1} \|\mathscr{U}(t,s)\| = K < \infty$.

To the family $\mathscr{U}$ we assign a linear operator

$$\mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}_{\mathscr{U}}) \subset \mathscr{F} = \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}.$$

The domain $D(\mathscr{L}_{\mathscr{U}})$ is defined as follows. A function $x \in \mathscr{F}$ belongs to $D(\mathscr{L}_{\mathscr{U}})$ if and only if there exists a function $f \in \mathscr{F}$ such that for almost all $s, t \in \mathbb{R}$ with $s \le t$ we have the relations

$$x(t) = \mathscr{U}(t,s)x(s) - \int_s^t \mathscr{U}(t,\tau) f(\tau)\, d\tau. \qquad (1)$$

In this case we set $\mathscr{L}_{\mathscr{U}} x = f$.

Thus, $\mathscr{L}_{\mathscr{U}} = -d/dt + A(t)\colon D(\mathscr{L}_{\mathscr{U}}) \subset \mathscr{F} \to \mathscr{F}$ is an abstract parabolic operator [1, p. 165 of the Russian edition] provided that $\mathscr{U}$ is the family of evolution operators for the linear differential equation

$$\dot{x}(t) = A(t)x(t), \qquad t \in \mathbb{R},$$

where $A(t)\colon D(A(t)) \subset X \to X$ is a family of closed linear operators that generate a correct Cauchy problem [2].

In the present paper, in the study of the linear operator $\mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}_{\mathscr{U}}) \subset \mathscr{F} \to \mathscr{F} = \mathscr{F}(\mathbb{R}, X)$ we systematically use the semigroup $\{T_{\mathscr{U}}(t),\ t \ge 0\}$ of difference operators belonging to the Banach algebra $\operatorname{End}\mathscr{F}$ and given by

$$(T_{\mathscr{U}}(t)x)(s) = \mathscr{U}(s, s-t)x(s-t), \qquad x \in \mathscr{F},\ s \in \mathbb{R},\ t \ge 0. \qquad (2)$$

The main results of the paper are related to the following four theorems.

Theorem 1. *The operator $\mathscr{L}_{\mathscr{U}}$ is the infinitesimal generator of the strongly continuous operator semigroup $\{T_{\mathscr{U}}(t),\ t \ge 0\}$ in any of the Banach spaces $L_p = L_p(\mathbb{R}, X)$, $p \in [1,\infty)$, and $C_0 = C_0(\mathbb{R}, X)$.*

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Theorem 2 (spectral mapping theorem). *The spectrum $\sigma(\mathscr{L}_{\mathscr{U}})$ of the operator $\mathscr{L}_{\mathscr{U}}$ and the spectrum $\{\sigma(T_{\mathscr{U}}(t))\}$ of the difference operator $T_{\mathscr{U}}$, $t>0$, are related as follows:*

$$\sigma(T_{\mathscr{U}}(t))\setminus\{0\}=\exp\sigma(\mathscr{L}_{\mathscr{U}})t=\{\exp\lambda t:\lambda\in\sigma(\mathscr{L}_{\mathscr{U}})\}.$$

In particular, the operator $\mathscr{L}_{\mathscr{U}}$ is invertible if and only if the difference operator $\mathscr{D}_{\mathscr{U}}=I-T_{\mathscr{U}}(1)$, which has the form

$$(\mathscr{D}_{\mathscr{U}}x)(s)=x(s)-\mathscr{U}(s,s-1)x(s-1),\qquad x\in\mathscr{F},\ s\in\mathbb{R},\tag{3}$$

is invertible.

Note that the assertion of Theorem 2 is not valid for arbitrary strongly continuous operator semigroups (see [3, p. 676 of the Russian translation]).

For a Banach space X, we denote by $\mathscr{F}(\mathbb{Z},X)$ one of the following Banach spaces of two-sided sequences in X:

$$l_p=l_p(\mathbb{Z},X)=\left\{x:\mathbb{Z}\to X\;\Big|\;\|x\|_p=\Big(\sum_{n\in\mathbb{Z}}\|x(n)\|^p\Big)^{1/p}<\infty\right\},\qquad p\in[1,\infty),$$

$$l_\infty=l_\infty(\mathbb{Z},X)=\left\{x:\mathbb{Z}\to X\;\Big|\;\|x\|_\infty=\sup_{n\in\mathbb{Z}}\|x(n)\|<\infty\right\},$$

$$c_0=C_0(\mathbb{Z},X)=\left\{x\in l_\infty\;\Big|\;\lim_{n\to\infty}\|x(n)\|=0\right\}.$$

In the following theorem we use a pair $(\mathscr{F}(\mathbb{R},X),\mathscr{F}(\mathbb{Z},X))$ of Banach spaces, which can be one of the pairs (S_p,l_∞), (L_p,l_p), where $p\in[1,\infty)$, (C,l_∞), and (C_0,c_0).

Theorem 3. *The linear operator $\mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}_{\mathscr{U}})\subset\mathscr{F}(\mathbb{R},X)\to\mathscr{F}(\mathbb{R},X)$ is invertible if and only if the difference operator $\mathscr{D}_0\colon\mathscr{F}(\mathbb{Z},X)\to\mathscr{F}(\mathbb{Z},X)$ defined by the relations*

$$(\mathscr{D}_0x)(n)=x(n)-\mathscr{U}(n,n-1)x(n-1),\qquad x\in\mathscr{F}(\mathbb{Z},X),\ n\in\mathbb{Z},\tag{4}$$

is invertible.

We say that a family of evolution operators $\{\mathscr{U}(t,s),\ s\le t\}$ from the algebra $\operatorname{End}X$ admits an *exponential dichotomy on $\mathbb{R}$ with exponent $\beta>0$ and coefficient $M>0$* if there exists a bounded strongly continuous projection-valued function $P\colon\mathbb{R}\to\operatorname{End}X$ such that 1) $\mathscr{U}(t,s)P(s)=P(t)\mathscr{U}(t,s)$ for $t\ge s$, $t,s\in\mathbb{R}$; 2) $\|\mathscr{U}(t,s)P(s)\|\le M\exp(-\beta(t-s))$ for $t\ge s$; 3) for $t\ge s$, the restriction $\mathscr{U}(t,s)|\operatorname{Im}Q(s)$ of the operator $\mathscr{U}(t,s)$ to the range $\operatorname{Im}Q(s)$ of the projection $Q(s)=I-P(s)$ (here and in the following I stands for the identity operator) is an isomorphism of the subspaces $\operatorname{Im}Q(s)$ and $\operatorname{Im}Q(t)$ (and we define the operator $\mathscr{U}(s,t)$ as the inverse mapping from $\operatorname{Im}Q(t)$ into $\operatorname{Im}Q(s)$); 4) $\|\mathscr{U}(t,s)Q(s)\|\le M\exp\beta(t-s)$ for $s\ge t$ (the norms are taken in $\operatorname{End}X$, and the operator $\mathscr{U}(t,s)Q(s)$ is regarded as an element of $\operatorname{End}X$).

If $P=0$ or $Q=0$, then we say that for $\mathscr{U}$ we have a *trivial* dichotomy.

Theorem 4. *The following assertions hold.*

1) *The linear operator $\mathscr{L}_{\mathscr{U}}$: $D(\mathscr{L}_{\mathscr{U}})\subset\mathscr{F}\to\mathscr{F}$ is invertible if and only if the family $\mathscr{U}$ admits an exponential dichotomy.*

2) *The spectrum $\sigma(\mathscr{L}_{\mathscr{U}})$ of the operator $\mathscr{L}_{\mathscr{U}}$ does not depend on the choice of the space $\mathscr{F}(\mathbb{R},X)$.*

These results were announced in [4]. Difference operators of the form (3)–(4) were used to study differential equations in [1, 5–9].

§1. The Proof of the Main Statements

In the proof of many of our results, the following statement is useful (which is a consequence of (1)):

Remark 1. For any $\lambda\in\mathbb{C}$, the operator $\mathscr{L}_{\mathscr{U}}+\lambda I$ can be represented in the form $\mathscr{L}_{\mathscr{U}}+\lambda I=\mathscr{L}_{\mathscr{U}(\lambda)}$, where the family of evolution operators $\mathscr{U}(\lambda)$ from the algebra $\operatorname{End}X$ has the form $\mathscr{U}(\lambda)(t,s)=\exp\lambda(t-s)\mathscr{U}(t,s)$, $s\le t$.

It readily follows from the definition of an exponential dichotomy of the family $\mathscr{U}$ that the following assertion holds (see [1, p. 167 of the Russian edition]).

Lemma 1. *Let a family $\mathscr{U}$ admit an exponential dichotomy on $\mathbb{R}$. Then the corresponding linear operator $\mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}_{\mathscr{U}}) \subset \mathscr{F} = \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}$ is (continuously) invertible, and the inverse $\mathscr{L}_{\mathscr{U}}^{-1} \in \operatorname{End}\mathscr{F}$ is given by the formula*

$$(\mathscr{L}_{\mathscr{U}}^{-1} f)(t) = \int_{\mathbb{R}} G(t, s) f(s)\, ds, \qquad f \in \mathscr{F},\ t \in \mathbb{R}, \tag{5}$$

where the Green function $G\colon \mathbb{R}^2 \to \operatorname{End} X$ has the form

$$G(t, s) = \begin{cases} -\mathscr{U}(t, s) P(s), & s \le t, \\ \mathscr{U}(t, s) Q(s), & s > t. \end{cases} \tag{6}$$

Note that for $\mathscr{F} = C$ or C_0, the equivalence between the conditions that $\mathscr{L}_{\mathscr{U}}$ is invertible and that $\mathscr{U}$ admits an exponential dichotomy was established in [1, Chap. X].

Corollary. *There exists an $\alpha \in \mathbb{R}$ such that the operator $\mathscr{L}_{\mathscr{U}} - \alpha I\colon D(\mathscr{L}_{\mathscr{U}}) \subset \mathscr{F} \to \mathscr{F}$ is invertible.*

To prove this assertion, it suffices to note that properties 3) and 4) of the family $\mathscr{U}$ imply the existence of constants $C > 0$ and $\beta_0 \in \mathbb{R}$ such that $\|\mathscr{U}(t, s)\| \le C \exp \beta_0 (t - s)$ for $s \le t$, $s, t \in \mathbb{R}$. If $\alpha > \beta_0$, then it follows from Remark 1 that $\mathscr{L}_{\mathscr{U}} - \alpha I = \mathscr{L}_{\mathscr{U}(-\alpha)}$. For the family $\mathscr{U}(-\alpha)$, we have a trivial dichotomy (since $\|\mathscr{U}(-\alpha)(t, s)\| \le C \exp(\beta_0 - \alpha)(t - s)$, $s \le t$), and hence, the operator $\mathscr{L}_{\mathscr{U}} - \alpha I$ is invertible.

Proof of Theorem 1. Let $\mathscr{A}$ be the infinitesimal generator of the strongly continuous operator semigroup $\{T_{\mathscr{U}}(t),\ t \ge 0\}$ defined in (2). It suffices to prove that, for some $\alpha \in \mathbb{R}$ that belongs to the intersection $\rho(\mathscr{A}) \cap \rho(\mathscr{L}_{\mathscr{U}})$ of the resolvent sets $\rho(\mathscr{A})$ and $\rho(\mathscr{L}_{\mathscr{U}})$ of the operators $\mathscr{A}$ and $\mathscr{L}_{\mathscr{U}}$, the operators $(\mathscr{A} - \alpha I)^{-1}$ and $(\mathscr{L}_{\mathscr{U}} - \alpha I)^{-1}$ coincide. The existence of such a number α follows from the corollary to Lemma 1 and from the fact that $\mathscr{A}$ is the infinitesimal generator of a strongly continuous operator semigroup [3]. We can always consider the operators $\mathscr{A} - \alpha I$ and $\mathscr{L}_{\mathscr{U}} - \alpha I$ instead of $\mathscr{A}$ and $\mathscr{L}_{\mathscr{U}}$, and so without loss of generality we can assume that $\alpha = 0$; in this case, we have $\|T_{\mathscr{U}}(t)\| = \sup_{s \in \mathbb{R}} \|\mathscr{U}(s, s - t)\| \le \operatorname{Const} \exp(-\gamma t)$, $t \ge 0$, for some $\gamma > 0$. Hence, for the family $\mathscr{U}$ we have a trivial dichotomy, and it follows from Lemma 1 that we have the representation

$$(\mathscr{L}_{\mathscr{U}}^{-1} f)(s) = -\int_{-\infty}^{s} \mathscr{U}(s, \tau) f(\tau)\, d\tau, \qquad f \in \mathscr{F}.$$

On the other hand, for the operator $\mathscr{A}^{-1}$, the following relations hold (see [3, p. 354 of the Russian translation]):

$$\begin{aligned} (\mathscr{A}^{-1} f)(s) &= -\int_0^{\infty} (T_{\mathscr{U}}(t) f)(s)\, dt = -\int_0^{\infty} \mathscr{U}(s, s - t) f(s - t)\, dt \\ &= -\int_{-\infty}^{s} \mathscr{U}(s, \tau) f(\tau)\, d\tau = (\mathscr{L}_{\mathscr{U}}^{-1} f)(s), \qquad f \in \mathscr{F}. \end{aligned}$$

This completes the proof of the theorem.

In the following two lemmas, the pair $(\mathscr{F}(\mathbb{R}, X), \mathscr{F}(\mathbb{Z}, X))$ of Banach spaces is the same as in Theorem 3.

Lemma 2. *If the operator $\mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}_{\mathscr{U}}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ is invertible, then the difference operator $\mathscr{D}_0$ (see formula(4)) is also invertible, and we have the estimate*

$$\|\mathscr{D}_0^{-1}\| \le 1 + 3K + \tfrac{9}{2} K^2 \|\mathscr{L}_{\mathscr{U}}^{-1}\|, \qquad K = \sup_{0 \le t - s \le 1} \|\mathscr{U}(t, s)\|.$$

Proof. Suppose that the operator $\mathscr{L}_{\mathscr{U}}$ is invertible. Let us prove that the operator $\mathscr{D}_0$ is also invertible and the inverse has the form

$$\mathscr{D}_0^{-1} x = \tilde{y} + x, \qquad x \in \mathscr{F}(\mathbb{Z}, X),$$

where $\tilde{y}$ is the restriction of the function $y = \mathcal{L}_{\mathcal{U}}^{-1} Bx \in C(\mathbb{R}, X)$ to $\mathbb{Z}$ (the inclusion $D(\mathcal{L}_{\mathcal{U}}) \subset C(\mathbb{R}, X)$ follows from (1)) and the linear operator $B \colon \mathcal{F}(\mathbb{Z}, X) \to \mathcal{F}(\mathbb{R}, X)$ is defined by the formulas

$$(Bx)(s) = -\varphi(s)\mathcal{U}(s, n-1)x(n-1), \quad x \in \mathcal{F}(\mathbb{Z}, X),\ n \in \mathbb{Z},\ s \in [n-1, n].$$

Here $\varphi \colon \mathbb{R} \to \mathbb{R}$ is a 1-periodic function such that $\varphi(s) = 6s(1-s)$ for $s \in [0, 1]$. The operator B is well defined and bounded, and $\|B\| \le \frac{3}{2}K$.

First, we prove that the operator $\mathcal{D}_0$ is injective. If $x_0 \in \operatorname{Ker} \mathcal{D}_0 = \{x \in \mathcal{F}(\mathbb{Z}, X) : \mathcal{D}_0 x = 0\}$, then $x_0(n) = \mathcal{U}(n, n-1)x_0(n-1)$ for any $n \in \mathbb{Z}$, and the function $x \in \mathcal{F}(\mathbb{R}, X)$ defined by the relations

$$x(t) = \mathcal{U}(t, n)x_0(n), \qquad t \in [n, n+1],\ n \in \mathbb{Z},$$

belongs to the kernel $\operatorname{Ker} \mathcal{L}_{\mathcal{U}}$ of the operator $\mathcal{L}_{\mathcal{U}}$. Therefore, $x = 0$, and hence, $x_0 = 0$.

Let us prove that the operator $\mathcal{D}_0$ is surjective. Let $g \in \mathcal{F}(\mathbb{Z}, X)$, $f = Bg \in \mathcal{F}(\mathbb{R}, X)$, and $x = \mathcal{L}^{-1} f \in D(\mathcal{L}) \subset C(\mathbb{R}, X)$. Then relations (1) imply

$$\begin{aligned} x(n) &= \mathcal{U}(n, n-1)x(n-1) + \int_{n-1}^{n} \varphi(s)\mathcal{U}(n, s)\mathcal{U}(s, n-1)g(n-1)\, ds \\ &= \mathcal{U}(n, n-1)x(n-1) + \mathcal{U}(n, n-1)g(n-1), \qquad n \in \mathbb{Z}. \end{aligned}$$

Hence, the relation $\mathcal{D}_0(\tilde{x} + g) = g$ holds provided that $\tilde{x} \in \mathcal{F}(\mathbb{Z}, X)$. Equations (1) imply the estimates

$$\|\tilde{x}(n)\| = \|x(n)\| \le \tfrac{3}{2}K(\|x(s)\| + \|g(n-1)\|), \qquad s \in [n-1, n],\ n \in \mathbb{Z}. \tag{7}$$

Therefore, $\tilde{x} \in l_\infty(\mathbb{Z}, X)$ for $\mathcal{F} = L_\infty$ and for $\mathcal{F} = C$, whereas $\tilde{x} \in C_0(\mathbb{Z}, X)$ for $\mathcal{F} = C_0(\mathbb{R}, X)$. It follows from (7) that

$$\|\tilde{x}\|_\infty = \sup_{n \in \mathbb{Z}} \|x(n)\| \le \tfrac{3}{2}K(\|x\|_\infty + \|g\|_\infty). \tag{8}$$

Now we assume that either $\mathcal{F}(\mathbb{R}, X) = L_p(\mathbb{R}, X)$, $p \in [1, \infty)$, or $\mathcal{F}(\mathbb{R}, X) = S_p(\mathbb{R}, X)$, $p \in [1, \infty)$. Since $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for any $a, b \ge 0$, we can integrate the pth powers of both sides of inequality (7) over the interval $[n, n+1]$ and obtain

$$\|x(n)\|^p \le \tfrac{1}{2}(3K)^p \left(\int_{n-1}^{n} \|x(s)\|^p\, ds + \|g(n-1)\|^p \right), \qquad n \in \mathbb{Z}. \tag{9}$$

Therefore, for $\mathcal{F}(\mathbb{R}, X) = L_p(\mathbb{R}, X)$ we have the estimate

$$\|\tilde{x}\|_p \le 3K(\|x\|_p + \|g\|_p). \tag{10}$$

If $\mathcal{F}(\mathbb{R}, X) = S_p(\mathbb{R}, X)$, then inequality (9) yields

$$\|\tilde{x}\|_\infty \le 3K(\|x\|_{S_p} + \|g\|_\infty). \tag{11}$$

Thus, it follows from estimates (7)–(11) that for any choice of the space $\mathcal{F}(\mathbb{R}, X)$, the function $\tilde{x}$ belongs to the space $\mathcal{F}(\mathbb{Z}, X)$. Hence, the operator $\mathcal{D}_0$ is invertible, we have $\tilde{x} + g = \mathcal{D}_0^{-1} g$, and the above estimates imply

$$\begin{aligned} \|\mathcal{D}_0^{-1} g\| &\le \|\tilde{x}\| + \|g\| \le 3K(\|x\|_{\mathcal{F}} + \|g\|) + \|g\| \\ &\le 3K(\|\mathcal{L}^{-1}\|\,\|B\|\,\|g\| + \|g\|) + \|g\| = (1 + 3K + \tfrac{9}{2}K^2\|\mathcal{L}^{-1}\|)\,\|g\|. \end{aligned}$$

This proves the lemma.

Lemma 3. *If the difference operator $\mathcal{D}_0 \in \operatorname{End} \mathcal{F}(\mathbb{Z}, X)$ is invertible, then the operator $\mathcal{L}_{\mathcal{U}} \colon D(\mathcal{L}_{\mathcal{U}}) \subset \mathcal{F} \to \mathcal{F} = \mathcal{F}(\mathbb{R}, X)$ is also invertible.*

Proof. Assume that the operator $\mathcal{D}_0$ (see (4)) is invertible. Since it is injective, it follows that the operator $\mathcal{L}_{\mathcal{U}}$ is also injective (if $\mathcal{L}_{\mathcal{U}} x = 0$, then the restriction $\tilde{x}$ of the function x to $\mathbb{Z}$ belongs to $\operatorname{Ker} \mathcal{D}_0$; hence, it follows from the relations $x(t) = \mathcal{U}(t, n)x(n) = 0$ for any $t \in [n, n+1]$ and $n \in \mathbb{Z}$ that $x = 0$).

Let us prove that the operator $\mathcal{L}_\mathcal{U}$ is surjective. For an arbitrary function $f \in \mathcal{F}$, we consider the function $f_d \in \mathcal{F}(\mathbb{Z}, X)$ of the form $f_d(n) = -\int_n^{n+1} \mathcal{U}(n,\tau) f(\tau)\, d\tau$, $n \in \mathbb{Z}$. Then there exists a function $x_0 \in \mathcal{F}(\mathbb{Z}, X)$ such that $\mathcal{D}x_0 = f_d$. We can readily verify that the function $x \in \mathcal{F}$ defined on any interval $[n, n+1]$ $(n \in \mathbb{Z})$ by the formula

$$x(s) = \mathcal{U}(s,n)x_0(n) - \int_n^s \mathcal{U}(s,\tau) f(\tau)\, d\tau, \qquad s \in [n, n+1],$$

belongs to $D(\mathcal{L}_\mathcal{U})$ and satisfies $\mathcal{L}_\mathcal{U} x = f$ (that is, relations (1) hold). This proves the lemma.

Proof of Theorem 3. Theorem 3 immediately follows from Lemmas 2 and 3.

Proof of Theorem 4. The sufficiency of the condition that the family $\mathcal{U}$ admits an exponential dichotomy was established in Lemma 1.

Now we assume that the operator $\mathcal{L}_\mathcal{U}$ is invertible. By Theorem 3, the corresponding difference operator $\mathcal{D}_0 \in \operatorname{End} \mathcal{F}(\mathbb{Z}, X)$ is also invertible. It follows from [11, Theorem 3] that the operator $\mathcal{D}_0$ is invertible in the Banach space $l_\infty(\mathbb{Z}, X)$. On applying Theorem 3 once more, we see that the operator $\mathcal{L}_\mathcal{U}$ is invertible in the space $C(\mathbb{R}, X)$, and hence it follows from the results of [1, Chap. X] that the family $\mathcal{U}$ admits an exponential dichotomy. This proves the theorem.

Remark 2. We can give another proof of the necessity of the assumption in Theorem 4 by considering the family of invertible operators $\{S(\alpha)\mathcal{L}_\mathcal{U} S(-\alpha);\ \alpha \in \mathbb{R}\}$, where $\{S(\alpha);\ \alpha \in \mathbb{R}\}$ is the group of isometric operators given by translations of the functions from $\mathcal{F}(\mathbb{R}, X)$ $((S(\alpha)x)(t) = x(t+\alpha),\ x \in \mathcal{F})$. It follows from Lemma 2 that the corresponding family of difference operators

$$(\mathcal{D}_\alpha x)(n) = x(n) - \mathcal{U}(n+\alpha, n+\alpha-1)x(n-1), \qquad \alpha \in \mathbb{R},\ x \in \mathcal{F}(\mathbb{Z}, X),$$

is uniformly invertible. Hence, these operators admit a discrete exponential dichotomy on $\mathbb{Z}$ (see [6, p. 250 of the Russian translation]). An exponential dichotomy of the family $\mathcal{U}$ follows from [6, p. 251 of the Russian translation].

Proof of Theorem 2. It follows from Theorem 3 in [11] that it suffices to carry out the proof for the case in which $\mathcal{F}(\mathbb{R}, X)$ is one of the Banach spaces $L_p(\mathbb{R}, X)$, $p \in [1, \infty)$, or $C_0(\mathbb{R}, X)$, on each of which the semigroup $\{T_\mathcal{U}(t),\ t \geq 0\}$ is strongly continuous.

The inclusion $\exp \sigma(\mathcal{L}_\mathcal{U})t \subset \sigma(T_\mathcal{U}(t)) \setminus \{0\}$ is known (this holds for an arbitrary strongly continuous semigroup of linear operators [3, Chap. XVI]). The opposite inclusion follows from the formula

$$((T_\mathcal{U}(t) - I)^{-1}x)(s) = \sum_{k \in \mathbb{Z}} G(s, s+kt)x(s+kt), \qquad t > 0, \tag{12}$$

which holds if the operator $\mathcal{L}_\mathcal{U}$ is invertible (here G is the Green function defined by formula (6)). Indeed, for any $x \in \mathcal{F}(\mathbb{R}, X)$, $s \in \mathbb{R}$, and $t > 0$ we have

$$\begin{aligned}
&(T_\mathcal{U}(t) - I)\Big(\sum_{k \in \mathbb{Z}} G(s, s+kt)x(s+kt)\Big) \\
&= \mathcal{U}(s, s-t)\Big[-\sum_{k \leq 0} \mathcal{U}(s-t, s+(k-1)t)P(s+(k-1)t)x(s+(k-1)t) \\
&\qquad + \sum_{k \geq 1} \mathcal{U}(s-t, s+(k-1)t)Q(s+(k-1)t)x(s+(k-1)t)\Big] \\
&\quad + \sum_{k \leq 0} \mathcal{U}(s, s+kt)P(s+kt)x(s+kt) - \sum_{k \geq 1} \mathcal{U}(s, s+kt)Q(s+kt)x(s+kt) \\
&= \mathcal{U}(s,s)(P(s) + Q(s))x(s) = x(s).
\end{aligned}$$

We can verify in a similar manner that the operator defined in (12) is a left inverse of $T_\mathcal{U}(t) - I$ as well. Thus, we have proved that if $0 \notin \sigma(\mathcal{L}_\mathcal{U})$, then $1 \notin \sigma(T_\mathcal{U}(t))$ for any $t > 0$. The case $\lambda \notin \sigma(\mathcal{L}_\mathcal{U})$ can be reduced to the case just considered by means of the family $\mathscr{U}(\lambda)$ (see Remark 1) and the operator $\mathscr{L}_{\mathscr{U}(\lambda)} = \mathscr{L}_{\mathscr{U}} - \lambda I$ and by applying the fact that $\mathscr{L}_{\mathscr{U}(\lambda)}$ is the infinitesimal generator of the operator semigroup $\{T_{\mathscr{U}}(t)\exp(-\lambda t),\ t \geq 0\}$. The second assertion of the theorem follows from the first assertion and from Lemma 1. This proves the theorem.

Corollary. *If the operator $\mathscr{L}_{\mathscr{U}}$ is invertible, then the operator $\mathscr{D}_0^{-1} \in \operatorname{End}\mathscr{F}(\mathbb{Z},X)$ has the form*

$$(\mathscr{D}_0^{-1}x)(n) = \sum_{m\in\mathbb{Z}} G(n,m)x(m), \qquad x \in \mathscr{F}(\mathbb{Z},X),\ n \in \mathbb{Z}. \tag{13}$$

Theorem 5. *The spectrum $\sigma(\mathscr{L}_{\mathscr{U}}) \subset \mathbb{C}$ of the operator $\mathscr{L}_{\mathscr{U}}$: $D(\mathscr{L}_{\mathscr{U}}) \subset \mathscr{F} \to \mathscr{F}$ is the union of a set of lines parallel to the imaginary axis $i\mathbb{R}$ and contains the line*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \chi_+(\mathscr{U})\}, \qquad \chi_+(\mathscr{U}) = \lim_{\tau\to\infty} \frac{1}{\tau} \ln \sup_{s\in\mathbb{R}} \|\mathscr{U}(s, s-\tau)\|;$$

moreover, $\sup_{\lambda\in\sigma(\mathscr{L}_{\mathscr{U}})} \operatorname{Re}\lambda = \chi_+(\mathscr{U})$. *If the dimension $\dim X$ of the Banach space X is finite, then $\sigma(\mathscr{L}_{\mathscr{U}})$ has the representation*

$$\sigma(\mathscr{L}_{\mathscr{U}}) = \bigcup_{k=1}^{m} \{\lambda \in \mathbb{C} : \alpha_k \leq \operatorname{Re}\lambda \leq \beta_k\}, \tag{14}$$

where $-\infty \leq \alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \dots < \alpha_m \leq \beta_m = \chi_+(\mathscr{U}) < \infty$ *and* $m \leq \dim X$.

In particular, the spectrum of the scalar differential operator $\mathscr{L} = -d/dt + a(t)\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R},\mathbb{C}) \to \mathscr{F}(\mathbb{R},\mathbb{C})$, where $a \in S_1(\mathbb{R},\mathbb{C})$, coincides with the set

$$\left\{\lambda \in \mathbb{C} : \lim_{\tau\to\infty} \frac{1}{\tau} \inf_{s\in\mathbb{R}} \int_s^{s+\tau} \operatorname{Re} a(\alpha)\,d\alpha \leq \operatorname{Re}\lambda \leq \lim_{\tau\to\infty} \frac{1}{\tau} \sup_{s\in\mathbb{R}} \int_s^{s+\tau} \operatorname{Re} a(\alpha)\,d\alpha \right\}. \tag{15}$$

Proof. By Theorem 2, it suffices to consider the difference operator $T_{\mathscr{U}}(1) \in \operatorname{End}\mathscr{F}(\mathbb{R},X)$ and obtain the corresponding assertions for its spectrum (treated as a union of concentric circles). Consider the group of isometric linear operators $\{V(\lambda),\ \lambda \in \mathbb{R}\}$ from the Banach algebra $\operatorname{End}\mathscr{F}(\mathbb{R},X)$ that have the form

$$(V(\lambda)x)(t) = (\exp i\lambda t)x(t), \qquad x \in \mathscr{F}(\mathbb{R},X),\ t, \lambda \in \mathbb{R}.$$

It follows from the relations $V(\lambda)T_{\mathscr{U}}(1)V(-\lambda) = (\exp i\lambda)T_{\mathscr{U}}(1)$, $\lambda \in \mathbb{R}$, which mean that the operator $T_{\mathscr{U}}(1)$ is similar to the operators $\gamma T_{\mathscr{U}}(1)$, $\gamma \in \mathbb{T}$, that the spectrum $\sigma(T_{\mathscr{U}}(1))$ of the operator $T_{\mathscr{U}}(1)$ is the union of a family of circles centered at 0.

Assume that some circle $S(\alpha) = \{z \in \mathbb{C} : |z| = \alpha\}$, $\alpha > 0$, belongs to the resolvent set $\rho(T_{\mathscr{U}}(1))$ of the operator $T_{\mathscr{U}}(1)$, and let $\sigma(T_{\mathscr{U}}(1))$ be the union of two spectral sets

$$\sigma_+ = \{\lambda \in \sigma(T_{\mathscr{U}}(1)) : |\lambda| > \alpha\}, \qquad \sigma_- = \{\lambda \in \sigma(T_{\mathscr{U}}(1)) : |\lambda| < \alpha\}.$$

Let $P(\sigma_+)$ and $P(\sigma_-)$ be the corresponding Riesz projections ($I = P(\sigma_+) + P(\sigma_-)$).

Without loss of generality we can assume that $\alpha = 1$. In this case, $\mathscr{L}_{\mathscr{U}}$ is invertible, and hence, by Theorem 4, the family $\mathscr{U}$ admits an exponential dichotomy. Let us represent the projection $P(\sigma_-)$ in the form [10, p. 33 of the Russian edition]

$$P(\sigma_-) = \frac{1}{2\pi i} \int_{\mathbb{T}} (\gamma I - T_{\mathscr{U}}(1))^{-1}\,d\gamma = \frac{1}{2\pi} \int_0^{2\pi} (e^{i\varphi} I - T_{\mathscr{U}}(1))^{-1} e^{i\varphi}\,d\varphi.$$

By applying the formula (which is an analog of (13))

$$((e^{i\varphi}I - T_{\mathscr{U}}(1))^{-1}x)(s) = \sum_{k\in\mathbb{Z}} e^{-i(k+1)\varphi} G(s, s+k)x(s+k),$$

and then by integrating the expression for $P(\sigma_-)$, we see that the projection $P(\sigma_-)$ is the multiplication operator of the form

$$(P(\sigma_-)x)(s) = G(s,s)x(s) = P(s)x(s), \qquad x \in \mathscr{F},\ s \in \mathbb{R},$$

where $P\colon \mathbb{R} \to \operatorname{End} X$ is the strongly continuous projection-valued function occurring in the definition of an exponential dichotomy of the family $\mathscr{U}$.

Let $\dim X < \infty$; then the function P is continuous (in the uniform operator topology), and hence, the dimension of ranges of the projections $P(s)$, $s \in \mathbb{R}$, is independent of s. This, together with the form of the Riesz projections, means that the number of connected spectral components of the set $\sigma(T_{\mathscr{U}}(1))$ is at most $\dim X$. Therefore, Theorem 2 implies a representation of the form (14).

The assertion that the line $\chi(\mathscr{U}) + i\mathbb{R}$ belongs to the set $\sigma(T_{\mathscr{U}}(1))$ and the relation $\chi_+(\mathscr{U}) = \sup_{\lambda\in\sigma(\mathscr{L}_{\mathscr{U}})} \operatorname{Re}\lambda$ readily follow from Theorem 2 and from the relation

$$r(T_{\mathscr{U}}(1)) = \lim_{n\to\infty} \|T_{\mathscr{U}}(n)\|^{1/n} = \lim_{n\to\infty} \sup_{s in \mathbb{R}} \|\mathscr{U}(s, s-n)\|^{1/n} = \exp\chi_+(\mathscr{U}),$$

which follows from the Gelfand formula for the spectral radius [3, p. 138 of the Russian translation].

If $\mathscr{L} = -d/dt + a(t)$, then $\dim X = 1$, and therefore, the set $\sigma(T_{\mathscr{U}}(1))$ is connected. In this case we have $\mathscr{U}(s, s-\tau) = \exp\int_{s-\tau}^{s} a(\lambda)\,d\lambda$, $s \in \mathbb{R}$, $\tau \geq 0$. Since the operator $T_{\mathscr{U}}(1)$ is invertible and its inverse coincides with $T_{\mathscr{U}}(-1)$, we can apply the Gelfand formula to $T_{\mathscr{U}}(-1)$ and obtain a representation of the form (15) for $\sigma(\mathscr{L})$. This completes the proof of the theorem.

Corollary 1 (Massera's theorem [5]). *Let $\mathscr{L} = -d/dt + a(t)\colon D(\mathscr{L}) \subset C(\mathbb{R}, \mathbb{C}) \to C(\mathbb{R}, \mathbb{C})$ be a differential operator with a continuous almost periodic function $a\colon \mathbb{R} \to \mathbb{C}$. Then $\mathscr{L}$ is invertible if and only if $\operatorname{Re} a_0 \neq 0$, where $a_0 \in \mathbb{C}$ is the mean value of the function a.*

Corollary 2. *If $\mathscr{U}(s, s-t_0)$, $s \in \mathbb{R}$, are invertible operators in $\operatorname{End} X$, for some $t_0 > 0$, and if $\sup_{s\in\mathbb{R}} \|\mathscr{U}(s, s-t_0)^{-1}\| < \infty$, then the semigroup $\{T_{\mathscr{U}}(t),\ t \geq 0\}$ can be embedded in a certain strongly continuous group of operators (for $\mathscr{F} = L_p$, $p \in [1, \infty)$, or for $\mathscr{F} = C_0$), and the following inclusion holds:*

$$\sigma(\mathscr{L}_{\mathscr{U}}) \subset \{\lambda \in \mathbb{C} : \chi_-(\mathscr{U}) \leq \operatorname{Re}\lambda \leq \chi_+(\mathscr{U})\},$$

where $\chi_-(\mathscr{U}) = -\lim_{\tau\to\infty} \tau^{-1} \ln \sup_{s\in\mathbb{R}} \|\mathscr{U}(s, s+\tau)\|$ and $\mathscr{U}(s, s+\tau) = \mathscr{U}(s+\tau, s)^{-1}$ for $\tau > 0$.

A linear operator $\mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}_{\mathscr{U}}) \subset \mathscr{F} \to \mathscr{F} = \mathscr{F}(\mathbb{R}, X)$ is said to be *periodic* (with period $\omega > 0$) if it commutes with the operator $S(\omega) \in \operatorname{End}\mathscr{F}$ or, which is the same, if $\mathscr{U}(t+\omega, s+\omega) = \mathscr{U}(t, s)$ for any $s \leq t$.

Theorem 6. *Let $\mathscr{L}_{\mathscr{U}}$ be a periodic operator (with period 1). Then*

$$\exp\sigma(\mathscr{L}_{\mathscr{U}}) \setminus \{0\} = \{\lambda \in \mathbb{C} : \exists\,\mu \in \sigma(\mathscr{U}(1,0)) \text{ such that } |\mu| = |\lambda|\}. \tag{16}$$

In particular, the operator $\mathscr{L}_{\mathscr{U}}$ is invertible if and only if the condition $\sigma(\mathscr{U}(1,0)) \cap \mathbb{T} = \varnothing$ is satisfied.

Proof. Since $\mathscr{U}(n, n-1) = \mathscr{U}(1, 0)$ for any $n \in \mathbb{Z}$, the assertion of the theorem readily follows from Theorems 2 and 3 (if we note that the spectrum of the difference operator $(Ux)(n) = \mathscr{U}(1,0)x(n-1)$ coincides with the set on the right-hand side in (16)). This proves the theorem.

Corollary. *If A is the infinitesimal generator of a strongly continuous semigroup $\{T(t),\ t \geq 0\} \subset \operatorname{End} X$ of linear operators, then the differential operator $\mathscr{L} = -d/dt + A\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ is invertible if and only if the condition $\sigma(T(1)) \cap \mathbb{T} = \varnothing$ holds. In particular, if X is a Hilbert space, then $\mathscr{L}$ is invertible only if (1) $\sigma(A) \cap i\mathbb{R} = \varnothing$, and (2) $\sup_{\lambda\in\mathbb{R}} \|(A - i\lambda I)^{-1}\| < \infty$.*

Proof. If X is a Hilbert space, then it follows from the Parseval formula in the Hilbert space $L_2(\mathbb{R}, X)$ that the above two conditions are equivalent to the invertibility of the operator $\mathscr{L} = -d/dt + A\colon D(\mathscr{L}) \subset L_2(\mathbb{R}, X) \to L_2(\mathbb{R}, X)$. It remains to apply Theorem 4. This proves the corollary.

Note that the corollary readily implies the well-known Herhard theorem [12, p. 95].

Theorem 7. *Let a family $\mathscr{U} \subset \operatorname{End} X$ admit an exponential dichotomy exponent $\beta > 0$ and coefficient $M > 0$. If a family of evolution operators $\mathscr{V} = \{\mathscr{V}(t, s),\ s \leq t\}$ in $\operatorname{End} X$ satisfies the condition*

$$\sup_{n\in\mathbb{Z}} \|\mathscr{U}(n, n-1) - \mathscr{V}(n, n-1)\| < \frac{1}{M}\,\frac{1-\gamma}{1+\gamma}, \tag{17}$$

where $\gamma = \exp(-\beta)$, then the operator $\mathscr{L}_{\mathscr{V}}\colon D(\mathscr{L}_{\mathscr{V}}) \subset \mathscr{F} \to \mathscr{F}$ is invertible.

Proof. It follows from Theorems 3 and 4 that the operator $\mathscr{D}_0 \in \operatorname{End}\mathscr{F}(\mathbb{Z}, X)$ is invertible, and formula (13) implies the estimate $\|\mathscr{D}_0^{-1}\| \le \sum_{k\in\mathbb{Z}} M\gamma^{|k|} = M(1+\gamma)(1-\gamma)^{-1}$. Hence, condition (17) yields the invertibility of the difference operator $(\mathscr{D}_0'x)(n) = x(n) - \mathscr{V}(n, n-1)x(n-1)$. By Theorem 3, the operator $\mathscr{L}_{\mathscr{V}}$ is invertible. This proves the theorem.

§2. Discussion of the Obtained Results. Examples and Remarks*

The above results (especially, Theorems 1–4) make it possible to apply the theory of semigroups of linear operators widely in the study of linear parabolic operators with variable coefficients and hence to partial differential operators. For example, our considerations cover the differential operator

$$\mathscr{L} = -d/dt + A(t)\colon D(\mathscr{L}) \subset L_p(\mathbb{R}, X) \to L_p(\mathbb{R}, X), \tag{18}$$

where $X = L_2(\Omega) = L_2(\Omega, \mathbb{C})$ and Ω is a bounded smooth domain in $\mathbb{R}^n$. Here the family of linear differential operators $A(t)\colon H_0^m(\Omega) \cap H^{2m}(\Omega) \subset L_2(\Omega) \to L_2(\Omega)$ ($H_0^m(\Omega)$ and $H^{2m}(\Omega)$ are the Sobolev spaces [6]), $t \in \mathbb{R}$, is defined by the family of differential expressions

$$(l_t y)(u) = \sum_{|\alpha|\le 2m} a_\alpha(t, u)(D^\alpha y)(u), \qquad u \in \Omega,\ t \in \mathbb{R},$$

and by the Dirichlet boundary conditions on the boundary $\partial\Omega$ of the domain Ω. The functions $a_\alpha\colon \mathbb{R} \times \Omega \to \mathbb{C}$, $|\alpha| \le 2m$, are assumed to be elements of the space $C(\mathbb{R}, C^k(\Omega))$ for some sufficiently large $k \in \mathbb{N}$ and satisfy the Lipschitz condition when regarded as functions of the first argument with values in $C^k(\Omega)$. In addition to the above-mentioned conditions, it is assumed that the family of differential expressions l_t, $t \in \mathbb{R}$, is uniformly elliptic.

It follows from these conditions that the elliptic operators $A(t)$, $t \in \mathbb{R}$, are the infinitesimal generators of analytic semigroups of bounded linear operators, and the assumptions of the Sobolevskii–Tanabe theorem [13, p. 589 of the Russian translation], which provides the correctness of the Cauchy problem on $\mathbb{R}$ (the existence of a family of evolution operators), are satisfied.

In conclusion, we note that in the papers [5, 9, 10], differential operators of the form $-d/dt + A(t)$ in the Banach space $C(\mathbb{R}, X)$ with a function $A\colon \mathbb{R} \to \operatorname{End} X$ were considered. Assertion (1) of Theorem 4 was obtained in [10, Theorem 3.3′] under certain additional assumptions, which are eliminated, for $\mathscr{F} = C$, in [1, Chap. X], where it is not required that $A(t) \in \operatorname{End} X$, $t \in \mathbb{R}$. In [9], it was proved that the exponential dichotomy condition for a family of evolution operators is equivalent to the condition $\sigma(T_{\mathscr{U}}(1)) \cap \mathbb{T} = \varnothing$. A similar assertion was presented in [15, Theorem 9.3] for elements of some C^*-algebras generated by dynamical systems.

Differential operators with unbounded operator coefficients were considered in [1, 6, 8]. In the paper [8], the equivalence of the conditions of the (uniform) injectivity for operators $\mathscr{L}_{\mathscr{U}}$ and $\mathscr{D}_0$ was proved. Of the results in [6] that are related to Theorems 1–4 most closely, we note Theorem 7.6.3, which states that the operator $\mathscr{L}_{\mathscr{U}}$ is invertible under some assumptions that include the presence of an exponential dichotomy of the family $\mathscr{U}$ and the assertion (see p. 251 of the Russian edition) on an exponential dichotomy of this family under the exponential dichotomy condition for the operator family $\{\mathscr{U}(t_0 + n, t_0 + n - 1), n \in \mathbb{Z}\}$ for all $t_0 \in \mathbb{R}$. We also note that it follows from [6, Theorem 7.6.5] and from Theorems 3 and 4 that a family $\mathscr{U}$ admits an exponential dichotomy whenever the family $\{\mathscr{U}(n, n-1),\ n \in \mathbb{Z}\}$ admits the discrete dichotomy. A discrete analog of assertion 1 of Theorem 4 is given in [6, Theorem 7.6.5], and in the same monograph (see p. 363 of the Russian translation, comments to Chap. IX), the problem to prove assertion 1 of Theorem 4 was posed.

References

1. B. M. Levitan and V. V. Zhikov, Almost Periodic Functions and Differential Equations, Izdat. Moskov. Univ., Moscow, 1978; English translation: Cambridge University Press, Cambridge, 1982.
2. S. G. Krein, Linear Differential Equations in a Banach space, Nauka, Moscow, 1967; English translation: Am. Math. Soc., Providence, R.I., 1971.

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3. E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, Am. Math. Soc., New York, 1957.
4. A. G. Baskakov, "Investigation of spectral properties of the differential operator $\mathscr{L} = -d/dt + A(t)$ by means of the semigroup $\exp \mathscr{L}t$, $t \geq 0$," in: Intern. Conf. on Functional Equations and Applications, Moscow, Russia, August 14–21, 1994, pp. 8–9.
5. J. L. Massera and J. J. Schäffer, Linear Differential Equations and Function Spaces, Academic Press, New York, 1966.
6. D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag, Berlin-Heidelberg-New York, 1981.
7. A. G. Baskakov, "Spectral criteria for the almost periodicity of solutions to functional equations," Mat. Zametki, **24**, No. 2, 195–205 (1978).
8. V. M. Tyurin, "On the invertibility of linear differential operators in certain function spaces," Sib. Mat. Zh., **32**, No. 3, 160–165 (1991).
9. P. P. Zabreiko and Nguen Van Min′, "The group of characteristic operators and its applications to the theory of linear ordinary differential equations," Dokl. Ross. Akad. Nauk, **324**, No. 1, 24–28 (1992).
10. Yu. L. Daletskii and M. G. Krein, Stability of Solutions of Differential Equations in a Banach Space, Nauka, Moscow, 1970; English translation: Am. Math. Soc., Providence, R.I., 1974.
11. A. G. Baskakov, "Abstract harmonic analysis and asymptotic estimates of the elements of inverse matrices," Mat. Zametki, **52**, No. 2, 17–25 (1992).
12. R. Nagel, One-Parameter Semigroups of Positive Operators, Lecture Notes in Math., Vol. 1184, Springer-Verlag, 1984.
13. K. Iosida, Functional Analysis, Springer-Verlag, Vol. 1, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958.
14. N. Dunford and J. T. Schwartz, Linear Operators. General Theory, Vol. I, Interscience Publ. Co., New York, 1958.
15. A. B. Antonevich, Linear Functional Equations: An Operator Approach [in Russian], Universitetskoe, Minsk, 1988.

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