Functional Analysis and Its Applications, Vol. 30, No. 2, 1996

Statistical Mechanics of Combinatorial Partitions, and Their Limit Shapes*

A. M. Vershik

UDC 514.172.45, 519.2

To Ya. G. Sinai on the occasion of his 60th birthday

§0. Introduction

In this paper, by a partition we mean a division of a positive integer or a vector with positive integral coordinates into unordered summands of the same kind. Since Euler's time, this object has been of interest to specialists in different areas of mathematics, including number theory, combinatorics, probability theory, and others. A new problem in asymptotical partition theory, which arose some time ago, is the question about the asymptotic (limit) shape, or configuration, of partitions with respect to some statistics. Here is the simplest setting of this problem (see §1). Let μ^n be the uniform measure on the set of all partitions of the number n: $\mu^n(\lambda) = p(n)^{-1}$, $\lambda \in \mathcal{P}_n$, where $p(\cdot)$ is Euler's function (partitio numerorum); the question is: whether one can normalize the partitions in such a way that, in some properly chosen space, the measures μ_n have a weak limit on generalized partitions, and whether this limit is singular (a δ-measure). In the last case, the limit measure is concentrated on a *limit shape*. An affirmative answer to these questions, as well as explicit formulas for limit shapes, for a number of statistics, are presented below. In this paper we continue the study of similar problems (see the survey [1]) and stress their close relationship with the statistical physics of ideal gas. The problem of limit configuration in this context is especially appropriate to traditional statistics (Bose, Fermi, and others), although it was not studied before, as far as I know. It can be solved by means of some variation of classical tools (see below) and is interpreted as a problem of limit distribution of the total energy of the system over the energy spectrum. Nevertheless, our passage to the limit is not literally equivalent to the case of thermodynamic limit (N-V-limit): in fact, we reduce the problem of limit distribution of energy to the case of fixed volume, as energy tends to infinity so that its growth does not necessarily match the growth of the number of particles. If the number of summands (particles) is not fixed (the case similar to the statistics of photons), then the natural growth of the number of particles is automatically determined by the statistics. The classical N-V-limit is also covered by our scheme by selecting the growth of the number of summands in a special way (see below), but the corresponding combinatorial problems are still poorly studied; they are also related to the behavior of the convolution of distributions on the semigroup of partitions.

From the mathematical point of view, we are studying the weak convergence of measures in an appropriate space of limit distributions (compactum $\mathcal{D}$), and the main statement is that, in some cases, the limit of measures of a special form (for example, multiplicative measures, in particular, uniform measures on partitions) on the set of partitions of a chosen number n (=energy) coincide with the limits of a mixture of these measures for various n, and thus their limit shapes coincide as well. Strictly speaking, this is the merit of using the macrocanonical ensemble with Gibbs' measure; its study is simpler than that of the microcanonical ensemble, because the occupation numbers become independent with respect to Gibbs' measure. This idea was implicitly used in combinatorics (see [2–4]), but the problem of limit shape was not posed. The above statement can be referred to as a weak equivalence of the macrocanonical ensemble and the microcanonical one, but, according to the remark on the difference between the limits, this assertion is more general than the equivalence of the ensembles in statistical mechanics. Nevertheless, the methods are similar: both the saddle point method (by Darwin–Fowler [5]) and the local limit theorem (A. Ya. Khinchin seems to be the first who used the latter way, see [6]) can be used in both cases to find the limits and verify the results. We must stress that the main obstacle in the proof is the same, although it is overridden differently. In problems of number theory, probabilistic methods in finding the asymptotic

*This work was partially supported by RFBR (grant 94-01-00921) and ISF (grant MVQ000).

Steklov Mathematical Institute, St. Petersburg Branch. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 30, No. 2, pp. 19–39, April–June, 1996. Original article submitted January 15, 1996.

 0016-2663/96/3002-0090 $15.00

behavior of number functions were used by G. Freiman [7]. An excellent example of applying probabilistic methods in a problem on vector partitions and convex curves, which was solved in [8, 9], was given by Ya. G. Sinai [10].

The problem of limit shape for partitions was first posed by the author in connection with the asymptotic theory of classical groups and their representations, in particular, the symmetric group [1], but it should be considered in a more general framework of known problems of the random growth of shapes. Note that combinatorial schemes in combinatorics, group theory, representation theory, and additive number theory are much more various than ordinary models in the statistical physics of the ideal gas. In particular, the ergodicity of macro- and microcanonical ensembles can fail in natural problems. The stimulating nature of these problems becomes apparent, in particular, in the interaction of different methods of their solution. For example, the variational principle (see [8, 16]) that suggests finding a limit shape as the minimum of some functional led to a new type of variational problems (see §6).

In this paper, we present the main statements and a series of results concerning the limit shape. Their detailed exposition will appear elsewhere. During his work, the author recalled his conversations with Ya. G. Sinai, who popularized ideas of bringing together statistical physics and traditional mathematics. The author also made use of discussions with R. A. Minlos and, most notably, consultations with outstanding mathematician R. L. Dobrushin before his untimely death in 1995, who repeatedly imparted his insight on statistical physics and adjacent mathematical theories to the author. Finally, thinking over some of these topics was stimulated by questions of listeners of the lecture course "Partitions," which was given by the author in autumn 1995 at St. Petersburg State University and included a survey of these problems.

§1. Setting of the Problem, Definitions, Terminology

Denote by $\mathcal{P}_n = \{\lambda : \lambda \vdash n\}$ the set of all partitions of a positive integer $n \geq 0$ into (unordered) positive integers, $\lambda = (\lambda_1, \dots, \lambda_N)$; let $n(\lambda)$ be the sum and let $\#\lambda$, or $N(\lambda)$, be the number of summands. The set of all partitions of n with N summands is denoted by $\mathcal{P}_{n,N}$; thus, $\mathcal{P}_n = \bigcup_N \mathcal{P}_{n,N}$. Finally, $\mathcal{P} = \bigcup_n \mathcal{P}_n$ is the disjoint union; it can be called the *macrocanonical ensemble of partitions*; we also call $\mathcal{P}_{n,N}$ ($\mathcal{P}_n$) the *microcanonical* (*canonical*) *ensemble.*

To $\lambda \in \mathcal{P}_n$ we assign a collection of numbers $r_1(\lambda), \dots, r_t(\lambda)$, where $r_k(\lambda) = \#\{j : \lambda_j = k\}$ is the number of summands equal to k in the partition λ; we call $r_k(\lambda)$ *occupation numbers*; obviously, $\{r_i(\lambda)\}$ fully determine the partition λ and can also be called the "distribution" of the partition. Clearly, $n(\lambda) = \sum_k kr_k(\lambda)$, $\#\lambda = N(\lambda) = \sum_k r_k(\lambda)$. Recall that $\#\mathcal{P}_n = p(n)$ is Euler's function with the generating function

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{n=0}^{\infty} p(n)x^n.$$

To a partition $\lambda \in \mathcal{P}_n$ we assign a function φ_λ on $[0, \infty)$ by the following rule:

$$\varphi_\lambda(t) = \sum_{k \geq t} r_k(\lambda), \qquad 0 \leq t < \infty. \tag{1.1}$$

Clearly, φ_λ is a step-function continuous on the right such that $\int_0^\infty \varphi_\lambda(t)\,dt = n(\lambda)$. The closure of the interior of its ordinate set is called the *Young diagram* of the partition λ. For $a > 0$, we say that the function

$$\widetilde{\varphi}_\lambda(t) = \frac{a}{n} \sum_{k \geq at} r_k(\lambda) = \frac{a}{n} \varphi_\lambda(at) \tag{1.2}$$

is the function φ_λ normed by a; we have $\int_0^\infty \widetilde{\varphi}_\lambda(t)\,dt = 1$; $\widetilde{\varphi}$ is the image of φ under the plane transformation $(t, \varphi) \mapsto (at, \varphi n/a)$. The question how to choose the *scaling* a will be discussed below. We denote the transformation $\lambda \mapsto \widetilde{\varphi}_\lambda$ by τ_a.

Introduce the space of triples $\mathcal{D} = \{(\alpha_0, \alpha_\infty, p(\cdot))\}$: here $\alpha_0, \alpha_\infty \in \mathbb{R}_+$, and $p \in L^1_+(\mathbb{R}_+)$ is a nonnegative, monotone nonincreasing function on $[0, \infty)$, with $\alpha_0 + \alpha_\infty + \int_0^\infty p(t)\,dt = 1$. One can regard the numbers α_0 and α_∞ as the charges of the measure at 0 and $+\infty$, respectively, and thus $\mathcal{D}$ consists of the measures $\alpha_0\delta_0+\alpha_\infty\delta_\infty+p\,dt$ on $[0,\infty]$ that have two possible atoms at the endpoints and a monotone density with respect to the Lebesgue measure on $(0,\infty)$. The measures δ_0 and δ_∞ are called the trivial elements of $\mathcal{D}$, the measures with $p\neq 0$ are called proper elements, and measures with $\alpha_0=\alpha_\infty=0$ are called continuous elements.

We endow $\mathcal{D}$ with the topology of uniform convergence on compacta on $(0,\infty)$ and the ordinary convergence of charges at 0 and ∞.

Lemma 1.1. *$\mathcal{D}$ is a metrizable compactum in this topology.*

The compactness follows from the uniform boundedness and the uniform absolute continuity of the monotone functions on intervals $[t_0,t_1]$, $t_0>0$.

Corollary. *An infinite set of normalized nonnegative Borel measures on $\mathcal{D}$ has a nonempty set of limit points in the weak topology of measures.*

Clearly, the image of $\mathcal{P}_n$ under the mapping $\tau_a\colon \mathcal{P}_n\to\mathcal{D}$ consists of measures with zero α_0 and α_∞ and with step density.

Assume that a probability measure (statistics) μ^n is defined on $\mathcal{P}_n$; then the formula $\tau_a^*\mu^n(A)\equiv\mu^n(\tau_a^{-1}A)$, $A\in\mathcal{D}$, determines the image of μ^n under the mapping τ_a. We study the following question: *given a sequence of measures μ^n on $\mathcal{P}_n$, whether there exists a numerical sequence a_n such that the images $\tau_{a_n}^*\mu^n$ are weakly convergent to a measure with nontrivial support* (*in other words, the limit measure is not a charge at* 0 *or* ∞). If this scaling exists and the limit measure is continuous, then the scaling is, in essence, unique. To be more precise, if two scalings $\{a_n\}$ and $\{a_n'\}$ lead to nontrivial limit measures, then, first, these scalings are equivalent, that is, $a_n/a_n'\to 1$, and, secondly, these measures coincide. If the limit measure is singular, i.e., if it is concentrated on a single continuous element $C\in\mathcal{D}$, then this element is called a *limit shape*, or a *limit configuration*, of the random partitions for the statistics $\{\mu^n\}$ and for the scaling $\{a_n\}$. Thus, given a sequence of measures μ^n on $\mathcal{P}_n$, the question is to find a numerical sequence a_n such that the scaled images $\tau_{a_n}^*\mu^n$ are weakly convergent to some measure $\overline{\mu}$ on $\mathcal{D}$ (or to prove that this scaling does not exist). An affirmative answer to this question makes it possible to express statements of the following type: "asymptotically, for almost all partitions, the sum of summands that exceed ta_n forms a given part of the whole sum" (in the ergodic case) or "the probability of the event that this part belongs to a given interval is a given number" (in the nonergodic case). A huge number of problems of asymptotic combinatorics can be reduced to this question. In the general setting, this question is extremely wide; we restrict ourselves to a special class of measures μ^n, namely, the class of multiplicative statistics on $\mathcal{P}$. In terms of statistical physics, a partition $\lambda\in\mathcal{P}_n$ is an assembly of particles (to be more precise, of their energies) such that its total energy is n $(=E)$, and the number of particles of energy k is the occupation number $r_k(\lambda)$. The energies of particles (and hence of the whole system) are positive integers; this is not restrictive in the theory of ideal gases, because these numbers are eigenvalues of the Laplace operator on a cube or on a torus. The total number of particles is $\sum r_k(\lambda)=N(\lambda)$. In the case of ideal gas, the system is completely determined by the impulses of all its particles, i.e., by the occupation numbers. The set $\mathcal{P}_{n,N}$ can be interpreted as the set of configurations λ with given total energy and number of particles, i.e., as the *microcanonical ensemble* (in the unit volume). Nevertheless, the set of particles with given energy is the set of points of some lattice on the sphere with given radius, and thus the statistics on partitions must take into account the multiplicity of the energies of particles (see below). A statistics (a measure) on $\mathcal{P}_n$ or $\mathcal{P}_{n,N}$ is chosen in accordance with the problem, and its support can be a part of $\mathcal{P}_n$, for instance $\mathcal{P}_{n,N(n)}$. The scaling strongly depends on the statistics under consideration and determines a relation between the total energy and the number of particles whose energy exceeds a given level. The limit shape, as an element of the compactum $\mathcal{D}$, in the ergodic case, is the *limit distribution of the energies of particles*; and this makes it possible to express assertions of the above type: what limit part of the total energy corresponds to the particles with given energies.

From the point of view of statistical physics, we deal with a system in the unit volume. In our model, the growth of the volume means that partitions into nonintegral summands must be considered; to be more precise, the summands must be proportional to some inverse power of volume (where the power depends on the dimension of the problem). By scaling we can reduce this partition to an ordinary one (in other words, the volume can be reduced to the unit). However, the number of summands must be re-counted in the new scale. Thus, the ordinary thermodynamic limit fits into our scheme with special measures on $\mathcal{P}_{n,N(n)}$, where the function $N = N(n)$ is determined by the dimension of the problem (see below). The case of vector partitions corresponds to the situation in which the system has other first integrals besides the energy.

§2. Multiplicative Statistics. Macrocanonical Ensemble

Now we introduce a class of measures on $\mathcal{P}_n$ and show that this class is that of restrictions, to $\mathcal{P}_n$, of measures on $\mathcal{P}$ that are said to be *multiplicative*.

Consider a sequence of functions $s = \{s_k\}$, $s_k\colon \mathbb{N} \to \mathbb{R}_+$, $k = 1, \dots$, and define a multiplicative function of a partition by $F_s(\lambda) \equiv F(\lambda) = \prod_k s_k(r_k(\lambda))$, where $r_k = \#\{i : \lambda_i = k\}$ are the occupation numbers. Let $Q_n = \sum_{\lambda \vdash n} F(\lambda)$ be the partition function; denote by $\mu^{s,n} \equiv \mu^n$ the following measure on $\mathcal{P}_n$:

$$\mu^n\{\lambda : r_k(\lambda) = r\} = Q_n^{-1} s_k(r), \qquad \mu^n(\lambda) = Q_n^{-1} F(\lambda). \tag{2.1}$$

Similarly,

$$\text{for} \quad Q_{n,N} = \sum_{\substack{\lambda \vdash n \\ N(\lambda)=N}} F(\lambda) \quad \text{we set } \mu^{n,N}(\lambda) = Q_{n,N}^{-1} F(\lambda).$$

Thus, we have defined a class of measures on the canonical and microcanonical ensembles. Consider now the *macrocanonical ensemble*, that is, the set of all partitions, and introduce on it the family of measures $\mu_{s,x} = \mu_x$ depending on a real parameter (or on several parameters) that also depend on the sequence $s = \{s_k\}_{k=1}^{\infty}$:

$$\mu_x(\lambda) = x^{n(\lambda)} \mathcal{F}(x)^{-1} \prod s_k(r_k(\lambda)) = x^{n(\lambda)} \mathcal{F}(x)^{-1} F(\lambda), \tag{2.2}$$

where $\mathcal{F}(x) = \sum_{n=0}^{\infty} Q_n x^n$ (big statistical sum). We assume that the last series is convergent for $0 \le x < x_0$.

By setting $\mathcal{F}_k(y) = \sum_{r=0}^{\infty} s_k(r) y^r$ we obtain

$$\mathcal{F}(x) = \sum_{n=0}^{\infty} \sum_{\{r_k\}} \prod_k s_k(r_k) x^n = \sum_{n=0}^{\infty} \sum_{\{r_k\}} \prod s_k(r_k) x^{\sum k r_k} = \prod_{k=1}^{\infty} \sum_{r=0}^{\infty} s_k(r) x^{r_k} = \prod_{k=1}^{\infty} \mathcal{F}_k(x^k).$$

Lemma 2.1. *The measures $\mu_{s,x} \equiv \mu_x$ are defined for $x \in [0, x_0)$ and have the following properties:*

(1) $\mu_x|_{\mathcal{P}_n}(\lambda) \equiv \dfrac{\mu_x(\lambda)}{\mu_x(\mathcal{P}_n)} = \mu^n(\lambda)$;

(2) *the occupation numbers $r_1, \dots$, regarded as functions on $(\mathcal{P}, \mu_x)$, form a sequence of independent random variables with respect to the measures μ_x, $x \in [0, x_0)$;*

(3) $\mu_x = \sum_{n=0}^{\infty} \mathcal{F}(x)^{-1} x^n Q_n \mu^n$, *i.e., μ_x is a convex combination of measures μ^n.*

All statements can be verified directly. The above families (with respect to x) of measures are said to be *multiplicative*. The generating function $\mathcal{F}(x)$, along with its decomposition $\mathcal{F}(x) = \prod_{k=1}^{\infty} F_k(x^k)$, completely determines such a family. A number of important measures on the partitions, including the Plancherel measure on the Young diagrams, do not belong to this class; however, as we shall see below, multiplicative families contain very important examples.

In order to embrace the general case that includes the fixation of the number of summands, we introduce a function of two variables $\mathcal{F}(z, x) = \prod_{k=1}^{\infty} \mathcal{F}_k(z x^k)$. In this case

$$\mathcal{F}(z, x) = \prod_{k=1}^{\infty} \sum_{r=0}^{\infty} s_k(r) z^r x^{kr}.$$

As above we set

$$Q_{n,N} = \sum_{\substack{n(\lambda)=n \\ N(\lambda)=N}} F_s(\lambda), \qquad F_s(\lambda) = \prod_k s_k(r_k(\lambda)),$$

and obtain

$$\mathcal{F}(z,x)=\sum_{n,N} Q_{n,N}x^n z^N.$$

Consider the family of measures $\mu_{x,z}$ defined by

$$\mu_{x,z}(\lambda)=x^{n(\lambda)}z^{N(\lambda)}\mathcal{F}(z,x)^{-1}F(\lambda).$$

As well as for family μ_x, we have the following assertion.

Lemma 2.2. (1) *The measures $\mu_{x,z}$ are defined in the real domain of convergence of the series $\mathcal{F}(z,x)$ and $\mu_{x,1}=\mu_x$.*

(2) *The restrictions of $\mu_{x,z}$ to the microcanonical ensemble are the measures $\mu^{n,N}$ (see above).*

(3) $\mu_{x,z}=\sum_{n,N}\mathcal{F}(z,x)^{-1}x^n z^N Q_{n,N}\mu^{n,N}$.

At the same time, now the occupation numbers are not independent with respect to $\mu_{x,z}$.

Now we pass to the case of vector partitions. A partition of a vector with positive integral components $\mathbf{n}=(n_1,\dots,n_d)$ is its division into (unordered) vectors:

$$\mathbf{n}=\mathbf{n}^1+\dots+\mathbf{n}^k, \qquad \mathbf{n}^i=(n_1^i,\dots,n_d^i).$$

The set of all vector partitions of a d-dimensional vector $\mathbf{n}$ is denoted by $\mathcal{P}^d_{\mathbf{n}}$, $\mathbf{n}\in\mathbb{Z}^d_+$, and the set of partitions with N summands by $\mathcal{P}^d_{\mathbf{n},N}$. Let

$$\mathcal{P}^d=\bigcup_{\mathbf{n}\in\mathbb{Z}^d_+}\mathcal{P}^d_{\mathbf{n}}.$$

The following d-dimensional matrix is an analog of the occupation numbers: let

$$\lambda\in\mathcal{P}^d_{\mathbf{n}},\quad \lambda=(\mathbf{n}^1,\dots,\mathbf{n}^k),\quad \mathbf{n}^i\in\mathbb{Z}^d_+,\quad \mathbf{n}=\sum_i\mathbf{n}^i,\quad r_{k_1,\dots,k_d}(\lambda)=\#\{i:\mathbf{n}^i=(k_1,\dots,k_d)\},$$

and let the mapping φ_λ from §1 be replaced by

$$\varphi^d_\lambda(t_1,\dots,t_d)=\sum_{(k_1,\dots,k_d)\ge(t_1,\dots,t_d)} r_{k_1,\dots,k_d}(\lambda).$$

It remains to describe an analog of the mapping τ and of the compactum $\mathcal{D}$. The scaling is determined in this case by d scaling sequences $\{a_n^j: j=1,\dots,d\}$, $n\in\mathbb{N}$, and

$$\tilde{\varphi}^d_\lambda(t_1,\dots,t_d)=\frac{1}{n_1\dots n_d}a^1_{n_1}\dots a^d_{n_d}\varphi(a^1_{n_1}t_1,\dots,a^d_{n_d}t_d).$$

The compactum $\mathcal{D}^d$ is constructed similarly: its elements are formed by functions that are monotone nonincreasing with respect to any argument, being defined on coordinate octants and integrable with respect to the Lebesgue measure, and by charges at the infinities. The simplest way to describe $\mathcal{D}^d$ is to take the completion of the set of nonincreasing densities defined on the interior of $\mathbb{R}^d_+$ in the topology of uniform convergence on compacta in $\mathring{\mathbb{R}}^d_+$. The mapping

$$\tau^d\colon \mathcal{P}^g\to\mathcal{D}^d$$

makes it possible to pose the same questions on the weak convergence of images of the measures defined on $\mathcal{P}^d$.

Remarks. 1. The ordinate set of a function $\varphi^d_\lambda(t_1,\dots,t_d)$ is a many-dimensional partition (a many-dimensional Young diagram); however, not every diagram can be obtained in this way.

2. Another way of geometrizing vector partitions exists for $d=2$ (it was used in [8, 9]), namely, the ordering of the summands $\mathbf{n}^i=(k_1^i,k_2^i)$ in ascending order of ratios k_1^i/k_2^i.

The multiplicative families on $\mathcal{P}^d$ can also be determined by the generating functions

$$\mathcal{F}(x_1, \dots, x_d) = \prod_{k_1, \dots, k_d} \mathcal{F}_{k_1, \dots, k_d}(x_1^{k_1} \cdots x_d^{k_d}),$$

and the function of one variable

$$\mathcal{F}_{k_1, \dots, k_d}(y) = \sum_{r=0}^{\infty} s_{k_1, \dots, k_d}(r) y^r$$

is the generating function of the distribution of the occupation number $r_{k_1, \dots, k_d}$. If $\lambda \in \mathcal{P}^d_{\mathbf{n}}$, $\mathbf{n} = (n_1, \dots, n_d)$, then the measure $\mu_{x_1, \dots, x_d}$ is determined by

$$\mu_{x_1, \dots, x_d}(\lambda) = x_1^{n_1} \cdots x_d^{n_d} \mathcal{F}(x_1, \dots, x_d)^{-1} \prod_{k_1, \dots, k_d} s_{k_1, \dots, k_d}(r_{k_1, \dots, k_d}(\lambda)).$$

The measures $\mu_{x_1, \dots, x_d}$ are defined for real x in the domain of convergence for the series representing $\mathcal{F}(x_1, \dots, x_d)$, and the occupation numbers are independent with respect to these measures. We do not go into details and only note that formulas for the measures on the microcanonical ensemble $\mathcal{P}^d_{\mathbf{n}, N}$ are just the same as in the case $d = 1$.

§3. Examples of Multiplicative Measures

We begin with a series of examples of algebraic and combinatorial nature. In these examples, the measures are originally defined on some natural objects, for instance, on a symmetric group, on set partitions, and so on; then these measures are transferred to $\mathcal{P}_n$ for all n. Since these measures are multiplicative, it is possible to construct measures on the macrocanonical ensemble.

On the other hand, in physical examples, it is rather natural to introduce measures μ_x by means of a generating function which has the form of an Euler product or is a factorized (decomposed into an infinite product) function, and then to transfer these measures to canonical or microcanonical ensembles. We use the notation of §2.

1. Uniform statistics on $\mathcal{P}_n$ (an analog of the Bose statistics). Let $\mu^n(\lambda) = p(n)^{-1}$ be the uniform distribution on $\mathcal{P}_n$. The corresponding multiplicative function $F(\lambda)$ is identically one, and $s_k(r) \equiv 1$ for all $k = 1, \dots$. Thus, $\mathcal{F}_k(x) = 1/(1 - x)$,

$$\mathcal{F}(x) = \prod_{k=1}^{\infty} \mathcal{F}_k(x^k) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}, \tag{3.1}$$

$$\mu_x(\lambda) = x^{n(\lambda)} \mathcal{F}(x)^{-1} = x^{n(\lambda)} \prod_{k=1}^{\infty} (1 - x^k), \quad \mu_x\{\lambda : r_k(\lambda) = s\} = x^{ks}(1 - x^k).$$

As we shall see, this statistics is just the statistics of the two-dimensional Bose gas (see [5]); it is most natural from the number-theoretic point of view as well, because all partitions of n are equiprobable, and the partition function is Euler's function: $Q_n = p(n)$.

1a. However, if we consider a chosen number of summands, then the generating function is

$$\mathcal{F}(z, x) = \prod_{k=0}^{\infty} \frac{1}{1 - zx^k} = \sum_{n,N} p_N(n) x^n z^N. \tag{3.2}$$

Here the measure $\mu^{n,N}$ is the uniform distribution on the partitions of n with N summands (note that zero summands are also admitted here); in this case

$$\mathcal{F}_k(z, x) = \frac{1}{1 - zx}, \qquad \mu_{x,z}(\lambda) = x^{n(\lambda)} z^{N(\lambda)} \prod_{k=1}^{\infty} (1 - zx^k),$$

$$\mu_{x,z}(\lambda : r_k(\lambda) = r,\ N(\lambda) = N) = z^{kr} z^N (1 - zx^k).$$

The factor $1/(1 - z)$ in (3.2) corresponds to the distribution of the number of zero summands. It is substantial when explaining the condensation effect for the Bose gas.

2. Uniform distribution on partitions with different summands (analog of the Fermi statistics). Here

$$\mathcal{F}(x) = \prod_{k=1}^{\infty}(1+x^k) = \sum_{n=0}^{\infty} p_{\neq}(n)x^n, \tag{3.3}$$

$$\mu^n(\lambda) = p_{\neq}(n)^{-1}, \quad \mu_x(\lambda) = x^{n(\lambda)}\prod_{k=1}^{\infty}(1+x^k)^{-1}.$$

2a. If a number of summands is chosen, then we have

$$\mathcal{F}(z,x) = \prod_{k=1}^{\infty}(1+zx^k) = \sum_{n,N} p_{\neq}(n,N)x^n z^N, \tag{3.4}$$

$$\mu^{n,N}(\lambda) = p_{\neq}(n,N)^{-1}, \quad \mu_{z,x}(\lambda) = x^{n(\lambda)}z^N\prod_{k=0}^{\infty}(1+zx^k)^{-1}.$$

According to the tradition of partition theory, one can consider different classes of partitions (for example, partitions with given lower bound of the differences between summands, partitions with summands in a given subset of positive integers, and so on) with uniform statistics, but we restrict ourselves to the above cases and generalize them in a different direction.

3. Bell's statistics (see [11, 12]). As a rule, partitions appear as classes of some objects under some equivalence relation. Let Π_n be the set of all partitions of an n-tuple; let m_n be the uniform distribution on Π_n, and let $\pi_n\colon \Pi_n \to \mathcal{P}_n$ be the mapping that assigns to a partition $\Lambda \in \Pi_n$ the partition λ of n such that $\lambda = (\lambda_i)$ and λ_i are the cardinalities of the blocks of Λ. The image $\pi_n m_n = \beta_n$ is the measure on $\mathcal{P}_n$ which is called Bell's statistics. Thus, it is a statistics on the equivalence classes under the action of the symmetric group. Obviously,

$$\beta_n\{\lambda\} = 1\Big/\prod_k r_k(\lambda)!(k!)^{r_k(\lambda)}, \qquad \lambda \in \mathcal{P}_n.$$

The measures β_n are multiplicative, and the corresponding generating function is

$$\mathcal{F}_k(y) = e^{y/k!}, \qquad \mathcal{F}(x) = \prod \mathcal{F}_k(x^k) = e^{e^x-1}, \quad s_k(r) = \frac{1}{(k!)^r r!}.$$

This measure has been investigated in [12] in detail.

Choosing a number of blocks, we obtain the generating function

$$\mathcal{F}(z,x) = e^{ze^x}.$$

4. Haar's statistics and the Poisson–Dirichlet measures [11, 18]. We consider the uniform measure m_n on the symmetric group S_n and its deformation m_n^θ, $\theta \in \mathbb{R}_+$:

$$m_n^\theta(g) = \frac{1}{(n+\theta-1)\cdots\theta}\theta^{c(g)} = \frac{\theta^{c(g)}}{[\theta]^n}, \qquad [\theta]^n = \theta(\theta+1)\cdots(\theta+n-1),$$

where $c(g)$ is the number of cycles in the permutation $g \in S_n$. For $\theta = 1$, we obtain the uniform distribution on S_n.

The projection $\pi\colon S_n \to \mathcal{P}_n$ that maps any permutation to the collection of lengths of its cycles induces the following measure μ^θ on $\mathcal{P}_n$:

$$\mu^\theta(\lambda) = \frac{\theta^{N(\lambda)}}{\prod_k r_k(\lambda)!\,k^{r_k(\lambda)}[\theta]^{n(\lambda)}}, \qquad \theta \in [0,\infty).$$

For $\theta = 1$ we obtain the so-called Haar distribution on the partitions (it is induced by the Haar measure on S_n). All these measures are induced by multiplicative measures on $\mathcal{P}$; the big statistical sum is

$$\mathcal{F}(x) = \frac{1}{(1-x)^\theta} = e^{-\theta \ln(1-x)} = \prod_{k=1}^{\infty} e^{\theta x^k/k} = \prod_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\theta x^k}{n}\right)^n,$$

i.e., $\mathcal{F}_k(y) = e^{\theta y/k}$. Choosing a definite number of cycles we obtain

$$\mathcal{F}_k(zy) = e^{\theta z y/k}, \qquad \mathcal{F}(x, z) = \prod_{k=1}^{\infty} \mathcal{F}_k(zx^k) = \frac{1}{(1-x)^{z\theta}},$$

and we see that θ plays the role of the variable z; thus, we can assume $\mathcal{F}(x, z) = 1/(1-x)^z$.

5. Model of d-dimensional ideal gas [5]. Consider a model that is closer to quantum statistical physics. We restrict ourselves to the indication of the big statistical sum (that is, the generating function), since all other components can readily be reconstructed. Recall that in the case of ideal gas, the energy of the system is the sum $E = \sum_{\mathbf{p}} r_{\mathbf{p}} \varepsilon_{\mathbf{p}}$, where $\varepsilon_{\mathbf{p}}$ stands for the energy of a particle with momentum $\mathbf{p} \in \mathbb{R}^d$, i.e.,

$$\varepsilon_{\mathbf{p}} = \frac{1}{2m}\|\mathbf{p}\|^2 = \frac{1}{2m}\left(\frac{2\pi\hbar}{V^{1/d}}\right)^2 \|\mathbf{q}\|^2$$

(thus, $\varepsilon_{\mathbf{p}}$ are the eigenvalues of the Laplace operator on a Euclidean torus) [5], $\mathbf{q} \in \mathbb{Z}^d$, V is the volume, $\hbar$ is Planck's constant, m is the mass, d is the dimension, $r_{\mathbf{p}}$ is the number of particles with momentum $\mathbf{p}$, and $\|\mathbf{q}\|^2 = q_1^2 + \cdots + q_d^2$. We assume that the coefficient of $\|\mathbf{q}\|^2$ is equal to one (for the growing volume, see the remark at the end of §1).

In our notation we have the partition of n $(= E)$ into summands each of which is the sum of d squares of positive integers, where the summands are considered to be distinct, i.e., for the parameter of a summand we take the corresponding vector in $\mathbb{Z}^d$, and the summand is the square of the norm of this vector. This leads us to the following generating function for the Bose statistics:

$$\mathcal{F}(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^{j_d(k)}}, \tag{3.5}$$

where $j_d(k)$ is the number of representations of the number k as the sum of d squares. Recall that

$$\sum_{k=0}^{\infty} j_d(k) z^k = \theta(0, \tau)^d,$$

where $\theta(0, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2}$ is the theta function.

In the case of the Fermi statistics we obtain

$$\mathcal{F}(x) = \prod_{k=1}^{\infty} (1 + x^k)^{j_d(k)}.$$

If the number of particles or summands is chosen, then the corresponding generating functions are

$$\mathcal{F}_b(z, x) = \prod \frac{1}{(1 - zx^k)^{j_d(k)}}, \qquad \mathcal{F}_f(z, x) = \prod (1 + zx^k)^{j_d(k)}.$$

According to the scheme of §2, all these functions determine certain measures μ^n on the canonical ensemble and μ_x on macrocanonical one; the measures μ_x and $\mu_{x,z}$ are multiplicative, and hence we can apply our method to determine the corresponding limit shape.

6. Classical generating functions. The above examples are covered by a general function of the form

$$\mathcal{F}(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^{b_k}} \quad \text{or} \quad \mathcal{F}(x) = \prod_{k=1}^{\infty} (1 + x^k)^{b_k},$$

where $\{b_k\}$ is any sequence of positive integers. The corresponding distribution on $\mathcal{P}_n$ can be interpreted as follows: assume that there are b_k different types of summands equal to k (particles with energy k), and all partitions differ in the number and in the type of summands; the distribution is assumed to be uniform. For the later use, as well as for the calculation of the logarithmic asymptotics for the coefficients of $\mathcal{F}(x)$ (see Meinardus' theorem in [13]), only the following constant α determined by the sequence $\{b_k\}$ is substantial: let $\sum_{k=1}^{\infty} b_k/k^s$ be the corresponding Dirichlet series and let it converge for all s such that $\operatorname{Re} s > \alpha > 0$. Note that $\alpha = 1$ for Example 1 ($b_k \equiv 1$), $\alpha = 1/2$ for Example 5 with $d = 1$ ($j_1(k) = 1$ if k is an exact square and 0 otherwise). According to the well-known theorem (see Siegel [14]), $\alpha = d/2$ for $j_d(\cdot)$.

Thus, in the d-dimensional model of the ideal gas, the generating function can be replaced by

$$\prod \frac{1}{(1 - zx^k)^{[k^{\beta(d)}]}}, \quad \text{where } \beta(d) = \frac{d-2}{2}, \ \alpha = \beta + 1, \ d \geq 2.$$

In particular, our fundamental case $\prod 1/(1 - x^k)$ corresponds to $\beta(d) = 0$, or $d = 2$.

7. Fixed sizes of partitions. We only mention uniform distributions on partitions with fixed growth of the number of summands and their sizes. In this case, the corresponding diagrams belong to a rectangle with fixed growth of its sides; we must consider a sequence of multiplicative measures instead of a single one to apply this technique. The same situation occurs in the case of quasi-Boltzmann statistics (see [4]). We consider an example below.

8. Vector partitions. Let $d \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{Z}_+^d$, let $\mathcal{P}_{\mathbf{n}}^d$ be the set of all partitions of the vector $\mathbf{n} = (n^1, \dots, n^d)$ into summands with positive integer coordinates, and let $\mathcal{P}^d = \bigcup \mathcal{P}_{\mathbf{n}}^d$.

The generating function

$$\mathcal{F}(x_1, \dots, x_d) = \prod_{(k_1, \dots, k_d)} \frac{1}{1 - x_1^{k_1} \cdots x_d^{k_d}} = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} p(\mathbf{n}) x^{\mathbf{n}}$$

corresponds to the multiplicative measure that induces the uniform measure on the set $\mathcal{P}_{\mathbf{n}}^d$ of vector partitions.

Other multiplicative measures can be generalized to the d-dimensional case as well; for example,

$$\mathcal{F}(x) = \prod_{\mathbf{k} \in \mathbb{Z}_+^d} (1 + x^{\mathbf{k}}), \quad \text{and so on.}$$

A special class is formed by strict partitions (the product is taken over all vectors $(k_1, \dots, k_d) \in \mathbb{Z}_+^d$ with coprime (in the whole) coordinates, see [8]). Recall that for $d = 2$, there exist two different geometric interpretations of vector partitions (see §2). As mentioned above, in statistical physics, vector partitions correspond to systems with additional first integrals.

§4. Main Results

1. Weak equivalence of ensembles. Here we consider only ordinary (one-dimensional) partitions and do not choose the number of particles. Let μ_x be a multiplicative measure on $\mathcal{P}$, $x \in (0, x_0)$. According to §1, first we must find a scaling sequence $\{a_n\} = a$. Then we can construct the mapping $\tau_a \colon \mathcal{P} \to \mathcal{D}$.

Recall that $\mu_x = \sum_n \mathcal{F}(x)^{-1} Q_n x^n \mu^n$.

Theorem 4.1 (scaling). *For the multiplicative measure induced by the generating function*

$$\mathcal{F}(x) = 1 \Big/ \prod_k (1 - x^k)^{[k^\beta]}, \qquad 0 < x < 1, \ \beta \geq 0,$$

as the scaling $\{a_n\}$ such that a nontrivial limit of measures $\lim_{x \to 1} \tau_a^* \mu_x$ exists in the space $\mathcal{D}$ of generalized diagrams we can take

$$a_n = n^{1/(2+\beta)}.$$

In particular, for $\beta = 0$ *we have* $a_n = \sqrt{n}$.

The same scaling is appropriate for $\mathcal{F}(x) = \prod(1 + x^k)^{[k^\beta]}$.

Theorem 4.2 (singularity of the limit). *Under the scaling of Theorem* 1, *the limit* $\lim_{x\to 1} \tau_a \mu_x$ *exists and is a singular measure concentrated on some continuous curve.*

Theorem 4.3 (weak equivalence of ensembles). *Under the conditions of Theorem* 1, *the macrocanonical ensemble is weakly equivalent to the canonical one, i.e.,*

$$\lim_{x\to 1} \tau^* \mu_x = \lim_{n\to\infty} \tau^* \mu^n.$$

2. Explicit answers.

Theorem 4.4 (limit shape for the generalized one-dimensional Bose statistics). *Let*

$$\mathcal{F}(x) = \prod_{k=1}^{\infty} \frac{1}{(1 - x^k)^{[k^\beta]}}$$

be the generating function and let μ_x *be the corresponding multiplicative measure.*

Let the measures μ^n *on* $\mathcal{P}_n$ *be induced by the measures* μ_x. (*If* $\beta = 0$, *then* μ^n *is the uniform distribution on* $\mathcal{P}_n$.) *Introduce the mapping* $\tau_a \colon \mathcal{P}_n \to \mathcal{D}$ *for the scaling* $a = \{a_n\}$, $a_n = n^{1/(2+\beta)}$, *and the functions*

$$\widetilde{\varphi}_\lambda(t) = n^{-(1+\beta)/(2+\beta)} \sum_{k \geq t n^{1/(2+\beta)}} r_\lambda(k).$$

(*Recall that* $r_k(\lambda)$ *is the number of summands* k *in the partition* λ.)

Then for any ε, $\varepsilon > 0$, *and* a, b, $0 < a, b < \infty$, *there exist* n_0 *such that for all* $n > n_0$ *we have*

$$\mu^n\{\lambda \in \mathcal{P}_n : \sup |\widetilde{\varphi}_\lambda(t) - C_\beta(t)| < \varepsilon\} > 1 - \varepsilon,$$

where $C_\beta(t)$ *is the function (probability density) defined by*

$$C_\beta(t) = \int_t^\infty u^\beta \frac{e^{-cu}}{1 - e^{-cu}} \, du, \qquad (*)$$

and c *is a constant defined by* $\int_0^\infty C_\beta(t)\, dt = 1$.

In particular, for $\beta = 0$ *we have* $C_\beta(t) = -(\sqrt{6}/\pi)\ln(1 - e^{(\pi/\sqrt{6})t})$, $c = \sqrt{\zeta(2)}$, *or, in a more symmetric form,*

$$e^{-(\pi/\sqrt{6})x} + e^{-(\pi/\sqrt{6})y} = 1. \qquad (**)$$

This expression was first derived by means of some formulas in [15] (see [16, 19]) by the author, who later obtained this result in a more natural way that was generalized to a wide range of problems in the present paper.

For $\beta = 1$, we have

$$C_1(t) = -t\ln(1 - e^{-ct}) + c^{-1}\,\mathrm{Li}_2(e^{-ct}),$$

where Li_2 is the dilogarithm,

$$\mathrm{Li}_2(u) = \sum_{n=1}^{\infty} \frac{u^n}{n^2}.$$

The function $C_\beta(t)$ can be expressed by the series of the partial Γ-function [17]:

$$C_\beta(t) = \int_t^\infty \frac{u^\beta e^{-cu}}{1 - t^{-cu}}\, du = \sum_{k=1}^{\infty} \int_t^\infty u^\beta e^{-cku} du = \sum_{k=1}^{\infty} \frac{1}{c^\beta k^{\beta+1}} \int_{ckt}^\infty u^\beta e^{-u} du$$
$$= \frac{1}{c^\beta} \sum_{k=1}^{\infty} \frac{1}{k^{\beta+1}} \Gamma(\beta + 1, ckt) = \frac{1}{c^\beta} \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \Gamma(\alpha, ckt)$$

($\alpha = \beta + 1 = d/2$ is a simple pole of the corresponding Dirichlet series, see §3).

Remarks. 1. The measure μ^n on $\mathcal{P}_n$ corresponding to $\beta = (d-2)/2$, $d \geq 2$, coincides with the Bose statistics of ideal gas for dimension d (the number of particles is not fixed for this "photon case"). Thus, the function $C_\beta(t)$ in Theorem 4.4 (formula $(*)$) describes the density of the distribution of energy over the spectrum of particles; the typical energy of a particle has the order of $E^{2/(d+2)}$ $(E = n)$ and is bounded for $\beta > 0$ $(d > 2)$.

2. An explicit formula for the measure μ^n on $\mathcal{P}_n$ can readily be obtained in terms of the Gauss coefficients.

3. Multiplicative measures on the microcanonical ensembles $\mathcal{P}_{n,N}$ (that is, ensembles of partitions with chosen number of summands) can be obtained from the functions

$$\mathcal{F}(x) = \prod_{k=1}^{\infty} (1 - zx^k)^{-[k^\beta]}.$$

To obtain the thermodynamic limit, we must choose the growth

$$N = N(n) = vn^{d/(d+2)} = vn^{1-2/(2\beta+3)}.$$

The corresponding limit shape can also be found by our method; for $d = 3$, we obtain a natural explanation of the Bose–Einstein condensation. However, another growth of $N(n)$ can be considered as well.

Similar results can be proved in the case of partitions into mutually distinct summands (the Fermi statistics). Let

$$\mathcal{F}(x) = \prod_{k=1}^{\infty} (1 + x^k)^{[k^\beta]};$$

we construct the corresponding measures μ_x and μ^n on the macrocanonical ensemble $\mathcal{P}$ and on the canonical ensemble $\mathcal{P}_n$. In the above notation, the answer can be written in the same terms.

Theorem 4.5 (the limit shape for the generalized Fermi statistics). *For any ε, $\varepsilon > 0$, and a, b, $0 < a, b < \infty$, there exists n_0 such that for $n > n_0$ we have*

$$\mu^n \Big\{ \lambda \in \mathcal{P}_n : \sup_{t \in [a,b]} |\widetilde{\varphi}_\lambda(t) - C_\beta^f(t)| < \varepsilon \Big\} > 1 - \varepsilon,$$

where $C_\beta^f(t)$ is the function (probability density) given by

$$C_\beta^f(t) = \int_t^\infty u^\beta \frac{e^{-cu}}{1 + e^{-cu}} \, du \qquad (***)$$

with c defined by

$$\int_0^\infty C_\beta^f(t) \, dt = 1;$$

in particular, for $\beta = 0$ (the uniform distribution on the partitions into mutually distinct summands) we have

$$C_\beta^f(t) = (\sqrt{12}/\pi) \ln(1 + e^{-(\pi/\sqrt{12})t}) \quad or \quad e^{-(\pi/\sqrt{12})y} - e^{-(\pi/\sqrt{12})x} = 1.$$

Consider the quasi-Boltzmann statistics (see [5, 6])

$$\mathcal{F}(x) = e^{x/(1-x)} = \prod_{k=1}^{\infty} e^{x^k}.$$

It corresponds to the distribution on $\mathcal{P}$ defined by the formula for a multiplicative measure:

$$\mu_x\{\lambda : r_k(\lambda) = i\} = \frac{x^{k^i}}{i!} e^{-x^k}.$$

Theorem 4.6. *The limit shape for random partitions with the statistics μ_x is given by formula*

$$\lim_{x\to 1} \mu_x\Big\{\lambda : \sup_{0<t<\infty} |\widetilde{\varphi}_\lambda(t) - e^{-t}| < \varepsilon\Big\} = 1.$$

The same curve $C(t) = e^{-t}$ is the limit shape for the measures μ^n (induced by μ_x on $\mathcal{P}_n$) as $n \to \infty$.

Another class of statistics arises if one bounds the growth of summands in the partition and the number of summands. We consider the function

$$\mathcal{F}_N(x) = \prod_1^{\theta\sqrt{N}} \frac{1}{1 - tx^k} = \sum_{n,m} q_N(n,m) t^m x^n$$

and try to find the limit shape as $n = N \to \infty$ under the condition $m = \rho\sqrt{N}$. This means that the corresponding distribution is uniform on the partitions of N whose number of summands is at most $\rho\sqrt{N}$ and any summand is at most $\theta\sqrt{N}$. The corresponding diagram belongs to the rectangle $\rho\sqrt{N} \times \theta\sqrt{N}$, $\rho\theta \geq 1$.

Theorem 4.7. *The limit shape is defined by*

$$\frac{1 - e^{c\theta}}{1 - e^{-c(\theta+\rho)}} e^{-cy} + \frac{1 - e^{-c\rho}}{1 - e^{-c(\theta+\rho)}} e^{-cx} = 1.$$

For $\rho = \theta$ (square), this function is

$$e^{-cy} + e^{-cx} = 1 - e^{-c\theta}$$

with constant c defined by the condition $\int_0^\infty y\,dx = 1$. As $\theta \to \infty$, this function tends to that from Theorem 4.4 ($\beta = 0$), formula (**).

Now we consider vector partitions. We present the simplest example related to the uniform distribution μ^n on the vector partitions $\mathcal{P}^d_{\mathbf{n}}$. Let

$$\mathcal{F}(x) = \prod_k \frac{1}{1 - \mathbf{x}^k} = \sum p_d(\mathbf{n}) x^{\mathbf{n}};$$

where $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{k} = (k_1, \dots, k_d)$, $\mathbf{n} = (n_1, \dots, n_d)$, and $\mathbf{x}^{\mathbf{k}} = \prod_{i=1}^d x_i^{k_i}$. We define the scaling for the case $\mathbf{n} = (n, \dots, n)$:

$$\mathbb{R}^d_+ \ni t = (t_1, \dots, t_d) \to n^{(d+1)^{-1}} t \quad \text{and} \quad \widetilde{\varphi}_\lambda(t) = \frac{1}{n^{d/(d+1)}} \varphi(n^{(d+1)^{-1}} t).$$

Theorem 4.8. *For any ε, $\varepsilon > 0$, and d, $d \geq 2$, we have*

$$\lim_{n\to\infty} \mu^n\Big\{\lambda : \sup_t |\widetilde{\varphi}_\lambda(t) - C_d(t)| < \varepsilon\Big\} = 1,$$

where

$$C_d(t) = \gamma^{-d} \operatorname{Li}_d\Big(\exp\Big(-\gamma\sum_{i=1}^d t_i\Big)\Big), \quad \gamma = \zeta(d+1)^{(d+1)^{-1}}, \quad \operatorname{Li}_d(x) = \sum_{n=1}^\infty \frac{x^n}{n^d}, \quad d \geq 2.$$

Remarks. 1. For $d = 2$, the density C_d is bounded on $\mathbb{R}^d_+$ unlike the case $d = 1$.

2. For $d = 2$, the vector partition $(n_1, n_2) = \sum_i (n_1^i, n_2^i)$ has another geometric interpretation, namely, as a convex broken line that connects $(0,0)$ and $\mathbf{n} = (n_1, n_2)$ with the vertices at integral points. This interpretation is related to the ordering of summands in ascending order of the ratio n_1^i/n_2^i. It remains to consider the partial sums of these vectors in the above ordering. The corresponding asymptotic problem was studied in [8–10]; the method of the present paper makes it possible to refine these results.

3. The ordinate set of φ_λ is a $(d+1)$-dimensional Young diagram; however, this diagram is not arbitrary, because it necessarily satisfies a certain positivity condition (since φ_λ is a distribution function in a sense). Thus, we have found the limit shape for the uniform distribution on special d-dimensional Young diagrams. The problem about the limit shape of uniformly distributed many-dimensional diagrams is still open, even in the three-dimensional case.

§5. Statements of Lemmas

We sketch the proof of the results of §4. We use the notation of §§1–3. For the sake of simplicity, we consider ordinary (not vector) partitions only and multiplicative measures on $\mathcal{P}$ and $\mathcal{P}_n$, respectively. Each of these measures can be described by a generating function that is analytic in the disk $|x| \le 1$ and expanded in an infinite product:

$$\mathcal{F}(x) = \prod_k \mathcal{F}_k(x^k), \quad \mathcal{F}_k(y) = \sum_{r=0}^{\infty} s_k(r) y^r, \qquad s_k(r) \ge 0.$$

The measure μ_x is defined by formula (2.1), and the measure μ^n by formula (2.2). The relation between these measures is given by

$$\mu_x = \sum_n x^n \mathcal{F}(x)^{-1} Q_n \mu^n, \tag{5.1}$$

where Q_n is the partition function (see Lemma 2.1). If the scaling $a = \{a_n\}$ is chosen, then the mappings $\tau_a \colon \mathcal{P} \to \mathcal{D}$ are well defined, and the measures μ_x and μ^n map to $\tau_a^* \mu_x$ and $\tau_a^* \mu^n$. We preserve the same notation for them (μ) without risk of confusion, since the scaling is chosen uniquely.

Formula (5.1) shows that μ_x is a convex combination of the measures μ^n. Thus, the following statement is obvious.

Lemma 5.1. *If the measures μ^n have weak limit in $\mathcal{D}$, then the measures μ_x have the same limit as $x \to 1$.*

By the ergodic case (point case) we mean that in which a limit measure for μ_x as $x \to 1$ exists and is singular, i.e., is concentrated on some element of $\mathcal{D}$ and is a δ-measure; this element is called the *limit shape* (configuration, distribution) of the problem. If the limit exists but is not a δ-measure, we speak of the nonergodic case. (This is the case, for example, for the Haar statistics.)

The inverse statement needs "Tauberian" arguments and is nontrivial in our situation.

Lemma 5.2. *If the multiplicative measures μ_x have weak limit in $\mathcal{D}$ as $x \to 1$ and if the limit measure is singular, then the measures μ^n have the same limit.*

The proof is substantially based on the fact that the measures μ_x are multiplicative; in fact, it is a Tauberian-type theorem. The problem of general Tauberian theorems in this setting (for the nonergodic case and for nonmultiplicative measures) is of great interest and, seemingly, has not been studied. Now we can restrict our investigations to Gibbs' measures μ_x. Since they are multiplicative, the problem reduces to that of probability theory (namely, to the local limit theorem). Lemma 2 states, in fact, the equivalence of the macrocanonical ensemble to the original canonical ensemble.

Below we consider the generating functions of the form

$$\mathcal{F}(x) = \prod_{k=1}^{\infty} (1 - x^k)^{-[k^\beta]}, \quad \mathcal{F}(x) = \prod_{k=1}^{\infty} (1 + x^k)^{[k^\beta]}, \qquad \beta \ge 0.$$

The case of functions of several arguments or that of functions $\mathcal{F}(z, x) = \prod_k (1 + z x^k)^{[k^\beta]}$ is only more cumbersome.

The scaling for these functions is completely determined by the exponent β, as follows.

Lemma 5.3. *Any multiplicative measure μ_x determined by a function of the above type and transferred to the compactum $\mathcal{D}$ by the mapping τ_a^* with the scaling $a = \{a_n\}$, $a_n = n^{(2+\beta)^{-1}}$, has a singular weak limit $C = C(\,\cdot\,) \in \mathcal{D}$, that is,*

$$\lim_{x \to 1} \mu_x \Big\{ \lambda : \sup_{b_1 \le t \le b_2} |\widetilde{\varphi}_\lambda(t) - C(t)| < \varepsilon \Big\} = 1,$$

where $[b_1, b_2] \subset (0, \infty)$ is arbitrary segment in $\mathbb{R}_+$, i.e.,

$$\operatorname*{w-lim}_{x \to 1} \mu_x = \delta_C,$$

and $C(\cdot)$ is the limit shape.

Lemma 5.3 is proved together with explicit determination of the weak limit. Namely, at this point we face the same difficulty as in the saddle-point method. In fact, it suffices to find the expectation of the measure μ_x on $\mathcal{D}$ and to estimate its variance as $x \to 1$. However, we must choose x for which we can apply the property of being multiplicative (independent). Consider the expression

$$E_x\widetilde{\varphi}_\lambda(t) = E_x \frac{a_n}{n} \sum_{k \geq ta_n} r_k(\lambda) = E_x n(\lambda)^{\gamma-1} \sum_{k \geq tn(\lambda)^\gamma} r_k(\lambda), \qquad \gamma = (2+\beta)^{-1}, \ E_x \equiv E_{\mu_x}. \tag{5.2}$$

Since $n(\lambda)$ depends on λ, we must first choose $x = x_n$ such that $n(\lambda)$ has the desired value.

Lemma 5.4. *We have*

$$E_x r_k(\lambda) = y\,[\ln \mathcal{F}_k(y)]'\big|_{y=x^k}, \qquad E_x \sum_k k r_k \equiv E_x n(\lambda) = x\,[\ln \mathcal{F}(x)]'.$$

The equation (for $\mathcal{F}$ of the above type)

$$E_x n(\lambda) = x\,[\ln \mathcal{F}(x)]' = n \in \mathbb{N}$$

has a unique solution $x_n \in (0,1)$, and we have

$$\lim_{n\to\infty} E_x\Big(\frac{n(\lambda)}{n} - 1\Big)^2 = 0.$$

The choice of x_n is just the choice of a saddle-point contour (critical point in the Laplace method, and so on) for estimation of the partition function. The advantage of the probabilistic method is that it suffices to evaluate the variance instead of troublesome estimations near the saddle point (see Lemma 5.6 below).

The above lemma makes it possible to replace $n(\lambda)$ by n in (5.2), and thus to find the expectation $E_x\widetilde{\varphi}_\lambda(t)$, which gives the value of the limit shape at the point t.

Lemma 5.5. *For $\mathcal{F}(x) = \prod_{k=1}^\infty (1-x^k)^{-[k^\beta]}$ we have*

$$E_{x_n}\widetilde{\varphi}_\lambda(t) \equiv C(t) = \int_t^\infty u^\beta \, \frac{e^{-cu}}{1-e^{-cu}}\, du, \tag{5.3}$$

where c is a constant.

The lemma follows from the fact that the sum

$$E_{x_n}\widetilde{\varphi}_\lambda(t) \simeq \frac{1}{n^{1-\gamma}} \sum_{k \geq tn^\gamma}{}' E_{x_n} r_k(\lambda)$$

in (5.2) is an integral sum for the right-hand side of (5.3).

The same manipulations can be used for the other above-mentioned functions.

The last lemma states that the limit measure is singular, i.e., that it is a δ-measure concentrated at the limit of the expectations $C(\cdot)$.

Lemma 5.6. $\lim_{n\to\infty} E_{x_n}[\widetilde{\varphi}_\lambda(t) - C(t)]^2 = 0.$

Realization of this scheme for more general measures (even for multiplicative ones) encounters analytic difficulties, but it seems that the above scheme is adequate to many problems of asymptotic combinatorics, to applications and additive problems of number theory, to some geometric applications and, possibly, to statistical physics.

§6. Variational Principle

A completely different approach to problems of asymptotic combinatorics and, in particular, to the problem of limit shape is possible; this approach has features of the functional integral method, the minimal action principle, and the large deviation method in probability theory. For the sake of definiteness, we suppose that a measure μ^n on the Young diagrams is given, where the Young diagrams are defined on the sequence of lattices $\frac{1}{a_n}\mathbb{Z}_+ \times \frac{a_n}{n}\mathbb{Z}_+ \equiv L_n$, and let Γ be an element in $\mathfrak{D}$ that belongs to $\mathbb{R}^2_+$. Consider its ε-neighborhood $U_\varepsilon(\Gamma)$ and find $\mu^n\{\lambda : \lambda \in U_\varepsilon(\Gamma)\}$. Under an appropriate scaling γ_n we can find the limit

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n^\gamma}\ln\mu^n\{\lambda : \lambda \in U_\varepsilon(\Gamma)\}.$$

For chosen measures μ^n, this limit (if it exists) defines a functional of action on curves in $\mathfrak{D}$. It is natural to conjecture that the limit curves for statistics μ^n as $n\to\infty$ (if these limit curves exist as well) provide the maximum for this functional. The separate question is whether the set of maximum points for this functional coincides with the set of possible limit curves. If they do coincide, then we say that the *variational principle holds* (for the problem under consideration). The maximal value itself can be regarded as the entropy of the sequence μ^n, and the value of the functional on any curve as the capacity of this curve; see [8]. We state here an assertion that shows the validity of the variational principle for all examples of §§3–5. Nevertheless, an explicit formula for action is sometimes rather cumbersome.

Consider the uniform distribution on the partitions (diagrams) that belong to a square; to be more precise, we choose the square $[0,1]^2$ and consider the set X_n of all monotone nonincreasing step-functions φ, $\varphi(0)=1$, $\varphi(1)=0$, such that the steps of φ belong to the nodes of the lattice $(\frac{1}{n}\mathbb{Z})^2$, and its integral $\int_0^1\varphi(t)\,dt$ takes a given value ρ, $0<\rho<1/2$. Let μ^n be the uniform distribution on X_n. Let

$$f\in C^1([0,1]),\qquad \int_0^1 f(x)\,dx=\rho<\frac{1}{2},\qquad f(0)=1,\ f(1)=0,\ f'(x)\le 0,\ x\in[0,1].$$

The following theorem formulates the variational principle for this problem; the Lagrangian looks like the entropy.

Theorem 6.1. *The relation*

$$\lim_{\varepsilon\to 0}\lim_{n}\frac{1}{\sqrt{n}}\ln\mu_n\{\varphi\in X_n : |\varphi-f|<\varepsilon\}=\int_0^1 G(\sqrt{2|f'(t)|})\,dt$$

holds, where

$$G(u)=(2/u+u)\ln(1-e^{-c(u+2/u)})-u\ln(1-e^{-cu})-(2/u)\ln(1-e^{-2c/u}),\tag{6.1}$$

and c is a critical point of the function G.

A similar variational principle can be formulated for the entire semiaxis. In this problem (of limit shape for the partitions whose diagrams belong to the square), the variational principle holds, that is, the limit shape is a unique solution of the variational problem indeed.

Theorem 6.2. *The limit shape from Theorem* 6.1 *is a unique solution of the variational maximization problem with Lagrangian* (6.1).

It is interesting to compare the Lagrangian (6.1) to that related to the additional assumption that the functions f are convex, i.e., $f''(x)\ge 0$, in which case we consider only the diagrams whose upper linear envelope is convex. In this case, the variational problem was stated in [8], and the Lagrangian has the form

$$\int_0^1 f''(t)^{1/3}dt \equiv \int_{\Gamma_f}\varkappa^{1/3}ds,$$

where $\varkappa$ is the curvature of the graph Γ_f of function f. It was proved in [8] that the variational principle holds in this case as well.

References

1. A. M. Vershik, Asymptotic Combinatorics and Algebraic Analysis, Proceedings of ICM, Vol. 2, Birkhäuser, Zurich (1995).
2. R. Arratia and S. Tavare, "Limit theorems for combinatorial structures," Rand. Structure Alg., **3**, 321–345 (1992).
3. B. Fristedt, "The structure of random partitions of large integers," Trans. Am. Math. Soc., **337**, 703–735 (1993).
4. J. Pimtan and R. Aldous, "Brownian bridge asymptotics for random mappings," Rand. Structure Alg., **5**, 487–512 (1994).
5. K. Huang, Statistical Mechanics, John Wiley & Sons, New York–London (1963).
6. A. Ya. Khinchin, Mathematical Foundations of Quantum Statistics [in Russian], Fizmatlit, Moscow–Leningrad (1951).
7. G. Freiman, New Analytical Results in Subset-Sum Problems, Combinatorics and Algorithms, Jerusalem (1988).
8. A. M. Vershik, "The limit form of convex integral polygons and related questions," Funkts. Anal. Prilozhen., **28**, No. 1, 17–25 (1994).
9. I. Barany, "The limit shape of convex lattice polygons," Discrete Comput. Geom., **13**, 279–295 (1995).
10. Ya. G. Sinai, "A probabilistic approach to the analysis of the statistics of convex polygonal lines," Funkts. Anal. Prilozhen., **28**, No. 2, 41–48 (1994).
11. A. M. Vershik, "Statistics of the set of naturals partitions," in: Probab. Theory Math. Stat., Vol. 2, 683–694, VNU Sci. Press.
12. Yu. Yakubovich, "Asymptotics of random partitions of a set," Zap. Nauchn. Semin. POMI, **223**, 227–250 (1995).
13. G. E. Andrews, The Theory of Partitions, Addison Wesley, London–Amsterdam (1976).
14. C. L. Siegel, Lecture on Advanced Analytic Number Theory, Tata Inst., Bombay (1961).
15. M. Szalay and R. Turan, "On some problems of statistical theory of partitions. I," Acta Math. Acad. Sci. Hungr., **29**, 361–379 (1977).
16. A. M. Vershik and S. V. Kerov, "Asymptotics of maximal and typical dimensionality of irreducible partitions of the symmetric group," Funkts. Anal. Prilozhen., **19**, No. 1, 25–36 (1985).
17. I. M. Ryzhik and I. S. Gradstein, Tables of Integrals, Sums, Series, and Products [in Russian], 4th ed., Fizmatlit, Moscow (1963).
18. A. M. Vershik and A. A. Shmidt, "Limit measures arising in the asymptotic theory of symmetric groups," Teor. Veroyatn. Primenen., **22**, No. 1, 72–88 (1978); **23**, No. 1, 42–54 (1979).
19. A. M. Vershik, "Statistical sum related to Young diagrams," Zap. Nauchn. Sem. LOMI, **164**, 20–29 (1987); English transl. in J. Soviet Math., **47**, 2379–2386 (1989).

Translated by Yu. Yakubovich