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RESAMPLING STUDENT'S t-TYPE STATISTICS

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Abstract. The present paper establishes conditional and unconditional central limit theorems for various resampling procedures for the t-statistic. The results work under fairly general conditions and the underlying random variables need not to be independent. Specific examples are then the $m(n)$ (double) bootstrap out of $k(n)$ observations, the Bayesian bootstrap and two-sample t-type permutation statistics. In case when $m(n)/k(n)$ is bounded away from zero and infinity necessary and sufficient conditions for the conditional central limit law of the bootstrap t-statistics are established. For high resampling intensity when $m(n)/k(n)$ tends to infinity the following general result is obtained. Without further other assumptions the bootstrap makes the resampled t-statistic automatically normal. The results are based on a general conditional limit theorem for weighted resampling statistics which is of own interest.

Key words and phrases: Student's t-statistic, Welch statistic, two-sample permutation statistic, weighted bootstrap, double bootstrap, Bayesian bootstrap, central limit theorem, conditional central limit theorem.

1. Introduction

Consider a triangular array of arbitrary real random variables $X_{n,i}$, $1 \leq i \leq k(n)$, on some probability space $(\Omega, \mathcal{A}, P)$ with $k(n) \to \infty$ as $n \to \infty$. The one-sample t-statistic is then

$$t_n = \frac{k(n)^{-1/2} \sum_{i=1}^{k(n)} X_{n,i}}{\left(\frac{1}{k(n)-1} \sum_{i=1}^{k(n)} (X_{n,i} - \overline{X}_n)^2 \right)^{1/2}} \tag{1.1}$$

given by the mean $\overline{X}_n = 1/k(n) \sum_{i=1}^{k(n)} X_{n,i}$.

Throughout we will discuss the limit behaviour of various resampling versions of t_n and tests of t-test type. Specific examples are all kind of bootstrap and permutation resampling statistics. The results are typically applied in testing statistical hypotheses when critical values of t_n are derived by resampling methods under a nonparametric null hypothesis, see Section 5 and Janssen and Pauls (2003). At this stage we like to emphasize that the $X_{n,i}$ may come from arbitrary alternatives which may no longer be independent or identically distributed. In a forthcoming paper the present results will be applied to establish power functions for resampling tests. Two-sample t-statistics are treated similarly in Section 4. All proofs can be found in Section 6.

In the next step let us draw $m(n)$ resampling variables $X_1^*, \dots, X_{m(n)}^*$ given the original data $X_{n,i}$. Below various different resampling schemes are discussed in detail.

The conditional resampling statistic of t_n is given by

$$t_n^* = \frac{m(n)^{1/2}(\overline{X}^*_{k(n),m(n)} - \overline{X}_n)}{((m(n)-1)^{-1}\sum_{i=1}^{m(n)}(X_i^* - \overline{X}^*_{k(n),m(n)})^2)^{1/2}} \tag{1.2}$$

where $\overline{X}^*_{k(n),m(n)} = \frac{1}{m(n)}\sum_{i=1}^{m(n)} X_i^*$ denotes the $m(n)$-resampling mean of $k(n)$ variables.

We like to focus the attention of the reader to the fact that the denominator of t_n is also subject of the resampling procedure. This has sometimes an interpretation as a variance correction. Such studentized resampling procedures are recommended in the guidelines for bootstrap testing, see Hall and Wilson (1991), Beran (1997), Bickel and Freedman (1981). For the bootstrap of i.i.d. variables we refer to Mason and Shao (2001) who established necessary and sufficient conditions for asymptotic normality of the bootstrap version (1.2). The present results also apply to the double bootstrap which is important for prepivoted test statistics, see Beran (1988). Another application is given for studentized permutation tests, see Neuhaus (1993), Janssen (1997) and Janssen and Mayer (2001) for earlier results.

In this paper we will investigate conditional and unconditional central limit theorems for t_n^* and its two-sample version. The conditional results rely on a metric d on the set of probability measures $\mathcal{M}_1(\mathbb{R})$ on $\mathbb{R}$ such that convergence in $(\mathcal{M}_1(\mathbb{R}), d)$ is equivalent to weak convergence.

Introduce a standard normal random variable Z and let I_n denote the set $I_n = \{\sum_{i=1}^{k(n)}(X_{n,i} - \overline{X})^2 > 0\}$. If the denominator of a statistics happens to become zero we then define the statistics to be zero (according to $0/0 = 0$). Throughout, we will consider the conditional distribution $\mathcal{L}(\cdot \mid \vec{X}_n)$ given the vector of variables $\vec{X}_n := (X_{n,1}, \ldots, X_{n,k(n)})$. We say that a conditional central limit theorem (CCLT) holds for t_n^* if

$$d(\mathcal{L}(t_n^* \mid \vec{X}_n), \mathcal{L}(Z\mathbf{1}_{I_n} \mid \vec{X}_n)) \to 0 \tag{1.3}$$

in P-probability as $n \to \infty$. The resampling step will be described by weight functions $W_{n,i}$ which are independent of the X's, see Section 2.

Recall that under the extra condition

$$P(I_n) \to 1 \quad \text{as} \quad n \to \infty \tag{1.4}$$

the CCLT then implies the unconditional central limit theorem (CLT)

$$t_n^* \to Z \tag{1.5}$$

in distribution as $n \to \infty$, confer Remark 1 below.

The following example motivates the treatment of the CCLT. The applications for composite null hypotheses are discussed in Section 5 in more detail. The present discussion about resampling tests refers to Janssen and Pauls (2003), Sections 2 and 6. The use of studentized resampling tests for Behrens-Fisher type problems was earlier pointed out in Janssen (1997).

Example 1. Suppose that the means $E_P(X_{n,i})$ always exist. We like to establish tests for various composite (nonparametric) null hypotheses $\mathcal{P}_0^{(\cdot)}$ given by heterogeneous distributions of the $X_{n,i}$ with the same expectation. Under the null hypothesis the $X_{n,i}$ are in general neither independent nor identical distributed.

(a) *One-sample problem.* The composite null hypothesis is here given by

$$\mathcal{P}_0^{(1)} \subset \{P : E_P(X_{n,i}) = 0 \text{ for all } i \leq k(n)\}. \tag{1.6}$$

One sided alternatives are given by positive means. Two sided tests can be treated similarly. A good test statistic is the t-statistic t_n defined in (1.1).

Suppose that for each $P \in \mathcal{P}_0^{(1)}$ the unconditional limit theorem

$$t_n \to Z \tag{1.7}$$

holds in distribution where Z is again standard normal. If $u_{1-\alpha}$ denotes the $(1-\alpha)$ standard normal quantile then

$$\varphi_n := \mathbf{1}_{(u_{1-\alpha},\infty)}(t_n) \tag{1.8}$$

defines tests of asymptotical level α tests under $\mathcal{P}_0^{(1)}$. Resampling tests can now be applied in order to establish more accurate critical values at finite sample size. The procedure work as follows.

- Keep the data $X_{n,i}(\omega)$ fixed.
- Take a resampling statistic t_n^* of type (1.2) and calculate the conditional $(1-\alpha)$-quantile $c_n^*(\alpha)$ of $\mathcal{L}(t_n^* \mid X_{n,1}, \ldots, X_{n,k(n)})$.
- The resampling test is given by

$$\varphi_n^* = \mathbf{1}_{(c_n^*(\alpha),\infty)}(t_n). \tag{1.9}$$

The CCLT, stated in (1.4), then implies

$$c_n^*(\alpha) \to u_{1-\alpha} \text{ in } P\text{-probability and } E_P(|\varphi_n - \varphi_n^*|) \to 0 \tag{1.10}$$

as $n \to \infty$ for $P \in \mathcal{P}_0^{(1)}$. Then we say that the resampling tests work well.

(b) *Two-sample problem*, confer with Section 4. Suppose that the $X_{n,i}$ are derived in two groups of sample sizes n_1 and n_2 with $n_1 + n_2 = k(n)$ and means

$$E_P(X_{n,i}) = b_1 \quad \text{for} \quad i \leq n_1 \quad \text{and} \quad E_P(X_{n,i}) = b_2$$

otherwise for $i > n_1$. The null hypothesis is a composite subset

$$\mathcal{P}_0^{(2)} \subset \{P : b_1 = b_2\} \tag{1.11}$$

which is tested against one sided alternatives with $b_1 > b_2$. If we replace t_n and t_n^* by their two-sided counterparts $t_{n,II}((X_{n,i})_i)$, given in Section 4, the same methodology as in (1.7)–(1.10) can be developed. Detail are again discussed in Section 5.

Note that permutation tests are special resampling tests of type (1.9) which yields good results also for some non i.i.d. null hypotheses (1.11), see Janssen (1997) for a discussion of the Behrens-Fisher problem. Of course permutation tests do not work for the one-sample problem discussed in part (a). Here various bootstrap procedures can be recommended.

In the next section we will derive very general results about CCLT for resampling statistics. At this stage two important special cases are mentioned first, the bootstrap and the double bootstrap. The $m(n)$ out of $k(n)$ bootstrap denotes as usually resampling with replacement from the data of size $m(n)$.

The $m(n)$-double bootstrap applies two bootstrap step. Draw first a $m(n)$ out of $k(n)$-bootstrap sample and take then a $m(n)$-bootstrap sample from the new variables.

We will see that the CCLT works under very weak assumptions. One important condition will be:

$$\max_{1\le i\le k(n)} \frac{(X_{n,i}-\overline{X}_n)^2}{\sum_{i=1}^{k(n)}(X_{n,i}-\overline{X}_n)^2} \to 0 \quad \text{in probability.} \tag{1.12}$$

THEOREM 1.1. (a) *Suppose that the condition* (1.12) *holds. Then the CCLT holds for the* $m(n)$ *out of* $k(n)$*-bootstrap if*

$$\liminf_{n\to\infty} \frac{m(n)}{k(n)} > 0 \qquad \textit{holds.} \tag{1.13}$$

(b) *If in addition to* (1.13) *the assumption*

$$\limsup_{n\to\infty} \frac{m(n)}{k(n)} < \infty \tag{1.14}$$

holds then condition (1.12) *is also necessary for the bootstrap CCLT.*

(c) (*High resampling frequency*) *The CCLT always holds for the* $m(n)$ *out of* $k(n)$*-bootstrap if* $\lim_{n\to\infty} m(n)/k(n) = \infty$.

(d) *Under the additional condition*

$$\frac{m(n)^2}{k(n)^3} \to 0 \tag{1.15}$$

the assertions (a) *and* (c) *remain true for the* $m(n)$*-double bootstrap.*

We see that in case (1.4) high resampling frequency makes the bootstrap statistic t_n^* automatically normal without any further assumptions. However, such a design can not approximate the true underlying finite sample size statistic in general. The reason for this result can be explained by the series expansion for the limits of resampling statistics given in the next section, where all terms of the series become normal.

Remark 1. The additional assumption (1.4) together with the CCLT implies the CLT. It is easy to see that conditional convergence implies unconditional convergence, see also the Appendix of Janssen and Pauls (2003).

Example 2. Let $X_{n,i}$ denote an array of rowwise independent random variables which are infinitesimal, i.e.

$$\max_{1\le i\le k(n)} P(|X_{n,i}| \ge \varepsilon) \to 0 \quad \text{for each} \quad \varepsilon > 0. \tag{1.16}$$

Let the suitably centered partial sum $\sum_{i=1}^{k(n)} X_{n,i} - a_n$, $a_n \in \mathbb{R}$, be convergent in distribution to some non-constant infinitely divisible random variable ξ. Suppose that (1.13) and (1.14) hold. As consequence of Theorem 1.1 we have: The CCLT holds for the $m(n)$ out of $k(n)$ bootstrap iff ξ is a normal random variable. In this case also (1.4) holds and the CLT is valid. To see this recall that the central limit theorem for triangular arrays implies (1.4) and (1.12), see Araujo and Giné (1980) for details.

Conversely, let the CCLT be true. Then it is easy to see that the condition (1.12) implies

$$\max_{1\le i\le k(n)} |X_{n,i}| \to 0$$

in probability. This corresponds to the case when the Lévy-measure of ξ vanishes. Since ξ is not constant ξ must be normal. This result was earlier proved by Mason and Shao (2001) for i.i.d. variables X_i and schemes $\delta_n \sum_{i=1}^n X_i - a_n$. The analogue for the numerator of (1.2), the non-studentized version, is due to Janssen and Pauls (2003).

2. Weighted resampling statistics

Up to some constants the numerator of t_n^* given by (1.2) can be written as weighted resampling statistic and it can be handled by the results of Janssen and Pauls (2003). However, the main results of that paper rely on the crucial assumption of the L_2-convergence of the weights which is sometimes hard to verify, for instance for the double bootstrap. Throughout, we like to present a device how to substitute the L_2-convergence by simpler conditions which are only based on distributional convergence of the weights.

Introduce a triangular array of random weights $W_{n,i} : (\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P}) \to \mathbb{R}$ for $1 \le i \le k(n)$. We will introduce the following conditions:

$$(W_{n,1},\dots,W_{n,k(n)}) \text{ is exchangeable,} \tag{2.1}$$

$$\max_{1\le i\le k(n)} |W_{n,i} - \overline{W}_n| \to 0 \text{ in } \tilde{P}-\text{probability,} \tag{2.2}$$

$$\sum_{i=1}^{k(n)} (W_{n,i} - \overline{W}_n)^2 \to 1 \text{ in } \tilde{P}-\text{probability.} \tag{2.3}$$

We will show that the L_2-convergence used in Janssen and Pauls (2003) can be substituted by the following condition. The weights

$$k(n)^{1/2}(W_{n,1} - \overline{W}_n) \to Z_1 \tag{2.4}$$

are convergent in distribution to some limit variable with $E(Z_1) = 0$ and $\operatorname{Var}(Z_1) = 1$. The resampling statistic corresponding to $\sum_{i=1}^{k(n)} X_{n,i}$ is then

$$T_n^* = k(n)^{1/2} \sum_{i=1}^{k(n)} W_{n,i}(X_{n,i} - \overline{X}_n). \tag{2.5}$$

Weighted resampling statistics of this kind were first studied by Mason and Newton (1992).

Example 3. (a) The $m(n)$ out of $k(n)$-bootstrap weights are

$$W_{n,i} = m(n)^{1/2}\left(\frac{1}{m(n)} M_{n,i} - \frac{1}{k(n)}\right) \tag{2.6}$$

given by a multinomial distributed random variable $(M_{n,1},\dots,M_{n,k(n)})$ with sample size $m(n)=\sum_{i=1}^{k(n)} M_{n,i}$ and equal success probability. The conditions (2.1)–(2.4) are valid whenever $\lim_{n\to\infty} m(n)/k(n)=c\in(0,\infty]$ exists. For $c=\infty$ the limit Z_1 is standard normal and for $c<\infty$ it is equal in distribution to $Z_1 \stackrel{\mathcal{D}}{=} c^{-1/2}(X-c)$ where X is a Poisson random variable with mean c. The details of the proof are given in Janssen and Pauls ((2003), (8.37)–(8.46)).

(b) The $m(n)$-double bootstrap can be described by the weights

$$W_{n,i}=\frac{m(n)^{1/2}}{\sqrt{2}}\left(\frac{1}{m(n)}M'_{n,i}-\frac{1}{k(n)}\right). \tag{2.7}$$

Here $(M'_{n,1},\dots,M'_{n,k(n)})$ denotes a conditional multinomial distributed variable with sample size $m(n)=\sum_{i=1}^{k(n)} M'_{n,i}$ and success probability $M_{n,i}/m(n)$ for the i-th cell given by the first step (a). The condition $m(n)/k(n)\to c>0$ together with assumption (1.15) implies that the conditions (2.1)–(2.4) hold with $\operatorname{Var}(Z_1)=1$. Details are discussed in Lemma 6.2.

The weight functions are always independent from the data. For these reasons let the corresponding random variables $Z_i,\tilde{Z}_j,Z^{(\omega)}:\tilde{\Omega}\to\mathbb{R}$ of Theorem 2.1 be first specified on $\tilde{\Omega}$ for fixed $\omega\in\Omega$ whereas $\zeta_i,\tilde{\zeta}_j,Y_{n,i},\Pi$ are random variables on Ω with $\Pi(\omega):=\omega$. All random variables can be defined in an obvious manner via projections on the joint probability space $(\Omega\times\tilde{\Omega},\mathcal{A}\otimes\tilde{\mathcal{A}},P\otimes\tilde{P})$. Let $X_{1:k(n)}\le X_{2:k(n)}\le\cdots\le X_{k(n):k(n)}$ denote the order statistics of the array $X_{n,i}$, $1\le i\le k(n)$, and set $X_{i:k(n)}=0$ whenever $i\notin\{1,\dots,k(n)\}$.

THEOREM 2.1. *Suppose that the conditions* (2.1)–(2.4) *hold for the weights. Choose two jointly independent copies* $(Z_i)_{i\in\mathbb{N}},(\tilde{Z}_j)_{j\in\mathbb{N}}$ *of the limit variable of* (2.4). *Let* $\overline{X}_n\to 0$ *holds in probability and suppose that we have distributional convergence*

$$\left((X_{i:k(n)})_{i\in\mathbb{N}},(X_{k(n)+1-j:k(n)})_{j\in\mathbb{N}},\left(\sum_{i=1}^{k(n)}(X_{n,i}-\overline{X}_n)^2\right)^{1/2}\right) \tag{2.8}$$
$$\xrightarrow{\mathcal{D}}((\xi_i)_{i\in\mathbb{N}},(\tilde{\xi}_j)_{j\in\mathbb{N}},\xi_0)$$

of the joint distributions on $\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}\times\mathbb{R}$. *Then*

$$d(\mathcal{L}(T_n^*\mid X_{n,1},\dots,X_{n,k(n)}),\mathcal{L}(\tilde{T}_0\mid\Pi))\to 0 \tag{2.9}$$

holds in P*-probability, where* $\tilde{T}_0$ *is given by*

$$\tilde{T}_0=\sum_{i=1}^{\infty}Z_i\xi_i+\sum_{j=1}^{\infty}\tilde{Z}_j\tilde{\xi}_j+\tilde{Z}^{(\Pi)},$$

and $\tilde{Z}^{(\omega)}$ *is a centered normal random variable with variance*

$$\tilde{\sigma}^2(\omega)=\xi_0^2(\omega)-\sum_{i=1}^{\infty}\xi_i^2(\omega)-\sum_{j=1}^{\infty}\tilde{\xi}_j^2(\omega)$$

which is independent of all other variables.

Example 4. (Bayesian bootstrap, Rubin (1981), Mason and Newton (1992)) Consider a sequence $(\eta_i)_{i\in\mathbb{N}}$ of i.i.d. standard exponential distributed random variables with $E(\eta_i) = 1$. The Bayesian bootstrap is given by the weights

$$W_{n,i} = k(n)^{1/2}\left(\frac{\eta_i}{\sum_{j=1}^{k(n)}\eta_j} - \frac{1}{k(n)}\right). \tag{2.10}$$

If $k(n) \to \infty$ it is easy to see that the conditions (2.1)–(2.4) hold with $Z_1 \stackrel{\mathcal{D}}{=} (\eta_1 - 1)$. More general weights of this type are discussed in Pauls (2002).

Remark 2. (a) A weaker version of the crucial assumption (2.4) may also be used to classify the limit distributions of resampling statistics and to reconsider Theorems 1 and 2 of Janssen and Pauls (2003). Let (2.4) hold with $\mathrm{Var}(Z_1) \le 1$. Notice that according to Lemma 2 of Janssen and Pauls (2003) weak convergence of (2.4) already implies $E(Z_1) = 0$. The weak accumulation points in Theorems 1 and 2 of that paper are then also given by independent copies Z_i and $\tilde{Z}_j$ of Z_1. The proof follows from Lemma 6.1. Notice that in case of $\mathrm{Var}(Z_1) < 1$ the additional variable $Z^{(\Pi)}$ may be much more complicated than in Theorem 2.1.

(b) The present paper benefits from the approach of Mason and Newton (1992). Bootstrap limit theorems for partial sums, similar to the numerator of (1.2), can also be found in del Barrio *et al.* (1999, 2002) and del Barrio and Matrán (2000).

3. t-statistics with one-sample

In this section general t-type statistics given by weights are resampled. The present approach has the advantage that various different resampling procedures like the ordinary, the double, and the Bayesian bootstrap can be treated simultaneously. Suppose that there exist further weight functions $W'_{n,i}$ with

$$W_{n,i} = W'_{n,i} - \frac{m(n)^{1/2}}{k(n)}, \qquad \overline{W}_n = 0. \tag{3.1}$$

As in the first section let $m(n)$ be an additional sequence which can be viewed as the resampling sample size. By definition let then

$$t^*_{n,W} := \frac{k(n)^{1/2}\sum_{i=1}^{k(n)} W_{n,i}X_{n,i}}{\left(\frac{k(n)}{m(n)^{1/2}}\left[\sum_{i=1}^{k(n)} W'_{n,i}X^2_{n,i} - \left(\frac{1}{m(n)^{1/4}}\sum_{i=1}^{k(n)} W'_{n,i}X_{n,i}\right)^2\right]\right)^{1/2}} \tag{3.2}$$

denote the weighted resampling t-statistic.

The (double) bootstrap t-statistics are special forms of (3.2). If the resampling procedure is given by resampling variables X_i^*, $1 \le i \le k(n)$, and if

$$\sum_{i=1}^{k(n)} W'_{n,i}X_{n,i} = m(n)^{-1/2}\sum_{i=1}^{m(n)} X_i^*, \qquad \sum_{i=1}^{k(n)} W'_{n,i}X^2_{n,i} = m(n)^{-1/2}\sum_{i=1}^{m(n)} X_i^{*2} \tag{3.3}$$

holds then t_n^* of (1.2) can be expressed by

$$t_n^* = \left(\frac{m(n)-1}{m(n)}\right)^{1/2} t_{n,W}^*. \tag{3.4}$$

In case of the Bayesian bootstrap we set $m(n) = k(n)$ and consider just (3.2) since (3.3) is not defined.

THEOREM 3.1. *Suppose that the conditions* (2.1)–(2.4), (3.1) *and* (1.13) *hold. Then*

(a) *The condition* (1.12) *implies the CCLT for* $t_{n,W}^*$.

(b) *If the limit variable* Z_1 *of* (2.4) *is standard normal and* $\lim_{n\to\infty} m(n)/k(n) = \infty$ *holds then the CCLT always holds for* $t_{n,W}^*$ *also when* (1.12) *is violated.*

4. Two-sample t-statistics

Let $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n_2}$ denote random variables from two different samples. Then two-sample t-type statistics are given by

$$t_{n,II} = \left(\frac{n_1 n_2}{n_1+n_2}\right)^{1/2} \frac{\overline{X}_{n_1} - \overline{Y}_{n_2}}{V_n^{1/2}} \tag{4.1}$$

and suitable variance estimators V_n where $t_{n,II} = t_{n,II}(X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})$. The choice of V_n depends on the underlying statistical problem. For instance, if the variances are assumed to be the same for both groups under some underlying null hypothesis then the variance estimator is $V_n = V_{n,1}$ with

$$V_{n,1} := (n_1+n_2-2)^{-1}\left[\sum_{i=1}^{n_1}(X_i - \overline{X}_{n_1})^2 + \sum_{i=1}^{n_2}(Y_i - \overline{Y}_{n_2})^2\right]. \tag{4.2}$$

If the variances are not homogeneous the choice of the estimator $V_n = V_{n,2}$ leads to a Welch-type statistics by (4.1) which is of interest for Behrens-Fisher type problems, see Janssen (1997) for further references and applications in statistics. In this case we define

$$V_{n,2} = \frac{n_1 n_2}{n_1+n_2}\left[\frac{1}{n_1(n_1-1)}\sum_{i=1}^{n_1}(X_i - \overline{X}_{n_1})^2 + \frac{1}{n_2(n_2-1)}\sum_{i=1}^{n_2}(Y_i - \overline{Y}_{n_2})^2\right]. \tag{4.3}$$

In this section $t_{n,II}$ is resampled where again the X- and Y-variables are given by triangular arrays where the second index is suppressed. The treatment is again based on the previous results where now $k(n) := n_1 + n_2$ denotes the total sample size and

$$X_{n,i} := X_i \quad \text{for } i \le n_1 \qquad \text{and} \qquad X_{n,n_1+i} := Y_i \quad \text{for } i \le n_2. \tag{4.4}$$

In addition to the bootstrap type resampling procedures permutation statistics are of strong interest. They are based on random uniformly distributed permutations $\sigma = (\sigma(i))_{i\le k(n)}$ of the index set $\{1, \ldots, k(n)\}$ which are mutually independent of the X's. The permutation resampling variables of (4.4) are now

$$X_i^* = X_{n,\sigma(i)} \qquad 1 \le i \le k(n) \tag{4.5}$$

and the permutation statistic is now

$$t^*_{n,II,p} = t_{n,II}((X_{n,\sigma(i)})_{i\le k(n)}). \tag{4.6}$$

Throughout, we will always assume that

$$0 < \liminf \frac{n_1}{n_2} \le \limsup \frac{n_1}{n_2} < \infty, \qquad \lim_{n\to\infty} k(n) = \infty, \tag{4.7}$$

holds. For convenience let $P(V_n > 0) \to 1$ always hold in any case which implies $\mathbf{1}_{I_n} \to 1$ in probability.

THEOREM 4.1. *Under the conditions* (1.12) *and* (4.7) *the CCLT holds for the permutation statistic* $t^*_{n,II,p}$ *for both variance estimators* $V_{n,1}$ *and* $V_{n,2}$.

Theorem 4.1 generalizes parts of Janssen ((1997), Section 3). The present two-sample permutation distributions are typically used to adjust conditional critical values for the test statistic $t_{n,II}$ at the null hypothesis

$$E(X_1) = E(Y_1), \tag{4.8}$$

see Janssen (1997) for further details and statistical applications. Other authors like Beran (1988) proposed the bootstrap or double bootstrap for two-sample statistics. If we have (4.8) in mind the bootstrap version of $t_{n,II}$ has a slightly different form than (4.6). Consider n_1 bootstrap or double bootstrap variables $X_1^*, \dots, X^*_{n_1}$ of the X's and similarly $Y_1^*, \dots, Y^*_{n_2}$ (double) bootstrap variables of the other group where the resampling sample size always coincides with the underlying sample size. The two-sample (double) bootstrap statistic is then given by

$$t^*_{n,II,b} = \left(\frac{n_1 n_2}{n_1+n_2}\right)^{1/2} \frac{(\overline{X}^*_{n_1} - \overline{X}_{n_1}) - (\overline{Y}^*_{n_2} - \overline{Y}_{n_2})}{V_n^{1/2}(X_1^*, \dots, X^*_{n_1}, Y_1^*, \dots, Y^*_{n_2})}. \tag{4.9}$$

Again our choice will be $V_n = V_{n,1}$ or $V_n = V_{n,2}$.

THEOREM 4.2. *Suppose that the conditions* (4.7) *and* (1.12) *hold for the pooled sample. Then the CCLT holds as well for the two-sample bootstrap as for the two-sample double bootstrap under the following conditions*:

(a) *For* $t^*_{n,II,b}$ *with denominator* $V_n^{1/2} = V_{n,2}^{1/2}$.

(b) *For* $t^*_{n,II,b}$ *with denominator* $V_n^{1/2} = V_{n,1}^{1/2}$ *whenever*

$$\frac{\frac{1}{n_1}\sum_{i=1}^{n_1}\left(X_{n,i} - \frac{1}{n_1}\sum_{j=1}^{n_1} X_{n,j}\right)^2}{\frac{1}{n_2}\sum_{i=1}^{n_2}\left(X_{n,n_1+i} - \frac{1}{n_2}\sum_{j=1}^{n_2} X_{n,n_1+j}\right)^2} \to 1 \tag{4.10}$$

in P-probability.

Remark 3. (a) Theorem 4.2 does not require the assumption (4.8).

(b) Condition (4.10) reflects the variance homogeneity which is used to motivate the denominator $V_{n,1}^{1/2}$.

5. Applications to resampling tests

The CCLT applies to the general testing problems stated in Example 1. First of all let us recall the meaning of the studentization given by the denominator of the statistics for resampling, see also Janssen (1997) for a discussion of the concrete Behrens-Fisher problem. Under the present heterogeneous null hypotheses (1.6) and (1.11) the numerator of (1.1) alone (i.e. the means or the difference of the means, respectively) may have the wrong conditional resampling variance. In case of the two-sample permutation statistic the numerator of (4.1) has the wrong permutation variance in general. This gap is corrected by taking studentized statistics into account where the denominator is also resampled. This procedure stands into accordance with general guidelines for bootstrap testing, see Hall and Wilson (1991) and Beran (1997).

Throughout, we will use the notation of Example 1 of Section 1. We will compare the t-type tests φ_n of asymptotic level α with their conditional counterparts φ_n^* under the null hypotheses $\mathcal{P}_0^{(1)}$ and $\mathcal{P}_0^{(2)}$. Observe that the CCLT under each $P \in \mathcal{P}_0^{(\cdot)}$ implies

$$E_P(|\varphi_n - \varphi_n^*|) \to 0 \quad \text{as} \quad n \to \infty, \tag{5.1}$$

which follows from Janssen and Pauls ((2003), Lemma 1). Note whenever (5.1) holds the conditional tests φ_n^* can be recommended and

- the asymptotic level of the resampling test φ_n^* is α under $P \in \mathcal{P}_0^{(\cdot)}$,
- φ_n and φ_n^* have the same asymptotic power under local alternatives.

It is well known that resampling procedures like φ_n^* may have some advantages at finite sample size. For instance the bootstrap produces a bias reduction. In order to discuss the advantages of φ_n^* in comparision with φ_n we will first consider the two-sample problem.

Example 5. (Example 1(b) continued) For the two-sample problem with $\mathcal{P}_0^{(2)}$ of (1.11) the test statistic $t_{n,II}$ of Section 4 with variance estimators (4.2) or (4.3) are adequate. Their conditional distributions of $t_{n,II,p}^*$ given by (4.6) are called the permutation distribution of $t_{n,II}$. The corresponding resampling test φ_n^* of (1.9) is the so called permutation test.

(a) Suppose for a moment that $\mathcal{P}_0^{(2)}$ is a restricted null hypothesis so that $X_{n,1}, \dots, X_{n,k(n)}$ are exchangeable, i.e. with invariant joint distributions under any permutation of the indices. By a proper randomization the permutation test φ_n^* can then be made finite sample distribution free under $\mathcal{P}_0^{(2)}$ of exact level α whereas φ_n will not be distribution free and we have no control about the level of φ_n on the composite null hypothesis $\mathcal{P}_0^{(2)}$ at finite sample size in general. Together with (5.1) these facts suggest the choice of permutation tests. Finite sample Monte Carlo studies also confirm the quality of permutation tests for various null hypotheses which are larger than that of exchangeability, see Janssen (1997).

(b) Suppose that the conditions of Theorem 4.1 hold under the null hypothesis $\mathcal{P}_0^{(2)}$. Then (1.10) holds for the permutation tests φ_n^* and they work well.

(c) Suppose that under $\mathcal{P}_0^{(2)}$ the variable $X_{n,1}, \dots, X_{n,k(n)}$ are as in Example 2 row-wise independent and infinitesimal, i.e. (1.16) holds. Let $\sum_{i=1}^{n_1} X_{n,i}$ and $\sum_{i=n_1+1}^{n} X_{n,i}$ be asymptotically normal for both samples. Then it is easy to see that $t_{n,II}$ with the variance estimator $V_{n,2}$ of (4.3) is asymptotically standard normal. This follows from

Raikov's theorem, see Janssen and Pauls ((2003), Lemma 7). As explained in Example 2 our Theorem 4.1 can be applied if condition (4.7) holds. Thus (1.10) follows.

If the distributions are homogeneous under $\mathcal{P}_0^{(2)}$ then the denominator $V_{n,2}$ can also be replaced by $V_{n,1}$ of (4.2).

• As conclusion we remark that the present permutation tests are asymptotically equivalent to their unconditional counterparts whenever the CLT holds for both partial sums of rowwise independent variables.

(d) Under the conditions of Theorem 4.2 the two-sample (double) bootstrap also works well. However, personally I prefer permutation tests for two-sample problems since they are finite sample distribution free for exchangeable variables, see (a). Monte Carlo simulations also support the permutation tests, see Janssen (1997) and Janssen and Pauls (2004). Bootstrap tests do not have here exact level α in general. The null hypothesis of exchangeability is often the core of some wider nonparametric null hypotheses $\mathcal{P}_0^{(2)}$.

Example 6. (Example 1(a) continued) (a) Recall that permutation tests have no power when we are testing $\mathcal{P}_0^{(1)}$ against one-sample alternatives given by exchangeable distributions. At this stage bootstrap tests can be helpful and different kind of bootstrap procedures will be discussed. Again bootstrap tests work well, i.e. (1.10) holds whenever t_n of (1.1) and t_n^* are asymptotically standard normal. Sufficient conditions were discussed in Sections 1 and 3.

(b) A special case is that of rowwise independent variables $X_{n,i}$ under $\mathcal{P}_0^{(1)}$. Roughly speaking the same assertions hold as in Example 5(c):

• Under the unconditional CLT of the partial sums $\sum_{i=1}^{k(n)} X_{n,i}$ the bootstrap tests work well.

(c) Throughout different bootstrap procedures will be compared. The main assumption is condition (1.12).

If (1.13) and (1.14) hold the $m(n)$ out of $k(n)$ bootstrap can be recommended where typically $m(n) = k(n)$ are equal. For computational reasons $m(n)$ and $k(n)$ may be different (for instance $m(n) < k(n)$), but the high resampling frequency with $m(n)/k(n) \to \infty$ should be avoided. In this case t_n^* will be always asymptotically normal which can not reflect the finite sample distribution of t_n.

Beran (1988) pointed out that the double bootstrap leads to some improvements for prepivoted test statistics. Under our conditions the double bootstrap tests also work well. Other authors prefer the Bayesian bootstrap, see Example 4. It is shown that the general CCLT can here also be applied.

So far the discussion is concerned with a general scheme of random variables. A specific example is the case of heteroscedastic random variables $X_{n,i}$ which include the nonparametric extension of the Behrens-Fisher problem, confer Janssen (1997).

Suppose that under the null hypothesis $T_{n,i}$, $1 \le i \le k(n)$, denote rowwise i.i.d. random variables with finite mean. Let $\sigma_{n,i} > 0$ be an array of standard deviations. Then

$$X_{n,i} := \sigma_{n,i} T_{n,i} \tag{5.2}$$

fits into the present framework and we are again testing the means under inhomogeneous standard deviations. One-sample alternatives may be specified by positive means of the $T_{n,i}$. For two-sample alternatives the $T_{n,i}$ may be devided in two different groups of i.i.d.

variables with different means. Again the core of the model consists of equal $\sigma_{n,i}$ within the row but the statistician can often not be sure that this assumption is satisfied.

As pointed out above the resampling tests work well for (5.2) whenever the unconditional CLT holds for the partial sums of (5.2).

6. Technical results and proofs

The present results rely on Theorem 2.1 which extends Theorem 3 of Janssen and Pauls (2003). The next lemma clarifies the meaning of assumption (2.4). Recall from Janssen and Pauls (2003) that the subspace

$$S := \{\varphi \in L_2(0,1) : \varphi \text{ nondecreasing}, \ \|\varphi\|_2 \leq 1\} \tag{6.1}$$

of nondecreasing functions on the unit interval becomes a metric space $(S, \|\cdot\|_1)$ or $(S, \|\cdot\|_2)$ if the metric is induced by the L_1- or L_2-norm given by the uniform distribution. By Lemma 2 of Janssen and Pauls (2003) the metric space $(S, \|\cdot\|_1)$ is compact. Let $\psi : \Omega \to S$ and $U : \Omega \to (0,1)$ be independent random variables where U is uniformly distributed. Then $\omega \mapsto \psi(\omega, U(\omega))$, briefly $\psi(U)$, defines a new mean zero random variable with $\mathrm{Var}(\psi(U)) \leq 1$. The random weights (2.1)–(2.4) define random step functions $\varphi_n : \tilde{\Omega} \to S$,

$$\varphi_n(\cdot, u) := \frac{(k(n)-1)^{1/2}}{(\sum_{i=1}^{k(n)}(W_{n,i} - \overline{W}_n)^2)^{1/2}}(W_{1+[k(n)u]:k(n)} - \overline{W}_n) \tag{6.2}$$

via the order statistics $W_{1:k(n)} \leq W_{2:k(n)} \leq \cdots \leq W_{k(n):k(n)}$ of the weights and the entire function $[\cdot]$ for $u \in (0,1)$. The crucial assumptions of the L_2-convergence for the random functions $(\varphi_n)_{n\in\mathbb{N}}$ can be discussed again in the light of the following lemma.

LEMMA 6.1. *Let $\varphi : (0,1) \to \mathbb{R}$ be some function, $\varphi \in S$, and let U be a uniform distributed random variable.*

(a) *For each scheme of weights with* (2.1)–(2.3) *the following conditions are equivalent:*

(i) $k(n)^{1/2}(W_{n,1} - \overline{W}_n) \to \varphi(U)$ *in distribution.*

(ii) *For each distributional accumulation point $\psi : \tilde{\Omega} \to S$ of $(\varphi_n)_{n\in\mathbb{N}}$ in $(S, \|\cdot\|_1)$ the equality in distribution $\psi(U) \stackrel{\mathcal{D}}{=} \varphi(U)$ holds.*

(b) *In addition to* (i) *let* $\mathrm{Var}(\varphi(U)) = 1$ *hold. Then every subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ has a further distributional convergent subsequence in $(S, \|\cdot\|_2)$.*

PROOF. The proof is based on the results of Janssen and Pauls ((2003), Section 7). Let $\overline{W}_n = 0$ hold without further restrictions.

(a) (i) $\Rightarrow$ (ii). Consider $X_{n,i} = \mathbf{1}_{\{1\}}(i)$ for $i \leq k(n)$. Then $\sum_{i=1}^{k(n)}(X_{n,i} - \overline{X}_n)^2 = k(n)^{-1}(k(n)-1)$ holds and the statistics S_n of (7.8) of Janssen and Pauls (2003)

$$S_n := \frac{k(n)^{1/2} W_{n,1}}{(\sum_{i=1}^{k(n)}(W_{n,i} - \overline{W}_n)^2)^{1/2}} \to \varphi(U) \tag{6.3}$$

is convergent by assumption (i). On the other hand S_n can be treated by the methods of Section 7 there. Since $(S, \|\cdot\|_1)$ is compact we may consider distributional accumulation points $\varphi_n \to \psi$ in $(S, \|\cdot\|_1)$ along subsequences. Define $c_{n1} = 1 - k(n)^{-1}$ and $c_{ni} =$

$-k(n)^{-1}$ otherwise. Then the special construction (7.13) of Janssen and Pauls (2003) for suitable i.i.d. uniformly distributed random variables $U_{n,i}$ proves

$$S_n - \sum_{i=1}^{k(n)} c_{ni}\varphi_n(U_{n,i}) \to 0 \tag{6.4}$$

in probability. The weak law of large numbers implies

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} \varphi_n(U_{n,i}) \to 0$$

in probability. Thus $S_n - \varphi_n(U_{n,1}) \to 0$ holds in probability and (6.3) implies $\varphi(U) \stackrel{\mathcal{D}}{=} \psi(U)$.

(ii) $\Rightarrow$ (i). In addition to $\overline{W}_n = 0$ we may assume without restriction that $\sum_{i=1}^{k(n)} W_{n,i}^2 = 1$ holds. Define now the random variable

$$V_n = k(n)^{1/2} \sum_{i=1}^{k(n)} W_{i:k(n)} \mathbf{1}_{((i-1)/k(n),i/k(n)]}(U) \tag{6.5}$$

given by the step function (6.2). Due to the exchangeability of the weights we have

$$V_n \stackrel{\mathcal{D}}{=} k(n)^{1/2} W_{n,1}. \tag{6.6}$$

On the other hand we have distributional convergence along subsequences

$$\varphi_m(U) = \left(\frac{k(m)-1}{k(m)}\right)^{1/2} V_m \to \varphi(U) \tag{6.7}$$

where the limit distribution $\mathcal{L}(\varphi(U))$ is always the same by assumption (ii). This fact proves statement (i).

(b) is a consequence of part (a) and the compactness of $(S, \|\cdot\|_1)$ and Lemma 3 of Janssen and Pauls (2003).

PROOF OF THEOREM 2.1. We will apply Lemma 6.1(b) above and Theorem 3 of Janssen and Pauls (2003). Under condition (2.4) we have for each subsequence a further subsequence $\{m\} \subset \mathbb{N}$ such that $\varphi_m \to \psi$ in $(S, \|\cdot\|_2)$ for some limit variable $\psi : \Omega \to S$. By Lemma 6.1 we have $Z_1 \stackrel{\mathcal{D}}{=} \psi(U)$. Along the present subsequence the statement of Theorem 2.1 is proved in Theorem 3 of Janssen and Pauls (2003). However, the limit variable only depends on the distribution $\psi(U)$ and it is the same for all convergent subsequences. This implies the convergence.

This method of proof also applies to further cases, see Remark 2 above.

LEMMA 6.2. *Under the assumptions* (1.15) *and* $\lim_{n\to\infty} m(n)/k(n) = c \in (0, \infty]$ *the conditions* (2.1)–(2.4) *hold for the double bootstrap weights given by* (2.7). *The limit variable* Z_1 *of* (2.4) *can be specified as follows.*

(a) *Consider for* $0 < c < \infty$ *independent Poisson random variables* V *and* $V_1, V_2, \ldots,$ *with* $E(V_i) = 1$ *for all* $i \in \mathbb{N}$ *and* $E(V) = c$. *Then*

$$Z_1 \stackrel{\mathcal{D}}{=} \left(\sum_{i=1}^{V} V_i - c \right) / \sqrt{2c}$$

holds with $E(Z_1) = 0$ *and* $\mathrm{Var}(Z_1) = 1$.

(b) *For* $c = \infty$ *the limit variable* Z_1 *is standard normal.*

PROOF. The verification for the conditions (2.1)–(2.3) is straight forward. For $m(n) = k(n)$ we refer to Præstgaard and Wellner (1993). The general case can be outlined as follows. For details see also Pauls (2002). The condition (2.3) follows from the weak law of large numbers

$$\frac{1}{m(n)} \sum_{i=1}^{k(n)} \left(M'_{n,i} - \frac{m(n)}{k(n)} \right)^2 \to 2. \tag{6.8}$$

To prove this it is enough to verify the conditions

$$\frac{k(n)}{m(n)} E \left(\left(M'_{n,1} - \frac{m(n)}{k(n)} \right)^2 \right) \to 2 \tag{6.9}$$

$$\mathrm{Cov} \left(\left(M'_{n,1} - \frac{m(n)}{k(n)} \right)^2, \left(M_{n,2} - \frac{m(n)}{k(n)} \right)^2 \right) \leq 0 \tag{6.10}$$

$$\frac{k(n)}{m(n)^2} \mathrm{Var} \left(\left(M'_{n,1} - \frac{m(n)}{k(n)} \right)^2 \right) \to 0 \tag{6.11}$$

since the variables $(M'_{n,i} - \frac{m(n)}{k(n)})^2$ are exchangeable. The condition (2.3) obviously follows from

$$\tilde{P} \left(\max_{1 \leq i \leq k(n)} |W_{n,i}| \geq \varepsilon \right) \leq 4m(n)^{-2} \varepsilon^{-4} \sum_{i=1}^{k(n)} E \left(\left(M'_{n,i} - \frac{m(n)}{k(n)} \right)^4 \right)$$

where the upper bound asymptotically vanishes which is shown below.

The proof of (6.10) is similar to Præstgaard and Wellner (1993). The convergence of (6.9) can be proved by taking conditional expectations w.r.t. $M_{n,1}$. Observe that here

$$\begin{aligned} &E \left(\left(M'_{n,1} - \frac{m(n)}{k(n)} \right)^2 \right) \\ &\quad = E((M'_{n,1} - E(M'_{n,1} \mid M_{n,1}))^2) + E \left(\left(E(M'_{n,1} \mid M_{n,1}) - \frac{m(n)}{k(n)} \right)^2 \right) \\ &\quad = E((M'_{n,1} - M_{n,1})^2) + \mathrm{Var}(M_{n,1}) \end{aligned}$$

holds. If we multiply with $k(n)/m(n)$ then both terms converge to one.

The condition (6.11) can be verified by the calculation of fourth moments of conditional binomial distributed variables. For details we refer to Pauls (2002). Recall that if Y is a binomial random variable with n degrees of freedom and parameter p then

$$E(Y^4) \leq np + 7n^2p^2 + 6n^3p^3 + n^4p^4 \tag{6.12}$$

holds.

An upper bound of (6.11) is given by

$$\frac{k(n)}{m(n)^2} E\left(\left(M'_{n,1} - \frac{m(n)}{k(n)}\right)^4\right)$$
$$\leq \frac{k(n)}{m(n)^2}\left[E(M'^4_{n,1}) + 6E(M'^2_{n,1})\left(\frac{m(n)}{k(n)}\right)^2 + \left(\frac{m(n)}{k(n)}\right)^4\right].$$

We will see that each term is of the order $m(n)^2/k(n)^3$ which vanishes as $n \to \infty$. To see this we will take conditional expectations w.r.t. $M_{n,1}$ where $\mathcal{L}(M'_{n,1} \mid M_{n,1})$ is a $m(n)$-binomial random variable with parameter $p = M_{n,1}/m(n)$. Thus

$$E(M'^2_{n,1} \mid M_{n,1}) \leq 2M^2_{n,1} \quad \text{and} \quad E(M'^2_{n,1}) \leq 2\left(\frac{m(n)}{k(n)} + \frac{m(n)^2}{k(n)^2}\right).$$

The fourth moment can be treated similarly. Observe that

$$E(M'^4_{n,1} \mid M_{n,1}) \leq 15 M^4_{n,1}$$

holds and thus

$$E(M^4_{n,1}) = O\left(\left(\frac{m(n)}{k(n)}\right)^4\right)$$

follows from (6.12). These arguments establish (6.7) under condition (1.15).

Thus we may restrict ourselves to the proof of (2.4) which is done separately for $c < \infty$ and $c = \infty$.

(a) By Poisson's limit law we have $M_{n,1} \to V$ in distribution. On a suitably chosen probability space we have $M_{n,1}(\omega) \to V(\omega)$ a.e. Thus we may study the behaviour of the second resampling step under the condition $M_{n,1} = k$ for $k \geq 0$. For this purpose let $(N_{n,1}, \ldots, N_{n,m(n)})$ be new multinomial variables with $m(n)$ degrees of freedom and success probability $1/m(n)$ which are independent of the other variables.

Under the condition $M_{n,1} = k$ we have again by Poisson's limit law

$$M'_{n,1} \stackrel{\mathcal{D}}{=} N_{n,1} + \cdots + N_{n,k} \to V_1 + \cdots + V_k$$

in distribution. This fact implies the unconditional convergence

$$M'_{n,1} \to \sum_{i=1}^{V} V_i =: \xi.$$

It is well known that $E(\xi) = c$ and $\mathrm{Var}(\xi) = 2c$ holds.

(b) Suppose now that $m(n)/k(n) \to \infty$ holds. Then $M_{n,1}$ is binomial distributed with $m(n)$ degrees of freedom and parameter $p_n = 1/k(n)$. The central limit theorem of Lindeberg and Feller for binomial random variables yields

$$\frac{M_{n,1} - \dfrac{m(n)}{k(n)}}{\sqrt{\dfrac{m(n)}{k(n)}\left(1 - \dfrac{1}{k(n)}\right)}} \to \xi_1 \tag{6.13}$$

where ξ_1 is standard normal. In addition

$$\frac{k(n)}{m(n)} M_{n,1} \to 1 \tag{6.14}$$

holds in probability.

In order to describe the second bootstrap step let $\eta_{n,1}, \eta_{n,2}, \dots$ denote an i.i.d. sequence of uniformly distributed random variables on $[0, m(n)]$ mutually independent to the previous variables. The variable $M'_{n,1}$ has the representation

$$M'_{n,1} - \frac{m(n)}{k(n)} \stackrel{\mathcal{D}}{=} \left(\sum_{i=1}^{m(n)} \mathbf{1}_{[0,M_{n,1}]}(\eta_{n,i}) - M_{n,1} \right) + \left(M_{n,1} - \frac{m(n)}{k(n)} \right). \tag{6.15}$$

Without restrictions we may assume that (6.14) is a.e. convergent. Either we may take subsequences or the probability space is changed. On a set of probability one the following conditional central limit theorem given $M_{n,1}$ holds. There exists a standard normal random variable ξ_2 independent of ξ_1 with

$$\frac{\sum_{i=1}^{m(n)} \mathbf{1}_{[0,M_{n,1}]}(\eta_{n,i}) - M_{n,1}}{\sqrt{m(n) \frac{M_{n,1}}{m(n)} \left(1 - \frac{M_{n,1}}{m(n)} \right)}} \to \xi_2 \tag{6.16}$$

in distribution. This statement follows by the same arguments used in (6.13) for binomial random variables. Obviously, we also have unconditional convergence in (6.16) and thus

$$\frac{\sum_{i=1}^{m(n)} \mathbf{1}_{[0,M_{n,1}]}(\eta_{n,i}) - M_{n,1}}{\sqrt{\frac{m(n)}{k(n)}}} \to \xi_2 \tag{6.17}$$

converges in distribution. By (6.14) and (6.17) we then obtain

$$k(n)^{1/2} W_{n,1} \to \frac{1}{\sqrt{2}} (\xi_1 + \xi_2)$$

in distribution which establishes the result for the double bootstrap for $c = \infty$.

PROOF OF THEOREM 3.1. By the subsequence principle it is enough to prove that for each subsequence there exists a further subsequence which converges to the same limit. After selecting a subsequence we may assume that $m(n)/k(n) \to c$ holds for some $c > 0$. The proof runs parallel for both cases. Below we will indicate where are the differences under assumption (a) or (b).

Without restrictions we may assume that $\sum_{i=1}^{k(n)} (X_{n,i} - \overline{X}_n)^2 = 1$ and $\overline{X}_n = 0$ hold on the set I_n since the resampling statistic is homogeneous according to our assumptions. This can be seen as follows. Consider first for variables $X'_{n,i} = X_{n,i} - \overline{X}_n$. Since $\overline{W}_{k(n)} = 0$ holds the numerator $\sum_{i=1}^{k(n)} W_{n,i} X_{n,i} = \sum_{i=1}^{k(n)} W_{n,i} X'_{n,i}$ does not change.

Careful calculations show that the denominator does not change also

$$\sum_{i=1}^{k(n)} W'_{n,i} X'^2_{n,i} - \left(\frac{1}{m(n)^{1/4}} \sum_{i=1}^{k(n)} W'_{n,i} X'_{n,i} \right)^2$$
$$= \sum_{i=1}^{k(n)} W'_{n,i} X^2_{n,i} - \left(\frac{1}{m(n)^{1/4}} \sum_{i=1}^{k(n)} W'_{n,i} X_{n,i} \right)^2 .$$

The verification of that equality relies on the equations $\sum_{i=1}^{k(n)} W'_{n,i} = m(n)^{1/2}$ and

$$\sum_{i=1}^{k(n)} W'_{n,i}(X_{n,i} - \overline{X}_n) = \sum_{i=1}^{k(n)} W_{n,i} X_{n,i} = \sum_{i=1}^{k(n)} W'_{n,i} X_{n,i} - m(n)^{1/2} \overline{X}_n.$$

In a first step we will treat the numerator of $t^*_{n,W}$. Under assumption (1.12) we have by Theorem 2.1 the conditional central limit theorem

$$d\left(\mathcal{L}\left(k(n)^{1/2} \sum_{i=1}^{k(n)} W_{n,i} X_{n,i} \mid \vec{X}_n \right), \mathcal{L}(Z \mathbf{1}_{I_n} \mid \vec{X}_n) \right) \to 0 \tag{6.18}$$

in probability. Here we may turn to distributional convergent subsequences of $\mathbf{1}_{I_n}$ which is enough to prove (6.18). Under assumption (b) the same result holds since the series has also a normal distribution when Z_1 is standard normal. By tightness we may again turn to distributional convergent subsequences of (2.8) which always lead to the same type of limit variable. In the next step we will study the denominator. Recall that $\overline{X}_n = 0$ holds without restrictions. Consider now the new array $Y_{n,i} = (k(n)/m(n))^{1/2} X^2_{n,i}$ which is resampled throughout. We will prove that

$$k(n)^{1/2} \sum_{i=1}^{k(n)} W_{n,i} Y_{n,i} \to 0 \tag{6.19}$$

in probability. Again Theorem 2.1 can be applied. We have

$$\sum_{i=1}^{k(n)} (Y_{n,i} - \overline{Y}_{k(n)})^2 \le \frac{k(n)}{m(n)} \max_{i \le k(n)} X^2_{n,i} \sum_{i=1}^{k(n)} X^2_{n,i} \to 0 \tag{6.20}$$

since either (1.12) or $k(n)/m(n) \to 0$ holds. Thus (6.20) yields (6.19) which can be rewritten as follows:

$$k(n)^{1/2} \left(\sum_{i=1}^{k(n)} W'_{n,i} Y_{n,i} - \frac{m(n)^{1/2}}{k(n)} \sum_{i=1}^{k(n)} Y_{n,i} \right) \tag{6.21}$$

$$= \frac{k(n)}{m(n)^{1/2}} \sum_{i=1}^{k(n)} W'_{n,i} X^2_i - \sum_{i=1}^{k(n)} X^2_{n,i}. \tag{6.22}$$

Under $\overline{X}_n = 0$ we have $\sum_{i=1}^{k(n)} X_{n,i}^2 = 1$ on I_n and (6.19) implies

$$\frac{k(n)}{m(n)} \sum_{i=1}^{m(n)} X_i^{*2} - \mathbf{1}_{I_n} \to 0 \tag{6.23}$$

in probability. For centered X's the second term of the denominator asymptotically vanishes since

$$\frac{k(n)}{m(n)^{1/2}} \left(\frac{1}{m(n)^{1/4}} \sum_{i=1}^{k(n)} W'_{n,i} X_{n,i} \right)^2 = \left(\frac{k(n)^{1/2}}{m(n)^{1/2}} \sum_{i=1}^{k(n)} W_{n,i} X_{n,i} \right)^2$$

converges to zero. Thus the result follows from (6.18) and (6.23).

PROOF OF THEOREM 1.1. The assertion (a), (c), and (d) are special cases of Theorem 3.1. Only the converse result (b) requires some more effort. We show that (1.12) is also necessary for the CCLT. As above we may assume that $\overline{X}_n = 0$ and $\sum_{i=1}^{k(n)} X_{n,i}^2 = \mathbf{1}_{I_n}$ hold. Passing several times to subsequences, denoted again by $k(n)$, we may assume that $m(n)/k(n) \to \lambda \in (0, \infty)$ holds and (2.8) is distributional convergent on $[0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} \times [0,1]$ according to tightness. As in Janssen and Pauls (2003) we may change the probability space $(\Omega, \mathcal{A}, P)$ so that (2.8) converges almost surely. Thus the bootstrap statistics (2.5) with weights (2.6) has the conditional limit laws $\mathcal{L}(\tilde{T}_0 \mid \Pi = \omega)$ (2.9) with

$$Z_i = \lambda^{-1/2}(N_i - \lambda), \qquad \tilde{Z}_j = \lambda^{-1/2}(\tilde{N}_j - \lambda),$$

given by mutually independent Poisson random variables $N_i, \tilde{N}_j$ with mean λ, see Janssen and Pauls ((2003), (8.39) ff).

In the next step joint conditional convergence given $\vec{X}_n$ of the numerator and denominator is established. First it will be proved that

$$\begin{aligned} \mathcal{L}\left(T_n^*, \frac{k(n)}{m(n)} \sum_{i=1}^{m(n)} X_i^{*2} \mid \vec{X}_n(\omega) \right) \\ \to \mathcal{L}\left(\tilde{T}_0, \lambda^{-1} \left(\sum_{i=1}^{\infty} Z_i \xi_i^2 + \sum_{j=1}^{\infty} \tilde{Z}_j \tilde{\xi}_j^2 \right) + \tilde{\sigma}^2(\omega) \mid \Pi = \omega \right) \end{aligned} \tag{6.24}$$

holds for fixed ω, where $\tilde{\sigma}^2(\omega)$ is as in Theorem 2.1. Notice that $\xi_0(\omega)$ is always zero or one. Since (2.8) is almost surely convergent by our special construction we may restrict ourselves to the case $\xi_0(\omega) = 1$. Then we have finally $\sum_{i=1}^{k(n)} X_{n,i}^2(\omega) = 1$ and by (3.3):

$$\begin{aligned} &\left(T_n^*, \frac{k(n)}{m(n)} \sum_{i=1}^{m(n)} X_i^{*2} \right) \\ &= \left(k(n)^{1/2} \sum_{i=1}^{k(n)} W_{n,i} X_{n,i}, \frac{k(n)^{1/2}}{m(n)^{1/2}} \left(k(n)^{1/2} \sum_{i=1}^{k(n)} W_{n,i} X_{n,i}^2 \right) + 1 \right). \end{aligned} \tag{6.25}$$

Statement (6.24) is treated via deterministic coefficients $c_{ni} := X_{i:k(n)}(\omega)$ where finally $\sum_{i=1}^{k(n)} c_{ni}^2 = 1$ holds. For each coordinate of (6.25) now the substitution of the kind of (6.4) by rowwise independent variables can be applied and we have

$$T_n^* - \sum_{i=1}^{k(n)} c_{ni}\varphi_n(U_{n,i}) \to 0 \tag{6.26}$$

$$\frac{k(n)^{1/2}}{m(n)^{1/2}}\left(k(n)^{1/2}\sum_{i=1}^{k(n)} W_{n,i}X_{n,i}^2 - \sum_{i=1}^{k(n)} c_{ni}^2\varphi_n(U_{n,i})\right) \to 0 \tag{6.27}$$

both in probability for fixed ω. The details are figured out in Janssen and Pauls ((2003), (7.11)–(7.19)). The function φ_n is defined in (6.2). Up to convergent constants it is now enough to establish the limit variable of

$$\left(\sum_{i=1}^{k(n)} c_{ni}\varphi_n(U_{n,i}), \sum_{i=1}^{k(n)} c_{ni}^2\varphi_n(U_{n,i})\right) \tag{6.28}$$

in order to treat (6.25). The special construction of Janssen and Pauls (2003) allows to treat the extremes of (6.28) simultaneously for powers $k = 1, 2$. For each i, j we have convergence in probability

$$\begin{aligned} &c_{ni}^k\varphi_n(U_{n,i}) \to \xi_i^k(\omega)Z_i \quad \text{and} \\ &c_{n(n+1-j)}^k\varphi_n(U_{n,n+1-j}) \to \tilde{\xi}_j^k(\omega)\tilde{Z}_j, \qquad k = 1, 2. \end{aligned} \tag{6.29}$$

Thus for each $r \in \mathbb{N}$ we have joint distributional convergence of

$$\begin{aligned} &\left(\sum_{i=1}^{k(n)} c_{ni}\varphi_n(U_{n,i}), \sum_{\substack{i=1,\ldots,r \\ i=n,\ldots,n+1-r}} c_{ni}^2\varphi_n(U_{n,i})\right) \\ &\qquad \to \left(\tilde{T}_0(\omega,\cdot), \sum_{i=1}^{r}\xi_i^2(\omega)Z_i + \sum_{j=1}^{r}\tilde{\xi}_j^2(\omega)\tilde{Z}_j\right). \end{aligned} \tag{6.30}$$

It is easy to show that the middle part of the second quadratic component of (6.28) vanishes for $n \to \infty$, cf. Janssen and Pauls (2003), for related arguments. For this purpose consider sequences $r_n \uparrow \infty$, $s_n \uparrow \infty$. Then

$$\begin{aligned} \operatorname{Var}\left(\sum_{i=r_n}^{k(n)-s_n} c_{ni}^2\varphi_n(U_{n,i})\right) &\le \sum_{i=r_n}^{k(n)-s_n} c_{ni}^4 \\ &\le \max\{c_{ni}^2 : r : n \le i \le k(n) - s_n\} \to 0 \end{aligned} \tag{6.31}$$

since $i \mapsto c_{ni}$ is increasing and $\sum_{i=1}^{k(n)} c_{ni}^2 \le 1$.

Thus (6.30) and (6.31) together with standard arguments of Billingsley (1968), Theorem 4.2, prove that (6.28) converge in distribution to $(\tilde{T}_0, R)$ where

$$R = \sum_{i=1}^{\infty}\xi_i^2(\omega)Z_i + \sum_{j=1}^{\infty}\xi_j^2(\omega)\tilde{Z}_j \tag{6.32}$$

$$= \lambda^{-1/2}\left(\sum_{i=1}^{\infty}\xi_i^2(\omega)N_i + \sum_{j=1}^{\infty}\tilde{\xi}_j^2(\omega)\tilde{N}_j - \lambda\left(\sum_{i=1}^{\infty}\xi_i^2(\omega) + \sum_{j=1}^{\infty}\tilde{\xi}_j^2(\omega)\right)\right).$$

Now we may combine (6.25)–(6.26) and (6.24) is proved. At this stage the proof follows the lines of Mason and Shao (2001). We have

$$\mathcal{L}(t_n^* \mid \vec{X}_n(\omega)) \to \mathcal{L}\left(\frac{\tilde{T}_0}{(\lambda^{-1}(\sum_{i=1}^{\infty} Z_i\xi_i^2(\omega) + \sum_{j=1}^{\infty}\tilde{Z}_j\tilde{\xi}_j^2(\omega)) + \tilde{\sigma}^2(\omega))^{1/2}}\right) \tag{6.33}$$

in distribution. Observe that the second component of (6.24), the sum of the squares, is the leading term of the denominator. Similar arguments are carried out in the proof of Theorem 3.1 below (6.23). On the other hand $\mathcal{L}(t_n^* \mid \vec{X}_n(\omega))$ is standard normal when $\xi_0(\omega) = 1$ holds. The case $\xi_0(\omega) = 0$ is a trivial case which corresponds to $0 = \frac{0}{0}$. Now Proposition 2.2 of Mason and Shao (2001) implies that the limit law (6.33) can only standard normal if $\xi_1(\omega) = \tilde{\xi}_1(\omega) = 0$ holds. By tightness of (2.8) this holds for all cluster points of (2.8). This implies condition (1.12).

Proof of Theorem 4.1. Assume first that $n_1/k(n) \to \beta \in (0,1)$ converges as $n \to \infty$. Again we may assume that $\overline{X}_n = 0$ and $\sum_{i=1}^{k(n)} X_{n,i}^2 = 1$ hold on the set $\{V_n > 0\}$ since the resampling procedure (4.5), (4.6) is compatible with respect to normalization. The numerator of $t_{n,II}$ has now the form

$$T_n = \sum_{i=1}^{k(n)} c_{ni}X_{n,i} \tag{6.34}$$

given by the two-sample regression coefficients

$$c_{ni} = \left(\frac{n_1 n_2}{k(n)}\right)^{1/2}[n_1^{-1}\mathbf{1}_{\{i\le n_1\}} - n_2^{-1}\mathbf{1}_{\{n_1<i\le k(n)\}}]. \tag{6.35}$$

As in Janssen and Pauls (2003), Example 2, we may choose the weights $W_{n,i} = c_{n\sigma(i)}$. The resampling form of T_n is now $T_n^* = k(n)^{1/2}\sum_{i=1}^{k(n)} c_{n\sigma(i)}X_{n,i} \stackrel{\mathcal{D}}{=} k(n)^{1/2}\sum_{i=1}^{k(n)} c_{ni}X_i^*$. Under the condition (1.12) our Theorem 2.1 implies

$$d(\mathcal{L}(T_n^* \mid \vec{X}_n), \mathcal{L}(Z \mid \vec{X}_n)) \to 0 \tag{6.36}$$

in probability. In the next step let us resample the denominator $k(n)V_{n,1}$. As in (6.20) we have $\sum_{i=1}^{k(n)}(X_{n,i}^2 - \frac{1}{k(n)}\sum_{j=1}^{k(n)} X_{n,j}^2)^2 \to 0$ and the squares will be resampled first. The following convergence is understood as conditional convergence given $\vec{X}_n$. For the squares Theorem 2.1 yields degenerate limit laws. Thus

$$k(n)^{1/2}\sum_{i=0}^{k(n)} c_{ni}X_{n,\sigma(i)}^2 \to 0 \tag{6.37}$$

holds. This statistic can be rewritten as

$$\frac{k(n)}{(n_1n_2)^{1/2}}\sum_{i=1}^{k(n)}\left(\mathbf{1}_{\{i\le k(n)\}} - \frac{n_1}{k(n)}\right)X_{n,\sigma(i)}^2 \to 0. \tag{6.38}$$

Since $n_1/k(n)\sum_{i=1}^{k(n)} X_{n,\sigma(i)}^2 = n_1/k(n)$ holds we have

$$\frac{k(n)}{k(n)-2}\sum_{i=1}^{n_1} X_{n,\sigma(i)}^2 \to \beta. \tag{6.39}$$

It is easy to see that $n_1(1/n_1\sum_{i=1}^{n_1} X_{n,\sigma(i)})^2 \to 0$ holds. The second term of $V_{n,1}$ can be treated similarly. Thus

$$k(n)V_{n,1}(X_1^*,\dots,X_{k(n)}^*) \to \beta + (1-\beta) = 1. \tag{6.40}$$

The same assertion holds for $V_{n,2}$. However, we have

$$\frac{n_1 n_2}{n_1(n_1-1)}\sum_{i=1}^{n_1}\left(X_{n,\sigma(i)} - \frac{1}{n_1}\sum_{j=1}^{n_1} X_{n,\sigma(j)}\right)^2 \to 1-\beta \tag{6.41}$$

and the second term of this type now tends to β. These arguments complete the proof when $n_1/k(n) \to \beta$. The general case follows by the consideration of convergent subsequences of (4.7). Since the limit is always the same the statement then also holds.

PROOF OF THEOREM 4.2. This proof combines the arguments of the one-sample (double) bootstrap proof used in Theorem 3.1 and the treatment of the two-sample case of Theorem 4.1. Also we like to remind the reader to the following subsequence principle. Convergence of (1.3) in probability holds iff for each subsequence there exists a further subsequence such that (1.3) holds almost surely.

The present t-statistics (4.1) are homogenous with respect to affine transformations. Hence, we may assume that

$$\sum_{i=1}^{n_1} X_{n,i} = 0, \qquad \sum_{i=1}^{n_2} X_{n,n_1+i} = 0 \quad \text{and} \quad \sum_{i=1}^{k(n)} X_{n,i}^2 = 1 \tag{6.42}$$

holds on a set of asymptotic probability one. Assume also that $n_1/k(n) \to \beta$ for some $\beta \in (0,1)$.

(b) Define random variables

$$\sigma_{n,1} = \left(\sum_{i=1}^{n_1} X_{n,i}^2\right)^{1/2}, \qquad \sigma_{n,2} = \left(\sum_{i=1}^{n_2} X_{n,n_1+i}^2\right)^{1/2}.$$

Under (4.10) we have by (6.42)

$$\sigma_{n,1}^2 \to 1/2 \quad \text{and} \quad \sigma_{n,2}^2 \to 1/2 \tag{6.43}$$

in probability. Let now ξ_1 and ξ_2 denote two independent standard normal random variables. Then by (1.12)

$$d\left(\mathcal{L}\left(\sum_{i=1}^{n_1} X_i^* \mid \vec{X}_{k(n)}\right), \mathcal{L}(\sigma_{n,1}\xi_1 \mid \vec{X}_{k(n)})\right) \to 0 \tag{6.44}$$

and

$$d\left(\mathcal{L}\left(\sum_{i=1}^{n_2} Y_i^* \mid \vec{X}_{k(n)}\right), \mathcal{L}(\sigma_{n,2}\xi_2 \mid \vec{X}_{k(n)})\right) \to 0 \tag{6.45}$$

hold both in probability. Since the bootstrap procedures of the X- and Y-groups are conditionally independent given $\vec{X}_{k(n)}$ we have

$$\begin{aligned} d\Bigg(\mathcal{L}\left(\left(\frac{n_2}{n_1}\right)^{1/2}\sum_{i=1}^{n_1} X_i^* - \left(\frac{n_1}{n_2}\right)^{1/2}\sum_{i=1}^{n_2} Y_i^* \mid \vec{X}_{k(n)}\right), \\ \mathcal{L}\left(\left(\frac{n_2}{n_1}\right)^{1/2}\sigma_{n,1}\xi_1 + \left(\frac{n_1}{n_2}\right)^{1/2}\sigma_{n,2}\xi_2 \mid \vec{X}_{k(n)}\right)\Bigg) \to 0 \end{aligned} \tag{6.46}$$

in P-probability. By (6.43) we have

$$\frac{n_2\sigma_{n,1}}{n_1} + \frac{n_1\sigma_{n,2}}{n_2} \to 1 \tag{6.47}$$

in probability and we have the CCLT in (6.46). The denominator has to be multiplied by $k(n)^{1/2}$. Writing again

$$\sum_{i=1}^{n_1}(X_i^* - \overline{X}_{n_1}^*)^2 = \sum_{i=1}^{n_1} X_i^{*2} - n_1(\overline{X}_{n_1}^*)^2$$

we see that $\sum_{i=1}^{n_1} X_i^{*2}$ is the leading term.

As in the proof of Theorem 3.1 we have

$$\sum_{i=1}^{n_1} X_i^{*2} - \sigma_{n,1}^2 \to 0 \quad \text{and} \quad \sum_{i=1}^{n_2} Y_i^{*2} - \sigma_{n,2}^2 \to 0 \tag{6.48}$$

in probability. Thus

$$\sum_{i=1}^{n_1} X_i^{*2} + \sum_{i=1}^{n_2} Y_i^{*2} \to 1 \tag{6.49}$$

follows which proves the result.

(a) This proof is much the same as above. Assume first that $\sigma_{n,1}$ and $\sigma_{n,2}$ converge almost surely. In contrast to (6.47) the conditional limit (6.46) has now the asymptotic variance

$$\lim\left(\frac{n_2\sigma_{n,1}^2}{n_1} + \frac{n_1\sigma_{n,2}^2}{n_2}\right) =: \rho. \tag{6.50}$$

Next $k(n)V_{n,2}(X_1^*, \ldots, X_{n_1}^*, Y_1^*, \ldots, Y_{n_2}^*)$ will be treated similarly. The leading terms are

$$\frac{n_2}{n_1}\sum_{i=1}^{n_1} X_i^{*2} + \frac{n_1}{n_2}\sum_{i=1}^{n_2} Y_i^{*2} \to \rho \tag{6.51}$$

which converge in probability by (6.48). Now we can take ratios which implies the CCLT for the t_n^*-statistic.

The general case can be treated again by subsequence arguments. By (6.42) the sequence $(\sigma_{n,1}, \sigma_{n,2})$ is tight. Thus we may choose distributional convergent subsequences and then almost surely convergent versions of these random variables along the present subsequences. At this stage the first step of the proof applies and the result follows.

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