"CLASSICAL" PROPAGATOR AND PATH INTEGRAL IN THE PROBABILITY REPRESENTATION OF QUANTUM MECHANICS

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Abstract

In the probability representation of the standard quantum mechanics, the explicit expression (and its quasiclassical van-Fleck approximation) for the "classical" propagator (transition probability distribution), which completely describes the quantum system's evolution, is found in terms of the quantum propagator. An expression for the "classical" propagator in terms of path integral is derived. Examples of free motion and harmonic oscillator are considered. The evolution equation in the Bargmann representation of the optical tomography approach is obtained.

1.   Introduction

In [1, 2], the new formulation of quantum mechanics, in which a quantum state was described by the tomographic probability distribution (called "marginal distribution") instead of density matrix or the Wigner function was suggested and the quantum evolution equation of the generalized "classical" Fokker–Planck-type equation alternative to the Schrödinger equation was found. The physical meaning of the marginal distribution was elucidated as the probability distribution for the position measured in an ensemble of scaled and rotated reference frames in the classical phase space of the system under consideration. The "classical" propagator describing the position probability from an initial position labeled by parameters of the initial reference frame of the ensemble to a final position lebeled by parameters of the final reference frame of the ensemble was introduced [2].

The quantum propagator (Green function of the evolution equation for the density matrix) was expressed in terms of the "classical" propagator in [3]. Properties of the "classical" propagator and its relation to quantum time-dependent integrals of motion [4–9] were studied in [10]. An analog of the Schrödinger equation for energy levels in terms of the marginal distribution was discussed in [11]. Different examples like quantum dissipation and quantum top were considered in [12, 13]. An extension of the new formulation of quantum mechanics to the case of spin was given in [14–17]. An example of quantum diffraction in time [18] in the framework of the new (probability) representation of quantum mechanics was considered recently [19].

Feynman suggested the formulation of quantum mechanics by means of the path integral method [20]. In his formulation, the quantum transition-probability amplitude (Green function of the Schrödinger evolution equation for the wave function) is expressed as path integral determined by the classical action of the system under consideration. To use the path integral method in the new (probability) representation of quantum mechanics, one needs to have a formula for the "classical" propagator in terms of the quantum propagator,

which is inverse of the formula for the quantum propagator in terms of the "classical" propagator derived in [3]; till now, this formula was not available.

The aim of this paper is to derive the explicit relationship of the "classical" propagator (Green function of the evolution equation for marginal distribution) in terms of Green function of the Schrödinger evolution equation. Another goal of this study is to express, in view of the relationship obtained, the "classical" propagator as path integral determined by the classical action.

2. Marginal Distribution and Wave Function

Let a quantum state be described by the wave function $\Psi(x)$. The nonnegative marginal distribution $w(X, \mu, \nu)$, which desribes the quantum state, is given by the relationship [21, 19]

$$w(X,\mu,\nu) = \frac{1}{2\pi|\nu|} \left| \int \Psi(y) \exp\left(\frac{i\mu}{2\nu} y^2 - \frac{iX}{\nu} y \right) dy \right|^2, \tag{1}$$

where the random coordinate X corresponds to the particle's position and the real parameters μ and ν label the reference frame in the classical phase space, in which the position is measured. The density matrix of the pure state $\rho_\Psi(x, x')$ in the position representation is expressed in terms of the marginal distribution (see, for example, [22])

$$\begin{aligned} \rho_\Psi(x,x') &= \Psi(x)\,\Psi^*(x') \\ &= \frac{1}{2\pi} \int w(X,\mu,x-x') \exp\left[i\left(X - \mu\frac{x+x'}{2} \right) \right] d\mu\, dX. \end{aligned} \tag{2}$$

Thus, if one knows the density matrix of the pure state (or wave function), the marginal distribution is also known, in view of Eq. (1). Correspondingly, if one knows the marginal distribution, the density matrix is also known, in view of the inverse relationship (2).

3. "Classical" Propagator

The evolution of the marginal distribution $w(X, \mu, \nu, t)$ can be described by means of the "classical" propagator $\Pi(X_2, \mu_2, \nu_2, X_1, \mu_1, \nu_1, t_2, t_1)$, in view of the integral relationship [2]

$$w(X_2,\mu_2,\nu_2,t_2) = \int \Pi(X_2,\mu_2,\nu_2,X_1,\mu_1,\nu_1,t_2,t_1)\, w(X_1,\mu_1,\nu_1,t_1)\, dX_1\, d\mu_1\, d\nu_1. \tag{3}$$

Below, we will use also the notation $\Pi(X_2, \mu_2, \nu_2, X_1, \mu_1, \nu_1, t)$ for the "classical" propagator in the case of $t_1 = 0$, $t_2 = t$.

Having in mind the physical meaning of the "classical" propagator, we know that it must satisfy the nonlinear relationship

$$\begin{aligned} \Pi(X_3,\mu_3,\nu_3,X_1,\mu_1,\nu_1,t_3,t_1) &= \int \Pi(X_3,\mu_3,\nu_3,X_2,\mu_2,\nu_2,t_3,t_2) \\ &\otimes \Pi(X_2,\mu_2,\nu_2,X_1,\mu_1,\nu_1,t_2,t_1)\, dX_2\, d\mu_2\, d\nu_2. \end{aligned} \tag{4}$$

Formula (4) describes the obvious standard property of the transition probability from an initial point X_1 to a final point X_3 via an intermediate point X_2. The only pecularity of this formula consists in the fact that

the initial, final, and intermediate points X_1, X_3, X_2 are considered in their own reference frames labeled by the paramaters μ_1; ν_1, μ_3; ν_3, and μ_2; ν_2, respectively. Empoying this property one produces integration in Eq. (4) not only over points X_2 but also over the reference frames' parameters.

In view of Eq. (1), the marginal distribution $w(X, \mu, \nu)$ has the property [22]

$$w(aX, a\mu, a\nu) = \frac{1}{|a|} w(X, \mu, \nu). \tag{5}$$

Due to this, the "classical" propagator has the analogous property,

$$\Pi(bX, b\mu, b\nu, bX', b\mu', b\nu', t) = \frac{1}{|b|^3} \Pi(X, \mu, \nu, X', \mu', \nu', t). \tag{6}$$

Equation (6) provides the connection of two Fourier components, namely,

$$\Pi_F(1, \mu, \nu, X', \mu', \nu', t) = \int \Pi(X, \mu, \nu, X', \mu', \nu', t) e^{iX} dX \tag{7}$$

and

$$\Pi_F(k, \mu, \nu, X', \mu', \nu', t) = \int \Pi(X, \mu, \nu, X', \mu', \nu', t) e^{ikX} dX. \tag{8}$$

As a result, one has

$$\Pi_F(k, \mu, \nu, X', \mu', \nu', t) = k^2 \Pi_F(1, k\mu, k\nu, kX', k\mu', k\nu', t). \tag{9}$$

4. Green Function and "Classical" Propagator

The Green function $G(x, y, t)$ (quantum propagator) is determined by the relationship

$$\Psi(x, t) = \int G(x, y, t) \Psi(y, t = 0) \, dy. \tag{10}$$

Elaborating Eqs. (2) and (3) for arbitrary time t, one arrives at

$$\int G(x, y, t) G^*(x', z, t) \Psi(y, t = 0) \Psi^*(z, t = 0) \, dy \, dz = \frac{1}{2\pi} \int dX' \, d\mu' \, d\nu' dX \, d\mu \; w(X', \mu', \nu')$$
$$\otimes \Pi(X, \mu, x - x', X', \mu', \nu', t)$$
$$\otimes \exp\left[i\left(X - \mu \frac{x + x'}{2}\right)\right]. \tag{11}$$

For the wave function in (11), we used the notation

$$\Psi(y) \equiv \Psi(y, t = 0).$$

Taking into account relationship (2) for the product $\Psi(y, t = 0) \Psi^*(z, t = 0)$ and the relationship

$$w(X', \mu', y - z) = \int w(X', \mu', \nu') \delta(y - z - \nu') \, d\nu',$$

in view of relationship (9) between the Fourier components, one obtains

$$\Pi(X, \mu, \nu, X', \mu', \nu', t) = \frac{1}{4\pi^2} \int k^2 G\left(a + \frac{k\nu}{2}, y, t\right) G^*\left(a - \frac{k\nu}{2}, z, t\right) \delta(y - z - k\nu')$$
$$\otimes \exp\left[ik\left(X' - X + \mu a - \mu' \frac{y + z}{2}\right)\right] dk \, dy \, dz \, da. \tag{12}$$

Equation (12) can be rewritten in the form

$$\Pi\left(X,\mu,\nu,X',\mu',\nu',t\right) = \frac{1}{4\pi^2}\int k^2 G\left(a+\frac{k\nu}{2},z+k\nu',t\right)G^*\left(a-\frac{k\nu}{2},z,t\right)$$
$$\otimes\exp\left[ik\left(X'-X-\frac{k\mu'\nu'}{2}-\mu'z+\mu a\right)\right]dk\,dz\,da. \tag{13}$$

Relationships (12) and (13) provide the expression for the "classical" propagator in terms of Green function of the Schrödinger evolution equation. One can see that the "classical" propagator depends on the difference of positions $X - X'$.

5. "Classical" Propagator and Path Integral

The quantum propagator (Green function of the Schrödinger evolution equation) $G\left(x_2,\,x_1,\,t_2,\,t_1\right)$ can be presented as path integral [20]

$$G\left(x_2,x_1,t_2,t_1\right) = \int_{(x_1,t_1)}^{(x_2,t_2)}\exp\left\{iS\left[x\left(t\right)\right]\right\}\mathcal{D}\left[x\left(t\right)\right],\qquad \hbar = 1, \tag{14}$$

where the functional $S\left[x\left(t\right)\right]$ is the classical action

$$S\left[x\left(t\right)\right] = \int_{t_1}^{t_2}L\left(x(t),\,\dot{x}(t),\,t\right)dt, \tag{15}$$

with Lagrangian $L\left(x(t),\,\dot{x}(t),\,t\right)$ of the form $(m=0)$

$$L\left(x(t),\,\dot{x}(t),\,t\right) = \frac{\dot{x}^2}{2} - U\left(x,t\right). \tag{16}$$

The classical trajectory is given by the minimal action principle

$$\delta S = 0,$$

that is equivalent to the Lagrange–Euler equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}. \tag{17}$$

The action functional can be considered as a function of several variables

$$S\left[x\left(t\right)\right] \approx \sum_{n=1}^{N}\left[\frac{\left(x_n - x_{n-1}\right)^2}{2\,\Delta t} - U\left(x_n,\,t_n\right)\Delta t\right], \tag{18}$$

where

$$x_n \equiv x\left(t_n\right),\qquad t_2 - t_1 = N\Delta t,\qquad t_{n+1} - t_n = \Delta t.$$

Thus, path integral can be approximately considered as multidimensional integral over the variables $x_1,\,x_2,\,\ldots,\,x_N$.

In view of Eq. (14), relation (13) can be rewritten as the path integral representation for the "classical" propagator

$$
\Pi\left(X, \mu, \nu, X', \mu', \nu', t\right) = \frac{1}{4\pi^2} \int k^2 \int_{(x+k\nu', 0)}^{(a+k\nu/2, t)} \exp\left\{iS\left[x(t)\right]\right\} \mathcal{D}\left[x(t)\right]
$$

$$
\otimes \int_{(z, 0)}^{(a-k\nu/2, t)} \exp\left\{-iS\left[y(t)\right]\right\} \mathcal{D}\left[y(t)\right]
$$

$$
\otimes \exp\left[ik\left(X' - X - \frac{k\mu'\nu'}{2} - \mu'z + \mu a\right)\right] dk\, dz\, da. \tag{19}
$$

6. Free Motion and Harmonic Oscillator

For a free particle $(\hbar = m = 1)$, Green function is of the form

$$
G\left(x, y, t\right) = \frac{1}{\sqrt{2\pi i t}} \exp\left[\frac{i\left(x-y\right)^2}{2t}\right]. \tag{20}
$$

The "classical" propagator for free motion found in [3] has the appearance

$$
\Pi_f\left(X, \mu, \nu, X', \mu', \nu', t\right) = \delta\left(X - X'\right) \delta\left(\mu - \mu'\right) \delta\left(\nu - \nu' + \mu t\right). \tag{21}
$$

We can calculate the "classical" propagator (21) employing relation (12), where Green function (20) for free motion is used. As a result, we arrive at

$$
\Pi_f\left(X, \mu, \nu, X', \mu', \nu', t\right) = \frac{1}{4\pi^2} \int k^2 \frac{1}{2\pi t} \delta\left(y - z - k\nu'\right)
$$

$$
\otimes \exp\left\{\frac{i}{2t}\left(a + \frac{k\nu}{2} - y\right) - \frac{i}{2t}\left(a - \frac{k\nu}{2} - z\right)\right.
$$

$$
\left. +ik\left(X' - X + \mu a - \mu'\frac{yz}{2}\right)\right\} dk\, dy\, dz\, da. \tag{22}
$$

Integration in Eq. (22) over variable a gives a delta-function, namely,

$$
\int da \implies \delta\left(\frac{k\left(\nu + \mu t\right) + z - y}{t}\right).
$$

After introducing new variables

$$
y + z = s, \qquad y - z = m
$$

and integrating over variable s, one obtains another delta-function

$$
\int ds \implies \delta\left(\frac{k\left(\nu + \mu't\right) + z - y}{t}\right).
$$

After integrating first over variable m and then over variable k, one arrives at the expression

$$
\Pi_f\left(X, \mu, \nu, X', \mu', \nu', t\right) = |t|\, \delta\left(X - X'\right) \delta\left(\nu - \nu' + \mu t\right) \delta\left(\nu - \nu' + \mu't\right). \tag{23}
$$

In view of the relationships for delta-functions,

$$\delta\left(x+y\right)\delta\left(x+z\right) = \delta\left(x+y\right)\delta\left(y-z\right),$$
$$\delta\left(\frac{x}{t}\right) = |t|\,\delta\left(x\right),$$

one obtains the result (21).

Another example is the harmonic oscillator, for which Green function has the form ($m = \omega = \hbar = 1$)

$$G_{os}\left(x,y,t\right) = \frac{1}{\sqrt{2\,\pi i\sin t}}\,\exp\left[\frac{i}{2}\cot t\left(x^2+y^2\right)-\frac{ixy}{\sin t}\right]. \tag{24}$$

By using formula (12), we obtain the "classical" propagator for the harmonic oscillator,

$$
\begin{aligned}
\Pi_{os}\left(X,\mu,\nu,X',\mu',\nu',t\right) &= \frac{1}{4\pi^2}\int\frac{k^2}{2\pi\,|\sin t|}\,\delta\left(y-z-k\nu'\right)\\
&\otimes\exp\Bigg\{\frac{i}{2}\cot t\left[\left(a+\frac{k\nu}{2}\right)^2+y^2\right]-\frac{iy}{\sin t}\left(a+\frac{k\nu}{2}\right)\\
&\quad-\frac{i}{2}\cot t\left[\left(a-\frac{k\nu}{2}\right)^2+z^2\right]-\frac{iz}{\sin t}\left(a-\frac{k\nu}{2}\right)\\
&\quad-ik\left(X-X'\right)-i\mu'k\frac{y+z}{2}+ik\mu a\Bigg\}\,dk\,dy\,dz\,da. \tag{25}
\end{aligned}
$$

Integration in Eq. (25) over variable a gives a delta-function, namely,

$$\int da\Longrightarrow\delta\left(k\left(\nu\cot t+\mu\right)-\frac{y-z}{\sin t}\right).$$

Integration in Eq. (25) over variable $s = y + z$ results in another delta-function

$$\int ds\Longrightarrow\delta\left(\left(y-z\right)\cot t-k\mu'-\frac{k\nu}{\sin t}\right).$$

Integration over variable $m = y - z$ gives the term

$$\int dm\Longrightarrow\frac{k^2}{|\sin t|}\delta\left(k\left(\nu\cot t+\mu\right)-\frac{k\nu'}{\sin t}\right)\delta\left(k\nu'\cot t-k\mu'-\frac{k\nu}{\sin t}\right). \tag{26}$$

Employing properties of delta-function, after the last integration over variable k, one obtains

$$
\begin{aligned}
\Pi_{os}\left(X,\mu,\nu,X',\mu',\nu',t\right) &= |\sin t|\,\delta\left(X-X'\right)\delta\left(\nu\cos t+\mu\sin t-\nu'\right)\delta\left(\nu'\cos t-\mu'\sin t-\nu\right)\\
&= \delta\left(X-X'\right)\delta\left(\nu\cos t+\mu\sin t-\nu'\right)\delta\left(\mu\cos t-\nu\sin t-\mu'\right). \tag{27}
\end{aligned}
$$

Expression (27) is the "classical propagator" for the harmonic oscillator; it was derived using the different technique in [2, 3, 10].

7. Quasiclassical Approximation for the "Classical" Propagator

Formula (13) gives exact expression for the "classical" propagator in terms of the quantum Green function $G(x_2, x_1, t)$. On the other hand, there exists the quasiclassical van-Fleck formula for Green function of the Schrödinger equation

$$G^{(q)}(x_2, x_1, t) = \frac{C(t)}{\sqrt{|\mathcal{D}_x|}} e^{iS(x_2, x_1, t)}, \tag{28}$$

where

$$\mathcal{D}_x = \begin{vmatrix} \dfrac{\partial^2 S}{\partial x_1^2} & \dfrac{\partial^2 S}{\partial x_1 \partial x_2} \\[3mm] \dfrac{\partial^2 S}{\partial x_2 \partial x_1} & \dfrac{\partial^2 S}{\partial x_2^2} \end{vmatrix}.$$

In formula (28), $S(x_2, x_1, t)$ is the classical action, satisfying the classical Hamilton equation

$$\frac{\partial S}{\partial t} + \mathcal{H}(p, x_2)|_{p \to \partial S/\partial x_2} = 0 \tag{29}$$

for the classical system with the Hamiltonian $\mathcal{H}(p, x)$, and the function $C(t)$ is taken according to the Schrödinger equation for Green function.

One can use the quasiclassical approximation (28) for Green function in order to obtain the "classical" propagator in the quasiclassical approximation. We arrive at

$$\begin{aligned}
\Pi^{(q)}(X, \mu, \nu, X', \mu', \nu', t) = {} & \frac{1}{4\pi^2} \int k^2\, \delta(y - z - k\nu') \frac{|C(t)|^2}{\sqrt{|\mathcal{D}_x \mathcal{D}_y|}} \\
& \otimes \exp\left\{ iS\left(a + \frac{k\nu}{2}, y, t\right) - iS\left(a - \frac{k\nu}{2}, z, t\right) \right. \\
& \left. + ik\left[(X' - X) + \mu a - \mu' \frac{y+z}{2} \right] \right\} dk\, dy\, dz\, da,
\end{aligned} \tag{30}$$

where the notation

$$x_2 = a + \frac{k\nu}{2}, \qquad x_1 = y, \qquad y_1 = a - \frac{k\nu}{2}, \qquad y_2 = z$$

is used.

Quasiclassical formula (30) is the result of calculations of the "classical" propagator in terms of path integral (19) for the functional of the classical action of the form

$$S[x(t)] = S[\underline{x}(t)] + (S[x(t)] - S[\underline{x}(t)]), \tag{31}$$

where $\underline{x}(t)$ is the classical trajectory.

Then we used series expansion for the difference of the functional and its value at the classical trajectory. For linear systems with quadratic Hamiltonians, series (31) contains only two terms and the exact result for the "classical" propagator coincides with the result of the quasiclassical approximation.

8. Optical Tomography in Bargmann Representation

In previous sections, we used the symplectic tomography approach. In the optical tomography approach [23], the Wigner function is reconstructed, if one considers the marginal distribution $w(X, \varphi, t)$ of the homodyne observable X, the distribution being dependent on the rotation angle φ. Till now, the evolution equation was obtained for the marginal distribution $w(X, \mu, \nu, t)$ in the symplectic tomography approach. Since

$$w(X, \varphi, t) = w(X, \cos\varphi, \sin\varphi, t), \tag{32}$$

the possibility to derive the evolution equation for the optical marginal distribution arises. To do this, let us introduce complex variables

$$z = \mu + i\nu; \qquad \bar{z} = \mu - i\nu, \tag{33}$$

which are similar to the variables used in the Bargmann representation of coherent states.

The inverse of Eq. (33) reads

$$\mu = \frac{z + \bar{z}}{2}; \qquad \nu = \frac{z - \bar{z}}{2}. \tag{34}$$

The evolution equation for the marginal probability distribution has the appearance [1, 2]

$$\dot{w} - \mu\frac{\partial}{\partial\nu}w - i\left[V\left(-\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu} - i\frac{\nu}{2}\frac{\partial}{\partial X}\right) - V\left(-\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu} + i\frac{\nu}{2}\frac{\partial}{\partial X}\right)\right]w = 0. \tag{35}$$

Using for variables the notation determined by (33), we arrive at

$$\dot{w} - i\frac{z+\bar{z}}{2}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\bar{z}}\right)w - i\left\{V\left[-\frac{1}{\partial/\partial X}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\bar{z}}\right) - \frac{1}{4}(z - \bar{z})\frac{\partial}{\partial X}\right] - \text{c.c.}\right\}w = 0. \tag{36}$$

If one makes the substitution

$$z \Longrightarrow e^{i\varphi}, \qquad \bar{z} \Longrightarrow e^{-i\varphi},$$

$w(X, z, \bar{z}, t)$, being the solution to Eq. (36), is the marginal distribution of the optical tomography approach, namely,

$$w(X, \varphi, t) = w\left(X, e^{i\varphi}, e^{-i\varphi}, t\right). \tag{37}$$

Thus, if one obtains the solution to the evolution equation for the marginal distribution in the Bargmann representation, the marginal distribution of the symplectic tomography approach can be derived by means of Eq. (37), as well. Analogous substitutions can be applied for the "classical" propagator and its path integral representation.

9. Conclusion

Formulas (12) and (13), which give the expression for the "classical" propagator (determining the evolution of the quantum system in the probability representation of quantum mechanics) in terms of the quantum propagator (Green function of the Schrödinger evolution equation) are the main result of this study.

Another important result is given by formula (19) which desribes the "classical" propagator in terms of path integral determined by the functional of the classical action. Formula (13) obtained in this study is the inverse of the expression of the quantum propagator in terms of the "classical" propagator which was found in [3] (see also [10]).

Thus, the invertable map of density matrix onto marginal probability distribution, which determines completely the quantum state, is accomplished by the invertable map of the quantum propagator onto the "classical" propagator, which describes completely the evolution of the quantum system.

The discussed properties of the "classical" propagator can be used for studying evolution of states in the framework of the approach considered in [24].

Acknowledgments

References

1. S. Mancini, V. I. Man'ko, and P. Tombesi, *Phys. Lett.* A **213**, 1 (1996).
2. S. Mancini, V. I. Man'ko, and P. Tombesi, *Found. Phys.* **27**, 801 (1997).
3. Olga Man'ko and V. I. Man'ko, *J. Russ. Laser Research*, **18**, 407 (1997).
4. H. R. Lewis and W. B. Riesenfeld, *J. Math. Phys.*, **10**, 1458 (1968).
5. I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, *Phys. Lett.* A, **30**, 414 (1969).
6. I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, *Phys. Rev.* D, **2**, 1371 (1970).
7. I. A. Malkin and V. I. Man'ko, *Phys. Lett.* A, **32**, 243 (1970).
8. I. A. Malkin and V. I. Man'ko, *Dynamic Symmetries and Coherent States of Quantum Systems* [in Russian], Nauka, Moscow (1979).
9. V. V. Dodonov and V. I. Man'ko, *Invariants and Evolution of Nonstationary Quantum Systems*, Proceedings of the Lebedev Physical Institute, Nova Science, New York (1989), Vol. 183.
10. V. I. Man'ko, L. Rosa, and P. Vitale, *Phys. Rev* A, **58**, 3291 (1998).
11. V. I. Man'ko, *J. Russ. Laser Research*, **17**, 579 (1996).
12. V. I. Man'ko and S. S. Safonov, *Teor. Mat. Fiz.*, **112**, 1172 (1997).
13. V. I. Man'ko and S. S. Safonov, *Phys. Atom. Nucl.*, **61**, 585 (1998).
14. V. V. Dodonov and V. I. Man'ko, *Phys. Lett.* A, **229**, 335 (1997).
15. V. I. Man'ko and O. V. Man'ko, *JETP*, **85**, 430 (1997).
16. V. A. Andreev and V. I. Man'ko, *JETP*, **87**, 239 (1998).
17. V. A. Andreev, V. I. Man'ko, O. V. Man'ko, and S. S. Safonov, *J. Russ. Laser Research*, **19**, 340 (1998).
18. M. Moshinsky, *Phys. Rev.*, **88**, 625 (1952).
19. Vladimir Man'ko, Marcos Moshinsky, and Anju Sharma, "Diffraction in time in terms of Wigner distributions and tomographic probabilities," E-print quant-ph./9902075; *Phys. Rev.* A (1999, in press).
20. R. Feynman, *Rev. Mod. Phys.*, **20**, 367 (1948).
21. V. I. Man'ko and R. V. Mendes, "Noncommutative time–frequency tomography of analytic sugnals," Eprint LANL Physics/9712022 Data Analysis, Statistics, and Probability; *IEEE Signal Processing* (1998, in press).
22. V. I. Man'ko, L. Rosa, and P. Vitale, *Phys. Lett.* B, **439**, 328 (1998).
23. K. Vogel and H. Risken, *Phys. Rev.* A, **40**, 2847 (1989).
24. O. V. Man'ko, "Symplectic tomography of nonclassical states of a trapped ion," *Preprint* IC/96/39 (ICTP, Trieste, 1996); *Los Alamos Report* No. quant-ph/9604018; *J. Russ. Laser Research*, **17**, 439 (1996); *Phys. Lett.* A, **228**, 29 (1997); "Symplectic tomography of Schrödinger cat states of a trapped

ion," in: M. Ferrero and A. van der Merwe (eds.), *Proceedings of the Second International Symposium on Fundamental Problems in Quantum Mechanics* (Oviedo, Spain, July 1996), Kluwer Academic Press (1997), p. 225; "Symplectic tomography of nonlinear Schrödinger cats of a trapped ion," in: D. Han, J. Janszky, Y. S. Kim, and V. I. Man'ko (eds.), *Fifth International Conference on Squeezed States and Uncertainty Relations* (Balatonfüred, Hungary, May 1997), NASA Conference Publication, Goddard Space Flight Center, Greenbelt, Maryland (1998), Vol. NASA/CP-1998-206855, p. 309; "Tomography of a trapped ion," in: P. Kasperkovitz and D. Gran (eds.), *Proceedings of the 5th Wigner Symposium* (Vienna, August 1997), World Scientific, Singapore (1998), p. 413; "Optical tomography and measuring quantum states of an ion in a Paul trap and in a Penning trap," Contribution to the 16th International Conference on Coherent and Nonlinear Optics, Moscow, June–July 1998; *Proc. SPIE* (1999, in press).