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CLASSICAL REPRESENTATION OF A QUANTUM DAMPED OSCILLATOR — COMPARISON BETWEEN THE CALDIROLA–KANAI MODEL, THE KINETIC EQUATION, AND THE KOSTIN EQUATION

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Abstract

Solutions to a new quantum-mechanical kinetic equation for excited states of a damped oscillator are obtained explicitly. The difference between the position probability distributions, which determine (in the new formulation of quantum mechanics) states of the damped oscillator within the framework of the Caldirola–Kanai model, the kinetic equation with a collision term, and the nonlinear Kostin equation, is analyzed.

1. Introduction

The problem of motion with damping is a matter of no little interest in quantum mechanics. Within the framework of the quantum theory for describing dissipative quantum systems, different models were proposed. In [1, 2], within the framework of the linear Schrödinger equation for the wave function, a simple quantum Hamiltonian for the damping process was proposed and discussed in detail. For open (dissipative) quantum systems, the kinetic equation with a coefficient responsible for damping was proposed, for example, in [3, 4]. The nonlinear Kostin equation was also used to describe quantum friction (see, for example, [5]). Nevertheless, till now there exists a misunderstanding in comparative analysis of different models of quantum damping and classical damping. Attempts to compare different models describing damped quantum systems were made in a number of papers (see, for example, [6]), where the time dependence of the average (over period) energy of a damped oscillator in the classical and quantum cases was given), but these attempts did not result in a final understanding of the problem under discussion.

Recently, in [7–9] the probability representation of quantum mechanics was introduced and a new evolution equation was derived, which was a generalization of the results obtained in [10], where the role of the Wigner function was played by the probability distribution (marginal distribution) of the particle's position in an ensemble of rotated and scaled reference frames in the system's classical phase space (the classical representation of quantum mechanics). The probability representation of quantum mechanics uses the symplectic tomography procedure suggested for measuring quantum states in [11, 12]. The approach was developed in [13–26]. Within the framework of the classical representation, the quantum damped oscillator described by the Caldirola–Kanai model was considered in [17]. In view of the possibility of the classical treatment of a quantum system, we compare three different models for quantum damping, namely,

(1) the Caldirola–Kanai model with a Hamiltonian;
(2) the model where damping is taken into account by inserting a collision term into the kinetic equation;
(3) the model in which damping is described by the nonlinear Kostin equation (the Kostin model).

The aim of this paper is to derive the marginal distribution for the quantum damped oscillator using the kinetic equation with a collision term and to compare the expression obtained with the marginal distribution calculated for the system described by the Caldirola–Kanai model of [17] and for the system described by the nonlinear Kostin equation.

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2. Marginal Distribution for the Kostin Model

As was shown in [7], for the generic linear combination of the position q and momentum p, which is a measurable observable in the phase space ($\hbar = m = 1$)

$$\widehat{X} = \mu\widehat{q} + \frac{\nu}{\omega_1}\widehat{p}\,, \tag{1}$$

where ω_1 is the frequency of an unexcited oscillator, the marginal distribution $w\,(X,\,\mu,\,\nu)$ (normalized with respect to the variable X), depending on the two extra real parameters μ and ν, is related to the state of the quantum system expressed in terms of its Wigner function $W\,(q,\,p)$ as follows:

$$w\,(X,\mu,\nu) = \int \exp\left[-ik\left(X - \mu q - \frac{\nu}{\omega_1}p\right)\right] W\,(q,p)\,\frac{dk\,dq\,dp}{(2\pi)^2}\,. \tag{2}$$

The Kostin equation for the quantum damped oscillator has the form ($\hbar = m = 1$)

$$i\,\frac{\partial\Psi}{\partial t} = \widehat{H}\Psi\,, \tag{3}$$

where the Kostin Hamiltonian

$$\widehat{H} = \frac{\widehat{p}^2}{2} + \frac{\omega_1^2\widehat{q}^2}{2} - i\gamma_1\left[\ln\left(\frac{\Psi}{\Psi^*}\right) - \int \Psi^*\ln\left(\frac{\Psi}{\Psi^*}\right)\Psi dq\right] \tag{4}$$

depends not only on operators of the position $\widehat{q}$ and momentum $\widehat{p}$, but also on the wave function $\Psi\,(q,\,t)$ of the quantum damped oscillator.

The Kostin equation (3) with the Hamiltonian (4) for a quantum oscillator was solved, in particular, in [5], where it was shown that the wave function $\Psi\,(q,\,t)$ can be presented in Gaussian form:

$$\begin{aligned}\Psi\,(q,t) \;&=\; \left(\frac{\omega_1}{\pi}\right)^{1/4}\exp\left\{-\frac{\omega_1}{2}q^2 + e^{-\gamma_1 t}\left[\left(1 - i\,\frac{\gamma_1}{\omega_1}\right)\cos\Omega t - i\,\frac{\Omega}{\omega_1}\sin\Omega t\right]q\right.\\ &\left.-\frac{e^{-2\gamma_1 t}\cos^2\Omega t}{2\omega_1} + i\,\frac{\gamma_1 e^{-2\gamma_1 t}\cos^2\Omega t}{2\omega_1^2} + i\,\frac{\Omega\sin 2\Omega t}{4\omega_1^2} - i\,\frac{\omega_1 t}{2} + i\,\frac{\gamma_1}{4\omega_1^2}\right\},\end{aligned} \tag{5}$$

where

$$\Omega = \sqrt{\omega_1^2 - \gamma_1^2}.$$

Within the framework of the Kostin model, it is not difficult to show that at the initial time moment $t = 0$ the wave function of the quantum damped oscillator has the appearance of the coherent state's wave function

$$\Psi_\alpha\,(q,\,t=0) = \left(\frac{\omega_1}{\pi}\right)^{1/4}\exp\left[-\frac{|\alpha|^2}{2} - \frac{\omega_1}{2}\,q^2 + \sqrt{2\omega_1}\alpha q - \frac{\alpha^2}{2}\right], \tag{6}$$

with a fixed complex parameter α of the form

$$\alpha = \frac{1}{\sqrt{2\omega_1}}\left(1 - i\,\frac{\gamma_1}{\omega_1}\right). \tag{7}$$

At any fixed time moment t, the wave function $\Psi_\alpha\,(q,\,t=0)$ depends on the parameter α as follows:

$$\begin{aligned}\Psi_\alpha\,(q,t) \;&=\; \left(\frac{\omega_1}{\pi}\right)^{1/4}\exp\left[iF\,(t)\right]\exp\left\{-\frac{\omega_1}{2}\,q^2 + e^{-\gamma_1 t}\left[\sqrt{2\omega_1}\alpha\cos\Omega t - i\,\frac{\Omega}{\omega_1}\sin\Omega t\right]q\right.\\ &\left.-\left(\frac{|\alpha|^2}{2} + \frac{\alpha^2}{2}\right)e^{-2\gamma_1 t}\cos^2\Omega t\right\},\end{aligned} \tag{8}$$

where the phase F depends on time:

$$F(t) = \frac{\gamma_1 e^{-2\gamma_1 t}\cos^2 \Omega t}{2\omega_1^2} + \frac{\Omega e^{-2\gamma_1 t}\sin 2\Omega t}{4\omega_1^2} - \frac{\omega_1 t}{2} + \frac{\gamma_1}{4\omega_1^2}. \tag{9}$$

Now we derive the marginal distribution $w_\alpha(X, \mu, \nu, t)$ for the coherent state of the quantum damped oscillator within the framework of the Kostin model using the wave function $\Psi_\alpha(q, t)$ for the given coherent state of the oscillator under discussion. To do this, we use the relationship adopted, for example, from [17], which explicitly connects the marginal distribution $w_\alpha(X, \mu, \nu, t)$ with the wave function $\Psi_\alpha(q, t)$:

$$w_\alpha(X,\mu,\nu,t) = \int \exp(-ikX + ik\mu q)\,\Psi_\alpha\left(q + \frac{k\nu}{2}, t\right)\Psi_\alpha^*\left(q - \frac{k\nu}{2}, t\right)\frac{dk\,dq}{2\pi}. \tag{10}$$

Inserting the wave function $\Psi_\alpha(q, t)$ for the quantum damped oscillator in formula (10), after some algebra we obtain the marginal distribution $w_\alpha(X, \mu, \nu, t)$ for the coherent state in the form

$$\begin{aligned} w_\alpha(X,\mu,\nu,t) &= \frac{1}{\sqrt{\pi\omega_1^{-1}(\mu^2+\nu^2)}}\exp\left(-|\alpha|^2\right)\exp\left[-\frac{X'^2}{\omega_1^{-1}(\mu^2+\nu^2)}\right] \\ &\otimes \exp\left[\frac{\omega_1^{-1}(\mu - i\nu)^2 e^{-2\gamma_1 t}\cos^2\Omega t}{\omega_1^{-1}(\mu^2+\nu^2)}\alpha^2 + \frac{\sqrt{2}X'(\mu - i\nu)e^{-\gamma_1 t}\cos\Omega t}{\omega_1^{-1}(\mu^2+\nu^2)}\alpha\right] \\ &\otimes \exp\left[\frac{\omega_1^{-1}(\mu + i\nu)^2 e^{-2\gamma_1 t}\cos^2\Omega t}{\omega_1^{-1}(\mu^2+\nu^2)}\alpha^{*2} + \frac{\sqrt{2}X'(\mu + i\nu)e^{-\gamma_1 t}\cos\Omega t}{\omega_1^{-1}(\mu^2+\nu^2)}\alpha^*\right] \\ &\otimes \exp\left[-\left(e^{-2\gamma_1 t}\cos^2\Omega t - 1\right)|\alpha|^2\right], \end{aligned} \tag{11}$$

where

$$X' = X + \frac{\Omega e^{-\gamma_1 t}\sin\Omega t}{\omega_1}\nu. \tag{12}$$

Employing the fact that a coherent state is the generating function for the system's excited states (see, for example, [6]), i.e., the equality

$$w_\alpha(X,\mu,\nu,t) = \exp\left(-|\alpha|^2\right)\sum_{n,m=0}^{\infty}\frac{\alpha^n\alpha^{*m}}{\sqrt{n!m!}}w_{nm}(X,\mu,\nu,t) \tag{13}$$

is valid, and introducing the notation

$$w_n(X,\mu,\nu,t) = w_{nn}(X,\mu,\nu,t),$$

we obtain the marginal distribution for the excited Fock state $w_n(X, \mu, \nu, t)$ expressed through the Hermite polynomials of two variables $H_{nm}^{\{\beta\}}(x, y)$ (see, for example, [6]):

$$\begin{aligned} w_n(X,\mu,\nu,t) &= \frac{1}{\sqrt{\pi\omega_1^{-1}(\mu^2+\nu^2)}}\exp\left[-\frac{X'^2}{\omega_1^{-1}(\mu^2+\nu^2)}\right]\frac{e^{-2n\gamma_1 t}\cos^{2n}\Omega t}{n!} \\ &\otimes H_{nn}^{\{\beta\}}\left[\frac{\sqrt{2}X'}{(1+\beta)\sqrt{\omega_1^{-1}(\mu^2+\nu^2)}}, \frac{\sqrt{2}X'}{(1+\beta)\sqrt{\omega_1^{-1}(\mu^2+\nu^2)}}\right], \end{aligned} \tag{14}$$

where

$$\beta = 1 - \frac{\exp(2\gamma_1 t)}{\cos^2\Omega t}. \tag{15}$$

In view of the relationship between the Hermite polynomials of two variables and the usual Hermite polynomials [6]

$$H_{mn}^{\{\beta\}}(y_1, y_2) = m!\,n!\,2^{-(m+n)/2} \sum_{k=0}^{\min(m,n)} \frac{(-2\beta)^k}{k!\,(m-k)!\,(n-k)!} \otimes H_{m-k}\left(\frac{y_1+\beta y_2}{\sqrt{2}}\right) H_{m-k}\left(\frac{y_2+\beta y_1}{\sqrt{2}}\right), \tag{16}$$

we obtain the marginal distributiom for the excited Fock state in the form

$$w_n(X,\mu,\nu,t) = w_0(X,\mu,\nu,t)\,\frac{n!}{2^n}\,e^{-2n\gamma_1 t}\cos^{2n}\Omega t \otimes \sum_{k=0}^{n} \frac{(-2\beta)^k}{k!\,((n-k)!)^2} H_{m-k}^2\left[\frac{X+\nu\Omega\omega_1^{-1}e^{-\gamma_1 t}\sin\Omega t}{\sqrt{\omega_1^{-1}(\mu^2+\nu^2)}}\right], \tag{17}$$

where $w_0(X, \mu, \nu, t)$ is the marginal distribution for the system's ground state:

$$w_0(X,\mu,\nu,t) = \frac{1}{\sqrt{\pi\omega_1^{-1}(\mu^2+\nu^2)}} \exp\left[-\frac{\left(X+\nu\Omega\omega_1^{-1}e^{-\gamma_1 t}\sin\Omega t\right)^2}{\omega_1^{-1}(\mu^2+\nu^2)}\right]. \tag{18}$$

3. Marginal Distribution for the Model with the Kinetic Equation

As was shown in [7], for damped systems which are described by the kinetic equation for the density matrix with a collision term

$$\dot{\rho} = -i\omega_2\left[a^\dagger a, \rho\right] + \gamma_2\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right), \tag{19}$$

where ω_2 and γ_2 are the frequency and damping coefficient, respectively, for the model with the kinetic equation, $a^\dagger$ and a are the boson creation and annihilation operators, Eq. (19) can be rewritten (in the interaction representation) using the language of marginal distribution in the framework of the classical description of quantum mechanics, in the form of a new Fokker–Planck equation:

$$\dot{w} = \gamma_1\left[2 - \frac{\partial}{\partial\nu}\nu - \frac{\partial}{\partial\mu}\mu + \frac{1}{2\omega_2}\left(\mu^2+\nu^2\right)\frac{\partial^2}{\partial X^2}\right] w. \tag{20}$$

A solution to the evolution equation (20) is the marginal distribution for the coherent state of a damped system in the interaction representation [7]

$$w(X,\mu,\nu,t) = \sqrt{\frac{\omega_2}{\pi}}\,\frac{1}{\sqrt{\mu^2+\nu^2}} \exp\left\{-\frac{\left[X-\left(\mu q_0+\omega_2^{-1}\nu p_0\right)e^{-\gamma_2 t}\right]^2}{\omega_2^{-1}(\mu^2+\nu^2)}\right\}, \tag{21}$$

which, in view of Eq. (2), is related to the Wigner function of the damped harmonic oscillator

$$W(q,p) = 2\exp\left[-\omega_2\left(q-q_0e^{-\gamma_2 t}\right)^2 - \omega_2^{-1}\left(p-p_0e^{-\gamma_2 t}\right)^2\right]. \tag{22}$$

In Eqs. (21) and (22), the parameters of the initial coherent state q_0, p_0 can be written in the form

$$q_0 = \frac{\alpha + \alpha^*}{\sqrt{2\omega_2}}, \qquad p_0 = \frac{\alpha - \alpha^*}{i\sqrt{2}}\sqrt{\omega_2}, \tag{23}$$

where α is a complex number.

Since Eq. (21) was written in the interaction representation, in order to obtain the marginal distribution for the system's coherent state in the Schrödinger representation, it is sufficient to introduce the following notation:

$$\begin{aligned} \mu(t) &= \mu\cos\omega_2 t + \nu\sin\omega_2 t, \\ \nu(t) &= -\mu\sin\omega_2 t + \nu\cos\omega_2 t. \end{aligned} \tag{24}$$

By inserting (23) and (24) in (21), we obtain the marginal distribution for the coherent state of the damped oscillator, which is a solution to the new quantum equation of the Fokker–Planck-type (20) written in the Schrödinger representation:

$$\begin{aligned} w_\alpha(X,\mu,\nu,t) &= \sqrt{\frac{\omega_2}{\pi}}\exp\left(-\mid\alpha\mid^2\right)\frac{1}{\sqrt{\mu^2+\nu^2}}\exp\left[-\frac{X^2}{\omega_2^{-1}(\mu^2+\nu^2)}\right] \\ &\otimes\exp\left[-\frac{e^{-2\gamma_2 t}}{2(\mu^2+\nu^2)}\left\{(\mu\cos\omega_2 t-\nu\sin\omega_2 t)-i(\mu\sin\omega_2 t+\nu\cos\omega_2 t)\right\}^2\alpha^2\right. \\ &\left.+\frac{\sqrt{2}Xe^{-\gamma_2 t}}{\omega_2^{-1/2}(\mu^2+\nu^2)}\left\{(\mu\cos\omega_2 t-\nu\sin\omega_2 t)-i(\mu\sin\omega_2 t+\nu\cos\omega_2 t)\right\}\alpha\right] \\ &\otimes\exp\left[-\frac{e^{-2\gamma_2 t}}{2(\mu^2+\nu^2)}\left\{(\mu\cos\omega_2 t-\nu\sin\omega_2 t)+i(\mu\sin\omega_2 t+\nu\cos\omega_2 t)\right\}^2\alpha^{*2}\right. \\ &\left.+\frac{\sqrt{2}Xe^{-\gamma_2 t}}{\omega_2^{-1/2}(\mu^2+\nu^2)}\left\{(\mu\cos\omega_2 t-\nu\sin\omega_2 t)+i(\mu\sin\omega_2 t+\nu\cos\omega_2 t)\right\}\alpha^*\right] \\ &\otimes\exp\left[-\left(e^{-2\gamma_2 t}-1\right)\mid\alpha\mid^2\right]. \end{aligned} \tag{25}$$

Now we follow the treatment of Sec. 2, where the generating function was used for calculating the marginal distribution for the system's excited state. Since a coherent state is the generating function for the system's excited states (see, for example, [6]), i.e., the equality

$$w_\alpha(X,\mu,\nu,t) = \exp\left(-\mid\alpha\mid^2\right)\sum_{n,m=0}^{\infty}\frac{\alpha^n\alpha^{*m}}{\sqrt{n!\,m!}}w_{nm}(X,\mu,\nu,t) \tag{26}$$

is valid, introducing the notation

$$w_n(X,\mu,\nu,t) = w_{nn}(X,\mu,\nu,t),$$

we obtain the marginal distribution for the excited states $w_n(X,\mu,\nu,t)$ expressed in terms of the Hermite polynomials of two variables $H_{nm}^{\{\beta\}}(x,y)$ (see, for example, [6]):

$$\begin{aligned} w_n(X,\mu,\nu,t) &= \sqrt{\frac{\omega_2}{\pi}}\frac{1}{\sqrt{\mu^2+\nu^2}}\exp\left[-\frac{X^2}{\omega_2^{-1}(\mu^2+\nu^2)}\right]\frac{e^{-2n\gamma_2 t}}{n!} \\ &\otimes H_{nn}^{\{\beta\}}\left(\frac{X}{\sqrt{\omega_2^{-1}(\mu^2+\nu^2)}},\frac{X}{\sqrt{\omega_2^{-1}(\mu^2+\nu^2)}}\right), \end{aligned} \tag{27}$$

where

$$\beta = 1 - e^{2\gamma_2 t}.$$

In view of the relationship between the Hermite polynomials of two variables and the usual Hermite polynomials [6]

$$H_{nn}^{\{\beta\}}(y,y) = (n!)^2 2^{-n} \sum_{k=0}^{n} \frac{(-2\beta)^k}{k!\,[(n-k)!]^2} H_{n-k}^2 \left(\frac{(1+\beta)\,y}{\sqrt{2}} \right), \tag{28}$$

we obtain the marginal distribution for the excited Fock state in the form

$$\begin{aligned} w_n(X,\mu,\nu,t) &= w_0(X,\mu,\nu) \frac{n!}{2^n} e^{-2n\gamma_2 t} \\ &\otimes \sum_{k=0}^{n} \frac{(-2\beta)^k}{k!\,[(n-k)!]^2} H_{n-k}^2 \left(\frac{X}{\sqrt{\omega_2^{-1}(\mu^2+\nu^2)}} \right), \end{aligned} \tag{29}$$

where the marginal distribution for the system's ground state $w_0(X,\mu,\nu)$ reads

$$w_0(X,\mu,\nu) = \sqrt{\frac{\omega_2}{\pi}} \frac{1}{\sqrt{\mu^2+\nu^2}} \exp\left[-\frac{X^2}{\omega_2^{-1}(\mu^2+\nu^2)} \right]. \tag{30}$$

One can see that the calculations we produced in this section are similar to the ones treated in Sec. 2 in order to obtain the marginal distribution for the excited Fock state of the damped quantum oscillator within the framework of the Kostin model.

4. Comparison of the Three Models

As was shown in [17], for the quantum damped oscillator described by the Caldirola–Kanai Hamiltonian

$$\hat{H} = \frac{\hat{p}^2 e^{-2\gamma_1 t}}{2} + \frac{\omega_1^2 \hat{q}^2 e^{2\gamma_1 t}}{2},$$

the marginal distribution for the excited Fock state has the form

$$w_n(X,\mu,\nu,t) = w_0(X,\mu,\nu,t) \frac{1}{2^n n!} H_n^2 \left(\frac{X}{\sqrt{\varepsilon\varepsilon^*(a^2+b^2)}} \right), \tag{31}$$

where $w_0(X,\mu,\nu,t)$ is the marginal distribution for the oscillator's ground state

$$w_0(X,\mu,\nu,t) = \frac{1}{\sqrt{\pi\varepsilon\varepsilon^*(a^2+b^2)}} \exp\left(-\frac{X^2}{\varepsilon\varepsilon^*(a^2+b^2)} \right), \tag{32}$$

with

$$a = \frac{\exp(2\gamma_1 t)\,\nu\left(\varepsilon^*\dot{\varepsilon} + \varepsilon\dot{\varepsilon^*}\right)}{2\varepsilon\varepsilon^*} + \mu, \qquad b = \frac{\nu}{\varepsilon\varepsilon^*} \tag{33}$$

and

$$\varepsilon(t) = \frac{e^{-\gamma_1 t}}{\Omega} \left[(\gamma_1 \sin\Omega t + \Omega\cos\Omega t) + i\sin\Omega t \right], \qquad \Omega = \sqrt{\omega_1^2 - \gamma_1^2}. \tag{34}$$

It is worth noting that a particular feature of all three models is that the position mean values, within the framework of the Caldirola–Kanai model, the Kostin model, and the model with the kinetic equation, have the same classical appearance, namely, for the model with the Caldirola–Kanai Hamiltonian and the Kostin model, it reads

$$\langle \ddot{q} \rangle + 2\gamma_1 \langle \dot{q} \rangle + \omega_1^2 \langle q \rangle = 0, \tag{35}$$

and for the kinetic equation with a collision term it takes the form

$$\langle \ddot{q} \rangle + 2\gamma_2 \langle \dot{q} \rangle + \left(\omega_2^2 + \gamma_2^2\right) \langle q \rangle = 0\,. \tag{36}$$

Because of this, the quantum systems, which can be treated with the help of the three discussed models, can be conditionally considered as a quantum analog of the classical damped oscillator.

For the numerical comparison of the three models, we take the same values for damping coefficients and for the coefficients of $\langle q \rangle$ in the equation for the position mean value, namely,

$$\gamma = \gamma_1 = \gamma_2 = 0.9\,, \qquad \omega = \sqrt{\omega_2^2 + \gamma_2^2} = \omega_1 = 1.5\,.$$

The results of the numerical calculation of the marginal probability distributions for the kinetic equation with a collision term (29) and for the Caldirola–Kanai model (31) are presented in Figs. 1 and 2, and those for the Kostin model are shown in Fig. 3. Comparing the plots, one can draw several important conclusions:

(i) Dispersions of the state's marginal probability distributions for the two models of quantum damping (the Caldirola–Kanai model and the kinetic equation with a collision term) are different. This difference can be easily observed in the dispersion parameters of the state's marginal probability distributions for these two models of the quantum damped oscillator in Fig. 1, where marginal distributions for the ground state ($n = 0$) of the quantum damped oscillator for the model with the kinetic equation are shown in Fig. 1a, and those for the model with the Caldirola–Kanai Hamiltonian ($\mu = 1.5$, $\gamma t = 0.9$) are shown in Fig. 1b. The marginal distributions are plotted versus variables X and ν, with parameters μ and values of t and γ chosen to be constant. Within the framework of the Caldirola–Kanai model, the dispersion of the marginal probability distribution depends on parameters μ and ν as well as on values of γ and t; in the model with the kinetic equation, the dispersion of the marginal probability distribution is constant with respect to values of γ and t and depends only on parameters μ and ν.

(ii) From formulas (29) and (31) it follows that in the model of quantum damping with the kinetic equation one can observe zeros of the marginal probability distributions (zeros of the Hermite polynomials), but this is not the case for the marginal probability distribution within the framework of the Caldirola–Kanai model. This difference is shown in Fig. 2, where marginal distributions for the first excited state ($n = 1$) of the quantum damped oscillator with fixed parameters ($\mu = 1$, $\gamma t = 0.045$) are plotted for two models — for the model with the kinetic equation (Fig. 2a) and for the model with the Caldirola–Kanai Hamiltonian (Fig. 2b).

(iii) The third important conclusion can be drawn while studying the structure of formulas (29) and (31) from the viewpoint of the dependence on parameters μ and ν. As was mentioned before, the physical meaning of the parameters μ and ν consists of describing an ensemble of rotated and scaled reference frames in the system's phase space, i.e., they describe the reference frame where the position is measured [see formula (1)]. It is easy to see that the marginal probability distribution (29) for the model of quantum damping with the kinetic equation is symmetrical with respect to the parameters μ and ν, but the marginal probability distribution (31) for the Caldirola–Kanai model is asymmetrical with respect to the parameters μ and ν. This means that in symmetrical reference frames, i.e., in reference frames which are determined by symmetrical values of the parameters μ and ν, the marginal probability distributions for the quantum damped oscillator's state for the model with the kinetic equation have the same values in different symmetrical reference frames, but in the framework of the Caldirola–Kanai model, the marginal probability distributions for the quantum damped oscillator's state have different values in different symmetrical reference frames.

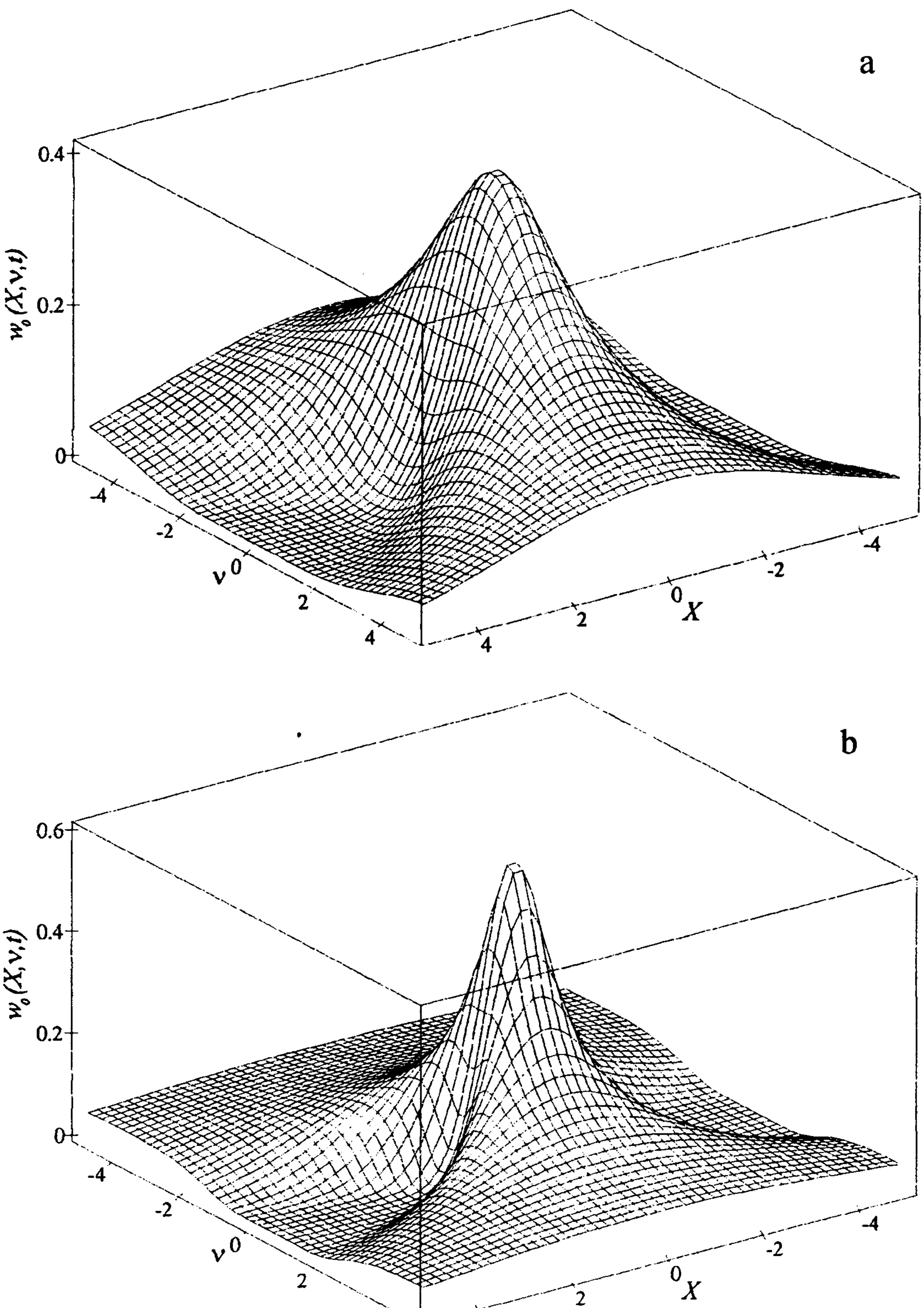

Fig. 1. Marginal probability distributions for two models of quantum damping — the kinetic equation with a collision term (a) and the Caldirola–Kanai model (b) for the ground state ($n = 0$) of the quantum damped oscillator versus variables X and ν, with parameters μ and values of t and γ chosen to be constant ($\mu = 1.5$, $\gamma t = 0.9$).

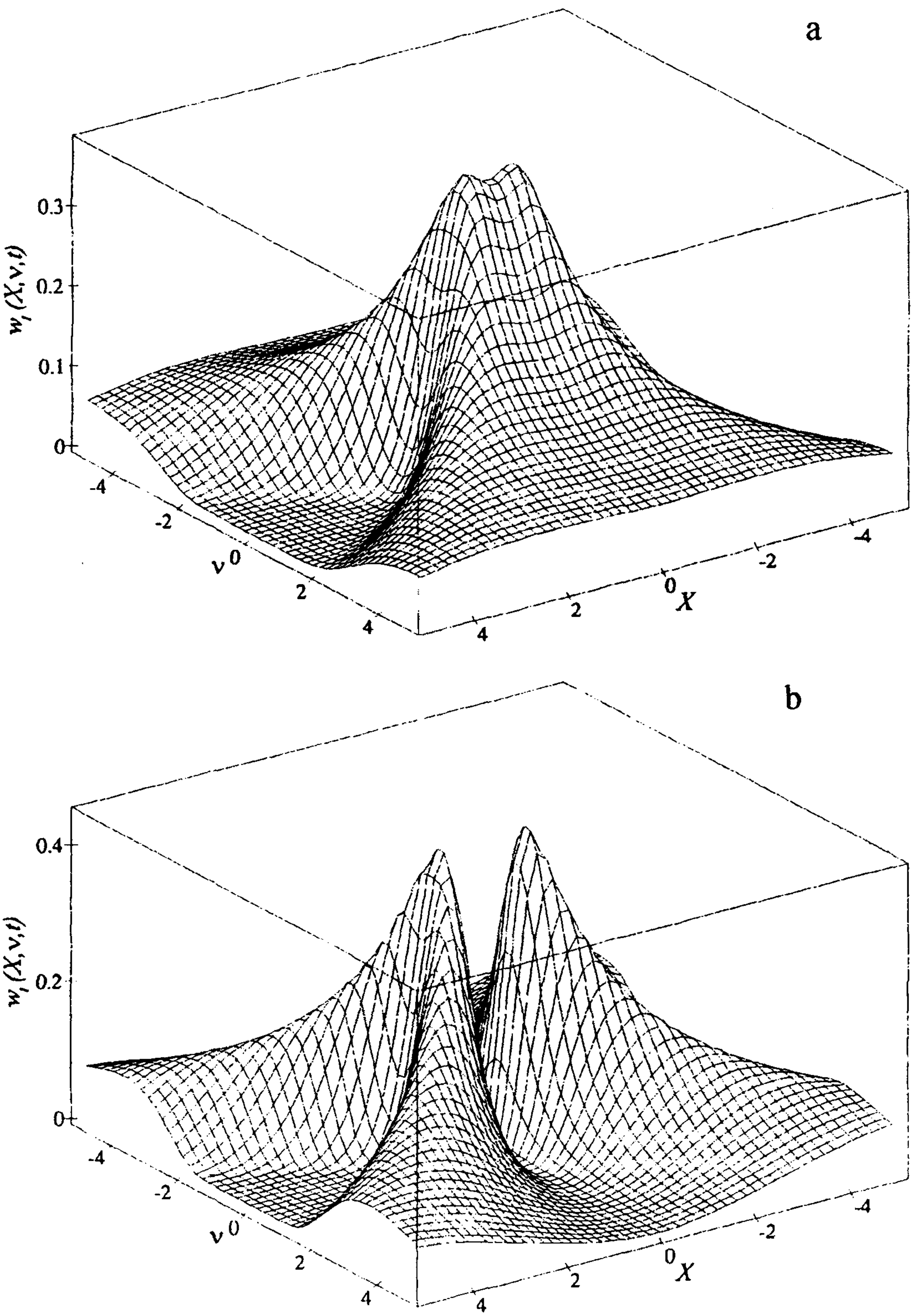

Fig. 2. Marginal probability distributions for two models of quantum damping — the kinetic equation with a collision term (a) and the Caldirola–Kanai model (b) for the first excited state ($n = 1$) of the quantum damped oscillator versus variables X and ν, with parameters μ and values of t and γ chosen to be constant ($\mu = 1.0$, $\gamma t = 0.045$).

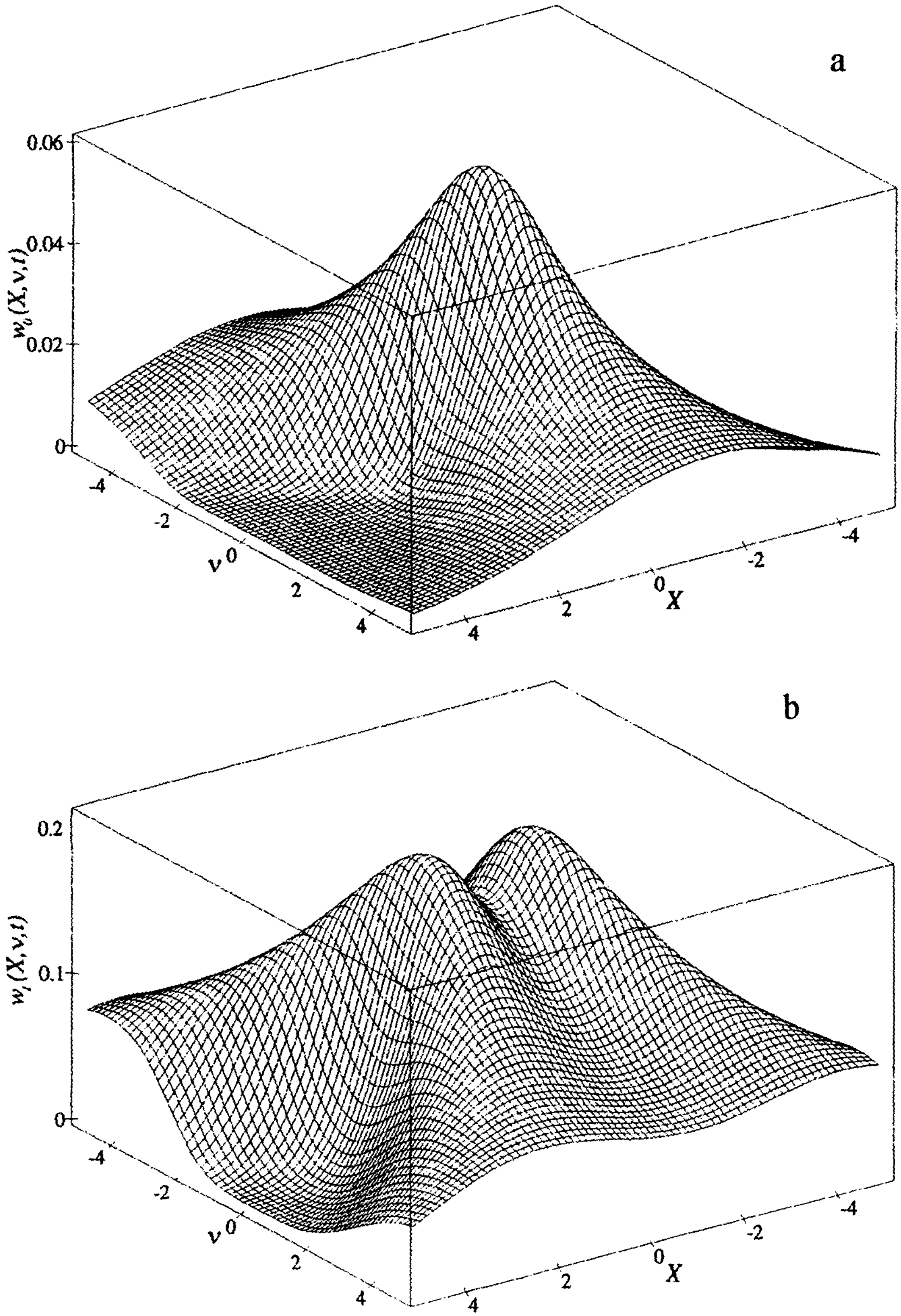

Fig. 3. Marginal probability distributions for the model of quantum damping in the framework of the nonlinear Kostin equation for the ground state ($n = 0$, $\mu = 1.5$, $\gamma t = 0.9$) (a) and for the first excited state ($n = 1$, $\mu = 2.0$, $\gamma t = 0.027$) (b) of the quantum damped oscillator versus variables X and ν.

(iiii) In Fig. 3, the marginal probability distributions for the ground state ($n = 0$, Fig. 3a) and for the first excited state ($n = 1$, Fig. 3b) of the quantum damped oscillator treated in the framework of the Kostin model are plotted. These two plots differ from the corresponding plots shown in Figs. 1 and 2, which is a demonstration of the difference between the three models of quantum damping considered in the paper.

5. Conclusions

In this paper, we showed that within the framework of the classical description of a quantum system it is possible to easily analyze the difference between the models of quantum damping — the model with Caldirola–Kanai Hamiltonian, the model with the kinetic equation with a collision term, and the model elaborating the nonlinear Kostin equation. The analysis shows that statistical properties of the quantum damped oscillator described within the framework of the three models considered differ significantly. The position dispersions and marginal distribution shapes calculated for the different models of the quantum damped oscillator depend on time and the parameters of the reference frames in a different manner. This observation gives the possibility of choosing the appropriate model of quantum damping (friction) by measuring the characteristics of the marginal probability distributions of the oscillator.

The analysis of damping presented in this paper can be extended for quantum systems with several degrees of freedom and also for quantum fields. As follows from the simple example of the one-dimensional oscillator considered in this study, one can expect a substantially different behavior of the damped field quanta depending on whether one uses the nonlinear field equations or the kinetic equation.

We hope that the technique presented can be successfully used for comparing other models of quantum damping.

Acknowledgments

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