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A Study on the External-Noise Input in Weber-Type Gravitational-Wave Antennas.

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Summary. — A very delicate problem in gravitational-wave experiments is that of insulation of the antenna from external, acoustic or seismic, disturbances. The mechanism of excitation in terms of time variable stresses exerted by the suspension on the lateral surface of the antenna itself has been studied and results useful for design purposes obtained. Measurements performed with a small model antenna are in agreement with calculations.

1. – Introduction.

Aim of the new-generation gravitational-wave antennas is to gain enough sensitivity to detect bursts produced in supernova explosions sufficiently far to collect significant samples in an acceptable period of time. To be more specific, candidates are the galaxies of the Virgo Cluster, which, on the basis of current estimates, would ensure a reasonable rate of about 10 events per year. The energy flow of the gravitational-radiation pulse on the Earth would lie in the range $\sim(10^4 \div 1)$ erg/cm^2 and the burst duration between 1 and 0.1 ms. The corresponding change in the amplitude of the vibrations of the antenna is, however, of the order of only $(10^{-17} \div 10^{-19})$ cm.

If we assume peıfect seismic and acoustic insulation, the main difficulties in achieving such sensitivities apparently lie in the construction of the electro-mechanical transducer, *i.e.* the sensor measuring the small displacements of the antenna. The above figures of displacement, however, show also how stringent the requirements for isolation from external disturbances are.

Even if the results recently obtained in Frascati and in Stanford demonstrate that this problem has been succesfully solved, for the present status of sensitivity, we think it worth-while a quantitative discussion of this subject to clarify the situation and to give the basis for future refinements.

The ultimate vehicle of noise energy transfer to the antenna is its suspension itself, of which various types have been devised, like cables, tuning forks, magnetic fields. In what follows we shall then restrict ourselves to the study of the influence of this support on the motions of the antenna, being the rest a mere, even if difficult, question of filtering.

As is well known, the almost totality of resonant antennas presently in use consists of big cylinders with masses ranging from some hundreds to some thousands kg. Only the odd longitudinal modes of vibration are of interest for our purposes. The zone of contact between suspension and antenna lies always on the lateral surface of the cylinder, consequently the stresses at these points will be responsible for the excitation, and what one wants to do is to keep the power flowing through this input below that of thermal fluctuations. A necessary condition for the excitation of a given mode is obviously that the stress has a time-variable component at the frequency of the mode. As will be made clear in the following and is, on the other hand, intuitive, this condition is, however, not sufficient. This remark indicates the possibility of optimizing the suspension.

In the first part of this paper we briefly describe the method of solution and the results of calculations; in the second, experimental results obtained with a small model antenna are reported and compared.

2. – The mechanism of excitation.

We shall proceed to the solution in two separate successive steps: after having determined the motion of the cylinder subjected to generic stresses on the lateral surface and with the ends free, the stresses due to the suspension are evaluated and then introduced in the general solution.

2'1. *Motion of the cylinder.* – Let us consider a homogeneous cylinder of length $2L$, radius a and density ϱ, referred to a cylindrical–co-ordinate system with the origin at the centre of mass and the z-axis along the axis of the cylinder.

The approximate solution for the free vibrations of a cylinder of finite length and in the absence of internal friction was obtained many years ago by POCHHAMMER and, independently, by CHREE [1]. We modify their equations by introducing the dissipation, as we want quantitative results on the excitation of the longitudinal modes caused by external forces. This is easily done by assuming that our viscoelastic body behaves like a Voigt solid, for which, as is known, the stress-strain relations maintain the same form as for a perfectly elastic solid by replacing the Lamè moduli λ and μ with the operators $\lambda + \lambda'(\partial/\partial t)$ and $\mu + \mu'(\partial/\partial t)$, respectively.

For longitudinal vibrations the θ-component, u_θ, of the displacement vanishes and one obtains for the other components, u_r and u_z, the equations

$$(1)\qquad \varrho\frac{\partial^2 u_r}{\partial t^2} = (\lambda + 2\mu)\frac{\partial\Delta}{\partial r} + (\lambda' + 2\mu')\frac{\partial}{\partial t}\frac{\partial\Delta}{\partial r} + 2\mu\frac{\partial\Omega_\theta}{\partial z} + 2\mu'\frac{\partial}{\partial t}\frac{\partial\Omega_\theta}{\partial z},$$

$$(2)\qquad \varrho\frac{\partial^2 u_z}{\partial t^2} = (\lambda + 2\mu)\frac{\partial\Delta}{\partial z} + (\lambda' + 2\mu')\frac{\partial}{\partial t}\frac{\partial\Delta}{\partial z} - \frac{2\mu}{r}\frac{\partial}{\partial r}(r\Omega_\theta) - \frac{2\mu'}{r}\frac{\partial}{\partial t}\frac{\partial}{\partial r}(r\Omega_\theta),$$

where Δ is the dilatation and Ω_θ the θ-component of the rotation

$$(3)\qquad \Delta = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z},$$

$$(4)\qquad \Omega_\theta = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right).$$

The procedure to solve these equations follows, whenever is possible, that of Pochhammer. By imposing a solution of the type

$$(5)\qquad u_r = U(r)\exp[i(\gamma z + \omega t)],$$

$$(6)\qquad u_z = W(r)\exp[i(\gamma z + \omega t)],$$

eqs. (1) and (2) reduce to

$$(1a)\qquad \frac{\partial^2\Delta}{\partial r^2} + \frac{1}{r}\frac{\partial\Delta}{\partial r} + h'^2\Delta = 0,$$

$$(2a)\qquad \frac{\partial^2\Omega_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\Omega_\theta}{\partial r} - \frac{\Omega_\theta}{r^2} + k'^2\Omega_\theta = 0,$$

where

$$(7)\qquad h'^2 = \varrho\omega^2[(\lambda + 2\mu) + i\omega(\lambda' + \mu')]^{-1} - \gamma^2$$

and

$$(8)\qquad k'^2 = \varrho\omega^2[\mu + i\omega\mu']^{-1} - \gamma^2.$$

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[1] See, *e.g.*, H. KOLSKY: *Stress Waves in Solids* (New York, N. Y., 1963).

By taking as the variable $h'r$ in (1a) and $k'r$ in (1b) in place of r, these equations become Bessel equations of order zero and one, respectively, giving the solution

$$\Delta = GJ_0(h'r) , \tag{9}$$

$$\Omega_\theta = HJ_1(k'r) , \tag{10}$$

where G and H are functions of z and t only. From this, and by taking into account (3) and (4), it follows that U and W must be of the form

$$U = \frac{A\partial}{\partial r} J_0(h'r) + C\gamma J_1(k'r) , \tag{11}$$

$$W = Ai\gamma J_0(h'r) + \frac{Ci}{r}\frac{\partial}{\partial r}[rJ_1(k'r)] \tag{12}$$

with A and C constants. A solution which approximately satisfies the conditions for free ends of the cylinder is hence obtained as

$$u_r = \left[A_n \frac{\partial J_0(h'r)}{\partial r} + \gamma_n C_n J_1(k'r)\right] \cos\gamma_n z \exp[i\omega_n t] , \tag{13}$$

$$u_z = [\gamma_n A_n J_0(h'r) + C_n k' J_0(k'r)] \sin\gamma_n z \exp[i\omega_n t] \tag{14}$$

with $\gamma_n = n\pi/2L$. When a is small compared to the wave-length of the vibrations, which is true for the lowest modes of all antennas presently in use, we have also

$$\omega_n = \frac{n\pi}{2L}\sqrt{\frac{E}{\varrho}} , \tag{15}$$

where $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ is Young's modulus.

The constants A_n and C_n are determined by the boundary conditions on the lateral surface, where we assume, without loss of generality, that the stresses $\sigma_{rr}(\theta, z)\exp[i\omega_n t]$ and $\sigma_{rz}(\theta, z)\exp[i\omega_n t]$ are applied. One then obtains the two equations

$$\left\{A_n\left[2\mu\frac{\partial^2 J_0(h'a)}{\partial a^2} - \lambda(h'^2 + \gamma_n^2)J_0(h'a)\right] + 2C_n\mu\gamma_n\frac{\partial}{\partial a}J_1(k'a)\right\} \cdot \cos\gamma_n z = \sigma_{rr}(\theta, z) , \tag{16}$$

$$-\mu\left[2A_n\gamma_n\frac{\partial J_0(h'a)}{\partial a} + C_n(\gamma_n^2 - k'^2)J_1(k'a)\right]\sin\gamma_n z = \sigma_{rz}(\theta, z) , \tag{17}$$

where $\partial(\ )/\partial a$ is used, for brevity, in place of $[\partial(\ )/\partial r]_{r=a}$.

For this system to be solvable, the left- and right-hand members in eqs. (16) and (17) must show the same dependence on the variables θ and z. This means that, to excite the n-th mode, only the components $\sigma_{rr}^{(n)}$ and $\sigma_{rz}^{(n)}$ having the appropriate space dependence in the Fourier expansion of σ_{rr} and σ_{rz}, intended as periodic functions of period 2π with respect to θ and of period $2L$ with respect to z, will be active. Conversely, in their absence the mode will not be excited, even if the corresponding frequency is present, as we had anticipated. These simple remarks may constitute an useful guide for a qualitative inspection on performances of suspensions.

One easily obtains

(18) $$\sigma_{rr}^{(n)} = \frac{1}{2\pi L}\int_{-L}^{+L}\int_{-\pi}^{+\pi}\sigma_{rr}(\theta, z)\cos\gamma_n z\,\mathrm{d}z\,\mathrm{d}\theta\,,$$

(19) $$\sigma_{rz}^{(n)} = \frac{1}{2\pi L}\int_{-L}^{+L}\int_{-\pi}^{+\pi}\sigma_{rz}(\theta, z)\sin\gamma_n z\,\mathrm{d}z\,\mathrm{d}\theta\,,$$

which must replace σ_{rr} and σ_{rz}, respectively, in eqs. (16) and (17). The solution of this system, in the same approximation in which (15) holds, is at the resonance

(20) $$u_r = -\sigma_{rr}^{(n)}\frac{Q}{\omega_n^2}\frac{2\nu^2}{\varrho}\gamma_n^2 r\cos\gamma_n z\exp[i\omega_n t]\,,$$

(21) $$u_z = \sigma_{rr}^{(n)}\frac{Q}{\omega_n^2}\frac{2\nu}{\varrho}\gamma_n\sin\gamma_n z\exp[i\omega_n t]$$

if $\sigma_{rz} = 0$,

(22) $$u_r = \sigma_{rz}^{(n)}\frac{Q}{\omega_n^2}\frac{2\nu}{\varrho}\gamma_n\frac{r}{a}\cos\gamma_n z\exp[i\omega_n t]\,,$$

(23) $$u_z = -\sigma_{rz}^{(n)}\frac{Q}{\omega_n^2}\frac{2}{\varrho}\frac{1}{a}\sin\gamma_n z\exp[i\omega_n t]$$

if $\sigma_{rr} = 0$. A linear combination (phases must be taken into account) of the above results gives the solution in general. Q, a function of λ, μ, λ' and μ', is the merit factor and ν the Poisson ratio. Note that the constant part of the coefficient of $\sigma_{rr}^{(n)}$ in (21) is the same as that of $\sigma_{rz}^{(n)}$ in (22) when $r = a$. This means that in general tangential stresses are much more efficient in exciting the antenna than radial, and must possibly be avoided.

Equations (20) through (23) can be easily extended, in analogy with the harmonic oscillator, to the case in which the stresses are random functions of time. This is formally done by taking the modulus square of both sides on the above equations and by interpreting $|u_r|^2$, $|\sigma_{rr}^{(n)}|^2, \ldots, |\sigma_{rz}^{(n)}|^2$ as power spectral densities.

Multiplication of the resulting right-hand sides by a Lorentzian $[1 + (\omega - \omega_n)^2(2Q/\omega_n)^2]^{-1}$ represents with very good approximation the behaviour of the resonance curve at frequencies ω close to ω_n.

2·2. *The stresses due to suspension.* – We consider the particular case of a cylinder suspended by a cable through its central section. Extension to other cases is straightforward. By eq. (19) σ_{rz} has now no effect and so only σ_{rr} is to be taken into account. One can easily show, considering friction, that the variable tension of the string is completely absorbed in a very small area at the surface of the cylinder, so that we assume the resulting force exerted at a pointlike contact. Such a force is due to both longitudinal and transverse vibrations of the cable and is given by $T\partial\xi/\partial x$, where ξ is the displacement and T the tension, which equals Mg (M = mass of the suspended body) in the transverse case, ES (Young's modulus times the area of the cross-section) in the longitudinal one. It is immediately seen that the ratio of the forces is of the order of the ratio of the corresponding propagation velocities. Consequently, the force due to transverse transmission is 1 or 2 orders of magnitude smaller. We shall then restrict ourselves only to the longitudinal vibrations of the string.

Let us consider a cable, parallel to the x-axis, of lenght l, at one end of which a mass M, much greater than the mass of the cable, is attached, whereas a sinusoidal force of circular frequency ω acts, in the x-direction, at the other end, which we assume as the origin of co-ordinates.

If $\omega \gg \omega_s = \sqrt{ES/Ml}$, $\omega c/\omega_s^2 l \gg \operatorname{ctg}(\omega l/c)$, where c is the velocity of the acoustic wave, the attenuation A is

$$A = \frac{\xi(l)}{\xi(0)} \simeq -\frac{\omega_s^2}{\omega^2}\frac{\omega l/c}{\sin(\omega l/c)} \tag{24}$$

and the force acting on the mass

$$\left[ES\frac{\partial\xi}{\partial x}\right]_{x=l} \simeq \frac{ES\,\xi(0)}{\sin(\omega l/c)}\frac{\omega}{c} = A\omega^2\xi(l)\,, \tag{25}$$

an obvious result, which, however, holds in general, in the sense that the force exerted by a generic suspension can be inferred from the acceleration of the centre of mass of the antenna.

We are now ready the evaluate the stresses in the real case, which is sketched in fig. 1. With reference to this figure, if we assume that the stimuli at points A' and B' are in phase and of the same amplitude, the radial component f_r of the force acting on the cylinder at point A or B, is easily found to be

$$f_r = (M/2)\omega^2\xi(l)\sin\Theta \tag{26}$$

by recalling that the variable part of the tension is now $M\omega^2\xi(l)/2\cos\Theta$.

In general the amplitudes and phases of stimuli at points A' and B' are arbitrary. Taking account of this fact, we obtain from (26), (24) and (18) for

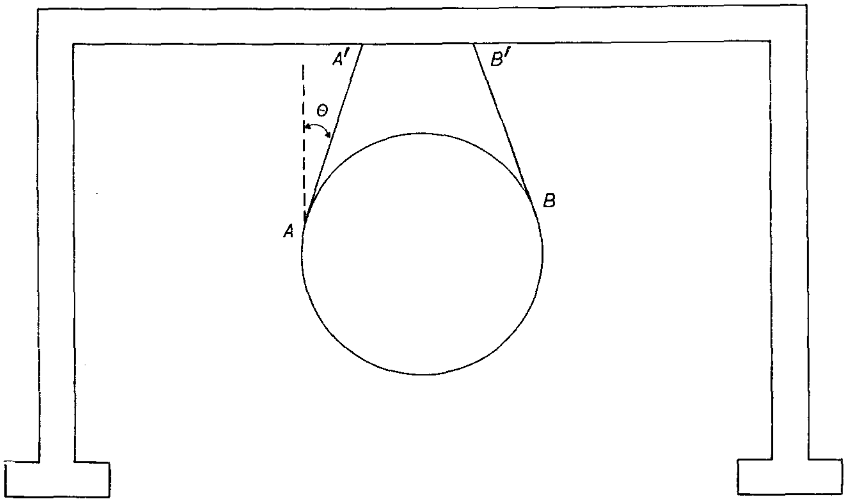

Fig. 1. - Sketch of the experimental set-up.

the stress we are interested in

$$\sigma_{rr}^{(n)} = \frac{1}{4\pi}\frac{\sin\theta}{La}M\omega^2 A\sqrt{S_A^2 + S_B^2 + 2S_A S_B\cos\varphi}\,, \tag{27}$$

where S_A and S_B are the amplitudes of displacement at points A' and B' and φ their phase difference.

Introducing (27) in (20) and (21), we obtain for the excitation of the n-th longitudinal mode of the antenna, at the resonance,

$$u_r = -\frac{1}{4\pi}\frac{\sin\Theta}{La}MA\sqrt{S_A^2 + S_B^2 + 2S_A S_B\cos\varphi}\,Q\frac{2\nu^2}{\varrho}\gamma_n^2 r\cos\gamma_n z\exp[i\omega_n t]\,, \tag{28}$$

$$u_z = \frac{1}{4\pi}\frac{\sin\Theta}{La}MA\sqrt{S_A^2 + S_B^2 + 2S_A S_B\cos\varphi}\,Q\frac{2\nu}{\varrho}\gamma_n\sin\gamma_n z\exp[i\omega_n t] \tag{29}$$

and, for the case stimuli are random in time (see the end of subsect. **2'1**),

$$|u_r(\omega)|^2 = \frac{(MAQ\nu^2\gamma_n^2 r\sin\Theta\cos\gamma_n z)^2}{(2\pi La\varrho)^2[1+(\omega-\omega_n)^2(2Q/\omega_n)^2]}|S(\omega)|^2\,, \tag{28a}$$

$$|u_z(\omega)|^2 = \frac{(MAQ\nu\gamma_n\sin\Theta\sin\gamma_n z)^2}{(2\pi La\varrho)^2[1+(\omega-\omega_n)^2(2Q/\omega_n)^2]}|S(\omega)|^2\,, \tag{29a}$$

where $|u_r|^2$, $|u_z|^2$ are the power spectral densities of displacements and $|S(\omega)|^2$ that of the net stimulus given by

$$|S|^2 = |S_A|^2 + |S_B|^2 + S_{AB} + S_{BA}\,,$$

S_{AB} and S_{BA} are cross spectra of stimulus A and stimulus B.

3. – Experimental results.

We have checked the previous results by performing measurements with a small model antenna of aluminium having $M = 16$ kg, $2L = 30$ cm, $a = 8$ cm.

The cylinder was suspended inside a vacuum chamber as sketched in fig. 1 at a rigid frame to which a small shaker imparted random vibrations in an interval of 5 Hz around the resonant frequency of the first longitudinal mode, which was approximately at 8443 Hz. In this way the full resonance curve was measured at one time. As transducers three Bruel and Kjaer accelerometers were used. The first two, fixed at points A' and B' of suspension of the system, measured the input noise, the third one was glued on the lateral surface at $z = 0$, allowing so us to determine the radial displacement u_r.

What in fact we measured was the transfer function $\mathscr{T}$ of this system, defined as $\mathscr{T} = \sqrt{|u_r|^2/|S|^2}$, that, with reference to eq. (28a), gives at resonance

$$\mathscr{T}(\omega_1) = \frac{MA}{4\pi La}\,Q\cdot 7.5\cdot 10^{-3}\sin\Theta\,. \tag{30}$$

In the present case $Q = 3.4\cdot 10^4$, $A = 7\cdot 10^{-5}$, so that

$$\mathscr{T}(\omega_1) = 0.19\,\sin\Theta\,. \tag{31}$$

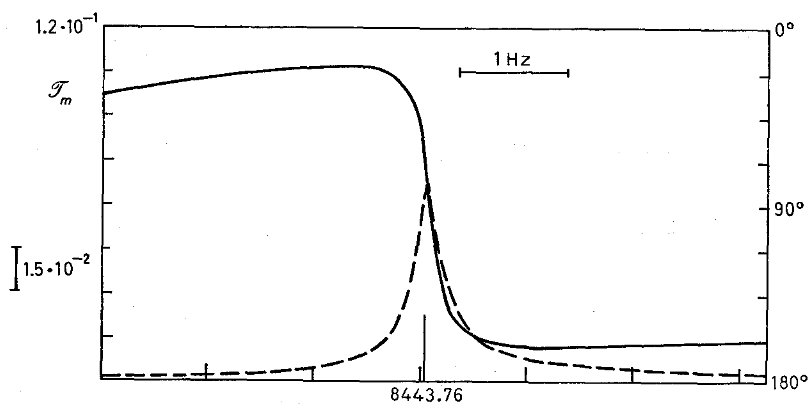

Fig. 2. – Measured transfer function (dashed line) for the case $\Theta = 20°$. The continuous line (scale on the right) represents the phase difference between stimulus and excitation of the antenna.

Measurements of $\mathcal{T}$ were performed at three different values of Θ(2°, 10°, 20°) by comparing outputs of the three accelerometers by means of a power spectrum analyser. An internal consistency test was obtained by checking that the phase difference between the resultant stimulus, as given by accelerometers A' and B', and the displacement of the antenna was as requested around resonance. Figure 2 gives an example of the curves we obtained. The results at the frequency of resonance are reported in table I ($\mathcal{T}_m$) and compared with the calculations ($\mathcal{T}(\omega_1)$) given by (31).

TABLE I.

Θ	$\mathcal{T}_m$	$\mathcal{T}(\omega_1)$
2°	$(5.4 \pm 1) \cdot 10^{-3}$	$6.6 \cdot 10^{-3}$
10°	$(2.8 \pm 0.5) \cdot 10^{-2}$	$3.2 \cdot 10^{-2}$
20°	$(6.4 \pm 0.6) \cdot 10^{-2}$	$6.5 \cdot 10^{-2}$

As can be seen, the agreement is satisfactory.

4. – Conclusions.

In the previous calculations no account was taken of the action on the suspension of the oscillations of the cylinder. This effect is very small and disregarding it does not affect the results for our purposes, whereas it may be of importance to determine the influence of suspensions on other parameters of the antenna, like the change in the frequency of resonance and in the Q. This has been done by FUJIMOTO (2) with a different method based on the development of the important quantities in eigenfunctions of the cylinder. The results FUJIMOTO obtains for the excitation of the antenna are very similar to ours.

As we have already pointed out, the results for the particular case of the suspension we have considered can be in a very simple way generalized to other types. Thus for the tuning fork, proposed by RICHARD (3), a radial force similar to that given by eq. (26) is obtained. To this a tangential stress is to be added. The magnetic suspension will also produce a radial force of the same type as above.

From the previous discussion it is apparent that, from the point of view of minimizing the noise input, the cable suspension at the central section is perhaps the best one. In fact, it does not apply tangential stresses and at the same time the radial ones can be further reduced by making Θ very small,

(2) M. K. FUJIMOTO: private communication.
(3) J. P. RICHARD: *Rev. Sci. Instrum.*, **47**, 423 (1976).

so that at last only transversal transmission will be important. This is, of course, impossible with the other types of suspension as, in these cases, Θ will always have a rather large value. In addition, one must also consider their ability to exert tangential stresses.

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● RIASSUNTO

Negli esperimenti sulle onde gravitazionali l'isolamento dell'antenna dai disturbi esterni, acustici e sismici, costituisce un problema molto delicato. Si è studiato il meccanismo di eccitazione dell'antenna in termini degli sforzi variabili esercitati sulla sua superficie laterale dalla sospensione ottenendo risultati utili per la progettazione. Tali risultati sono in accordo con le misure eseguite su un'antenna di prova.

Исследование внешнего шума на входе в антеннах гравитационных волн типа Вебера.

Резюме (*). — Самая чувствительная проблема в экспериментах с гравитационными волнами состоит в изоляции антенны от внешних, акустических или сейсмических возмущений. Исследуется механизм возбуждения в терминах изменяемых во времени напряжений, обусловленных подвесной системой, на боковой поверхности антенны. Полученные результаты оказываются полезными при конструировании. Измерения, проведенные с малой модельной антенной, согласуются с вычислениями.

(*) *Переведено редакцией.*