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QUANTITATIVE METHODOLOGY FOR STABILITY ANALYSIS OF NONLINEAR ROTOR SYSTEMS *

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Abstract: Rotor-bearings systems applied widely in industry are nonlinear dynamic systems of multi-degree-of-freedom. Modern concepts on design and maintenance call for quantitative stability analysis. Using trajectory based stability-preserving and dimensional-reduction, a quantitative stability analysis method for rotor systems is presented. At first, an n-dimensional nonlinear non-autonomous rotor system is decoupled into n subsystems after numerical integration. Each of them has only one-degree-of-freedom and contains time-varying parameters to represent all other state variables. In this way, n-dimensional trajectory is mapped into a set of one-dimensional trajectories. Dynamic central point (DCP) of a subsystem is then defined on the extended phase plane, namely, force-position plane. Characteristics of curves on the extended phase plane and the DCP's kinetic energy difference sequence for general motion in rotor systems are studied. The corresponding stability margins of trajectory are evaluated quantitatively. By means of the margin and its sensitivity analysis, the critical parameters of the period doubling bifurcation and the Hopf bifurcation in a flexible rotor supported by two short journal bearings with nonlinear suspensionare are determined.

Key words: nonlinear rotor system; bifurcation; stability margin; extended phase plane; dynamic central point; kinetic energy difference sequence

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Introduction

Rotor-bearings systems applied widely in industry are nonlinear dynamic systems of

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multi-degree-of-freedom. Synchronous vibration is its typical motion under unavoidable unbalance. Subharmonic, quasi-periodic and chaotic vibrations, caused by the loss of stability of synchronous motion in nonlinear rotor systems, are found in numerical analysis and practice. Therefore, stability study of periodic solution is of importance in both theory and practice.

Conventional analysis of periodic solution is based on approximate quantitative methods, such as KBM method, multiple scale method, average method and perturbation method. A strong point of these methods is that a general picture about the property of solution and its relationship with certain parameter can be got with a little amount of calculation. But their application is restricted to weak nonlinear and low dimensional systems. Numerical methods for analyzing the stability of periodic solution can be divided into two categories. One is to transfer the problem of ordinary differential equations (ODE) to that of nonlinear algebraic equations, such as Garlerkin method and Chebyshev method. The other is the numerical integration of ODE, whose representatives are shooting method and cell-to-cell mapping theory. Although above methods based on Floquet theory can be used to analyze the stability of periodic solution and determine bifurcation value, they need a large amount of calculation without margin concept. Moreover, it is difficult to calculate Floquet multiplier in engineering.

Modern design concepts for machines emphasize reliability and stability in addition to their functions, and the maintenance occasion changes from breakdown and fixed time period into state maintenance. This calls for quantitative stability analysis. In this paper, a method predicting bifurcation in nonlinear rotor systems using trajectory-based stability-preserving dimensional-reduction (TSPDR) is presented, and its application in dynamic behavior analysis of an unbalanced rotor system with two lubricated bearings is described.

1 TSPDR Method

TSPDR[1] is a linear transformation that maps the existing trajectory from R^n onto a set of R^2 or R^1, and ensures that the necessary and sufficient condition for stability is rigorously kept in one of the images. The R^1 observing space is separated from R^n integrating spaces via TSPDR transformation and a logical solution is to simulate disturbed trajectories in R^n and abstract quantitative information about dynamics in R^1.

For analyzing bounded stability, the complementary cluster center of inertia transformation ($R^n \rightarrow R^1$: CCCOI) and the complementary cluster energy-barrier criterion (CCEBC)[2] were proposed. The fully commercialized software package based on CCEBC, is the unique quantitative tool for assessing power system transient stability till now and has been widely used by utilities in China, France, Canada and USA for system planning, operation, as well as for preventive and emergency controls.

For analyzing bifurcation and nonlinear modes of absolute motion in individual dimensions, such as in rotor systems, coordinate-axis projection (CAP) method was proposed. It uses a unit matrix to realize the TSPDR transformation from R^n to R^1. Numerical integration simulates trajectories in R^n, then each image trajectory, namely, the trajectory with one-degree-of-freedom, is investigated individually, and quantitative information on the dynamics is abstracted from phase plane or the extended phase plane. Therefore, the task of studying bifurcation in higher-dimensional nonlinear rotor systems is transformed to that of investigating individual trajectory and abstracting quantitative information in one-dimensional space. In practice, only the trajectory in the concerned dimensions, not every individual one, is used to analyze the dynamic behavior in nonlinear rotor systems.

TSPDR transforms various difficulties, such as dealing with DAE, nonlinear factors, non-autonomous factors and dimensional problems, into a sole task, namely how to extract the dynamic information from trajectory in R^1 (or R^2). It possesses the advantages of both numerical integration method and (extended) phase plane method, and upgrades the conventional stability methods from heuristic qualitative study to rigorous quantitative study. Moreover, TSPDR can be used to assess the security of algebraic variables during the dynamics.

2 Characteristics of Nonlinear Rotor Systems in R^1

2.1 Kinetic energy difference sequence of dynamic central point

Dynamic central point (DCP) is the point through the horizontal axis during a bounded swing on force-position plane in R^1, such as O_1, O_2, O_3 and O_4 shown in Figs. 1(a) and 1(b), which are the points with extremum of kinetic energy in each swing[3]. DCP's kinetic energy sequence is composed of the value of kinetic energy of DCP at the same direction in turns. The kinetic energy of point O_1, O_2, O_3 and O_4 are denoted by $K(O_1)$, $K(O_2)$, $K(O_3)$ and $K(O_4)$, respectively. The absolute value of difference between each two elements in turns in DCP's kinetic energy series forms DCP's kinetic energy difference (KED) sequence, which is denoted by K_{13} and K_{24}.

2.2 Characteristics of nonlinear rotor systems in R^1

The general motions in nonlinear rotor systems are synchronous motion, period doubling and quasi-periodic vibration. The characteristics of trajectories and DCP's KED sequences of them in R^1 are shown in Fig. 1 to Fig. 3.

For different steady motions, there are two kinds of transient KED sequences. For synchronous vibration as steady motion, KED reaches the peak after a little period, then descends either in turns or in the way of fluctuation, at last it equals zero, shown in Figs. 3(a) and 3(d). For period doubling motion, one way is that it reaches the peak after a while, then descents until to a constant. The other way is in reverse order, that is, it reaches minimum value, and then increases until to a constant, shown in Figs. 3(b) and 3(e). For quasi-periodic motion, the KED sequence differs from that of period doubling one in steady state. The former is a fluctuating sequence while the latter is a constant, shown in Figs. 3(c) and 3(f).

Characteristic parameters of KED sequence for different steady motion are listed in Table 1. They are used to estimate if the vibration is steady and determine the type of trajectory.

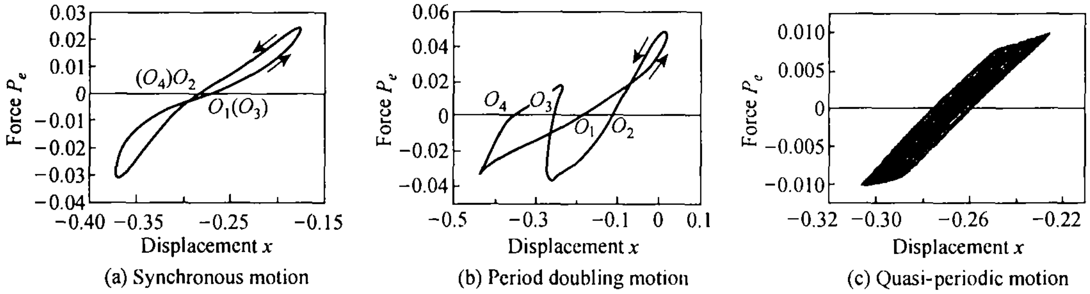

Fig. 1 Trajectories on extended phase plane

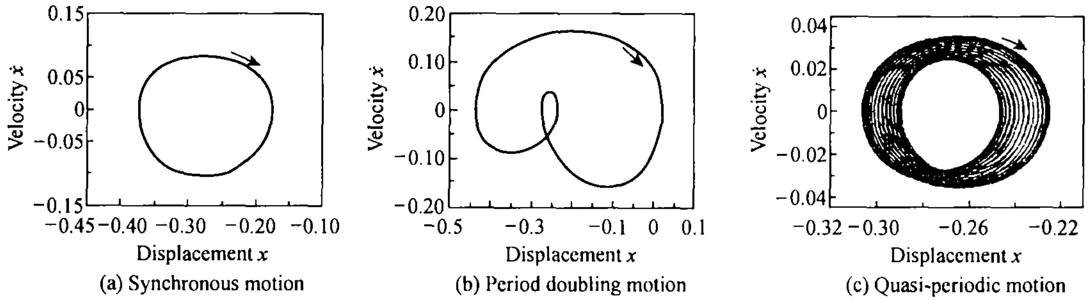

Fig. 2 Trajectories on phase plane

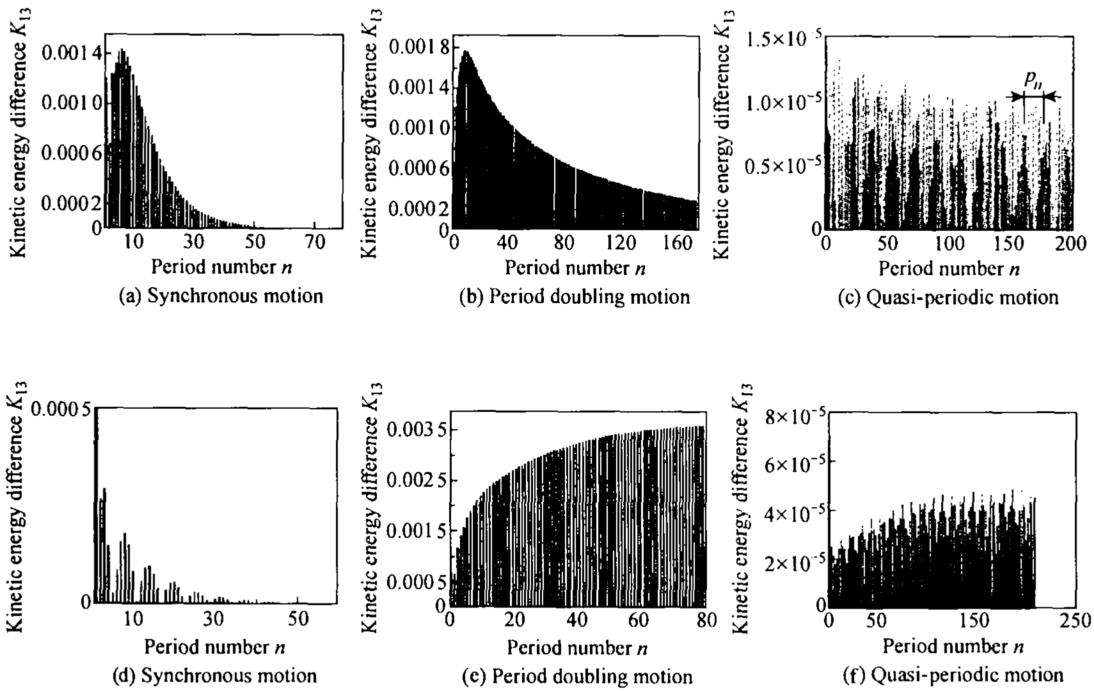

Fig. 3 KED sequence

Table 1 Characteristic parameters of KED sequence

Motions	K_{13}	ΔK_{13}	$E(K_{13})$	$D(K_{13})$	$\Delta E(K_{13})$
Synchronous	0	0	0	0	0
Period doubling	C_1	0	C_1	0	0
Quasi-Periodic	QW	QW	C_2	C_3	0

Notes: ΔK_{13} is the absolute value of difference of each two elements in K_{13} sequence; $E(K_{13})$ and $D(K_{13})$ are the mean value and unbiased variance of a sample in K_{13}, respectively; $\Delta E(K_{13})$ denotes the variety of $E(K_{13})$ series; C_1, C_2 and C_3 are the constants; QW means quasi-periodic wave.

3 Stability Margins

3.1 Stability margin of synchronous motion

Stability margin of synchronous motion is defined by the damped exponential of KED sequence, which is an index of systematic dampness. The larger the damped exponential is, the bigger the systematic dampness is, and the more stable the system is. At first, one should collect the transient KED in order to calculate the damped exponential and then determine the envelope of it. Using the following function to fit the envelop:

$$z = a\mathrm{e}^{-bn}, \tag{1}$$

where n is the period number, z the value of KED and b the damped exponential. As a result the stability margin for synchronous vibration is defined as follows:

$$\eta = \min(b_{K_{13}}, b_{K_{24}}), \tag{2}$$

where $b_{K_{13}}$ and $b_{K_{24}}$ are the damped exponentials of K_{13} and K_{24} series, respectively.

3.2 Stability margin of period doubling motion

With the variation in controlling parameter, synchronous vibration becomes period doubling one via period doubling bifurcation in nonlinear rotor systems. A distinct trait of it is that the value of steady KED is a constant C_1, which equals the absolute value of difference between the largest energy of the longest swing and that of the same direction swing in a period. It denotes the amount of low frequency element and the severity degree of vibration. So stability margin for period doubling vibration is defined as follows:

$$\eta = -C_1. \tag{3}$$

The bigger C_1 is, the smaller η is, which means the larger amount of sub-synchronous element and the severer vibration, *i. e.*, the worse stability of the system. When η equals zero, the system runs in the critical state.

3.3 Stability margin of quasi-periodic motion

The nonlinear rotor system may occur Hopf bifurcation with the variation in controlling parameter, and synchronous vibration becomes quasi-periodic one. Due to "lock model", N periodic motion may appear on occasion. Quasi-periodic motion has two irrationally related frequencies. Trajectory shows movement of a point with extreme period in a band picture on the extended plane and in a limit ring on phase plane, shown in Fig. 1(c) and Fig. 2(c), respectively. It results in KED sequence's fluctuation in steady state, shown in Figs. 3(c) and 3(f). N period motion differs from quasi-period one in rigorous period. It is a close curve with period NT on phase plane and the KED sequence fluctuates in period N.

Because of the non-periodic behavior for KED sequence of quasi-periodic motion, one must collect the right sample length p_n at first, which equals the length of KED sequence with a (or several) near period shown in Fig. 3(c). Abstracting each one with p_n elements from KED sequence in turns forms the sample series. The mean value $E(K_{13})$ and unbiased variance $D(K_{13})$ of each sample consist of mean value and unbiased variance series, respectively. The variation of mean value series can be expressed:

$$\Delta E(K_{13}) = \frac{|E_{i+1}(K_{13}) - E_i(K_{13})|}{E_i(K_{13})}, \tag{4}$$

where $E_{i+1}(K_{13})$ is next to $E_i(K_{13})$ in mean value series.

Non-periodic degree of trajectory for quasi-periodic motion can be expressed by the value of $E(K_{13})$ in steady state, which denotes the average variation of DCP's KED. The fluctuating degree of DCP's KED can be expressed by the value of $D(K_{13})$. Therefore, stability margin for quasi-periodic motion has two items:

$$\eta = (\eta_p, \eta_{pd}), \quad \eta_p = -E(K_{13}), \quad \eta_{pd} = -D(K_{13}), \tag{5}$$

where η_p is used to determine the critical parameter between synchronous and quasi-periodic motion, while η_{pd} between period doubling and quasi-periodic motion. The smaller η is, the worse the stability of the system is. When η equals zero the system runs in the critical state.

4 Prediction of Bifurcation in Nonlinear Rotor Systems

4.1 Sensitivity analysis based on stability margins

Stability margin for the system under critical parameters equals zero. Therefore, predicting bifurcation is to determine the value of controlling parameter with zero stability margins. Stability margin of the system varies with the controlling parameter. The influence of a tiny increase (or decrease) of parameter α on the stability margin η can be expressed by the sensitivity coefficients η for α. However complex the system is, the sensitivity coefficients for any parameter concerned can be calculated at all time by numerical perturbation method.

The stability margin $\eta(\alpha_0)$ for the initial parameter α_0 is known, then the $\eta(\alpha_0 + \Delta\alpha)$ can be calculated according to the perturbation of parameter $\Delta\alpha$. So the sensitivity coefficient η for α in one order is

$$S_\alpha = (\eta(\alpha_0 + \Delta\alpha) - \eta(\alpha_0))/\Delta\alpha. \tag{6}$$

Based on the hypothesis of linearity one can obtain the critical parameter as follows:

$$\alpha_{\lim} = \alpha_0 - \eta_0/S_\alpha. \tag{7}$$

Different motion with its own definition of stability margin results in a combination of sensitivity analysis and dichotomy to predict bifurcation in practice.

4.2 Examples

A flexible rotor supported by two oil film journal bearings in parallel with nonlinear springs is shown in Fig. 4. O_m is the center of rotor gravity; O_1, O_2, O_3 are the geometric centers of the bearing, the rotor and the journal, respectively; m is the mass of the rotor; ρ is the mass eccentricity of the rotor; K_p is the stiffness of the shaft; ϕ is the rotational angle; ω is the rotational speed of the shaft; m_0 is the mass of the bearing housing; K_1 and K_2 are the stiffnesses of the springs supporting the bearing housings; C_1 is the damping coefficient of the supporting structure; C_2 is the viscous damping of the rotor disk; R is the inner radius of the bearing housing, and r is the radius of the shaft. Figure 5 shows the cross section of the oil film journal bearing, where (X,Y) is the fixed coordinate and (e,φ) is the rotational coordinate, e the offset of the journal center and φ the attitude angle of X-coordinate. Dimensionless equations of the O_1, O_2, O_3 are as follows[4]:

$$\begin{cases} y''_1 + \frac{2\xi_1}{a_1}y'_1 + \frac{1}{a_1^2}y_1 + \frac{\alpha}{a^2}y_1^3 - \frac{1}{2C_{O_m}a^2}(y_2 - y_1 - \varepsilon\sin\varphi) + \frac{f}{a^2} = 0, \\ x''_1 + \frac{2\xi_1}{a_1}x'_1 + \frac{1}{a_1^2}x_1 + \frac{\alpha}{a^2}x_1^3 - \frac{1}{2C_{O_m}a^2}(x_2 - x_1 - \varepsilon\cos\varphi) = 0, \\ \varepsilon' = \frac{1}{(1/2)\pi^2(1+2\varepsilon^2) - 8\varepsilon^2}(\pi b(1-\varepsilon^2)^{5/2}((x_2 - x_1)\cos\varphi \\ \quad + (y_2 - y_1)\sin\varphi - \varepsilon) - 4\varepsilon b(1-\varepsilon^2)^2((x_2 - x_1)\sin\varphi - (y_2 - y_1)\cos\varphi)), \\ \varphi' = \frac{1}{2} - \frac{1}{2\varepsilon(1-\varepsilon^2)^{1/2}(\varepsilon^2 - (1/16)\pi^2(1+2\varepsilon^2))}(b\varepsilon(1-\varepsilon^2)^{5/2} \\ \quad \times ((x_2 - x_1)\cos\varphi + (y_2 - y_1)\sin\varphi - \varepsilon) - \frac{1}{4}\pi b(1+2\varepsilon^2)(1-\varepsilon^2)^2 \\ \quad \times ((x_2 - x_1)\sin\varphi - (y_2 - y_1)\cos\varphi)), \\ x''_2 + (2\xi_2/a)x'_2 + (1/a^2)(x_2 - x_1 - \varepsilon\cos\varphi) = \beta\cos\phi, \\ y''_2 + (2\xi_2/a)y'_2 + (1/a^2)(y_2 - y_1 - \varepsilon\sin\varphi) = \beta\sin\phi - f/a^2, \end{cases} \quad (8)$$

where "′" represents $\frac{d}{d\phi}$, $b = \frac{\Lambda}{a\sqrt{f}}$, $a_1^2 = C_{O_m}C_{p1}a^2$.

For the dimensionless parameters $\xi_1 = 0.01$, $\xi_2 = 0.02$, $f = 0.3$, $C_{p1} = 2.0$, $C_{O_m} = 0.2$, $\alpha = 1.0$, $\Lambda = 0.1$, $\beta = 0.05$, use the dimensionless parameter a as control parameter to predict the bifurcation value in x_1 observation space. When initial value a_0 is 0.2 and an increment Δa is 0.1, the critical value ($a_c = 0.6143$) is found through six steps, at which

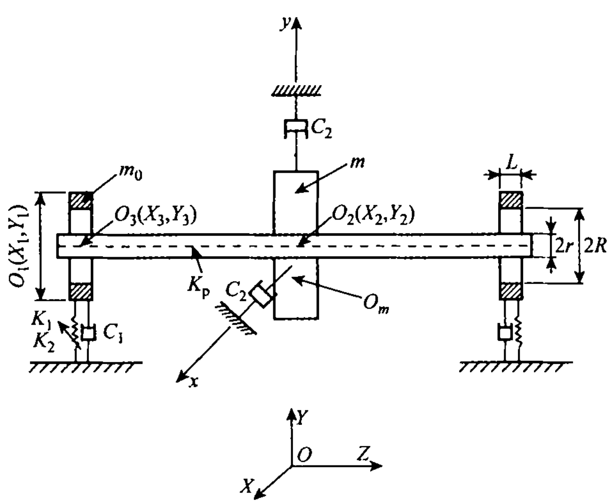

Fig. 4 Model of a flexible rotor supported by two oil journal bearings

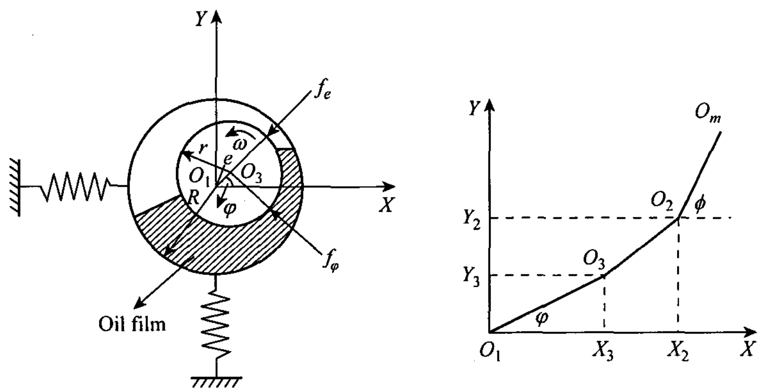

Fig. 5 Cross section of an oil journal bearing

period doubling bifurcation occurs in the rotor systems. The direct numerical calculation shows that critical parameter is 0.65[4].

For the dimensionless parameters $\xi_1 = 0.01$, $f = 0.3$, $C_{p1} = 2.0$, $C_{O_m} = 0.2$, $\alpha = 1.0$, $\Lambda = 0.1$, $\beta = 0.05$, $a = 2.0$, use the dimensionless parameter ξ_2 as control parameter to predict the bifurcation value in x_1 observation space. When initial value ξ_{20} is 0.02 and an increment $\Delta\xi_2$ is 0.02, the critical value ($\xi_{2c} = 0.0439$) is found through four steps, at which Hopf bifurcation occurs in the rotor systems. The direct numerical calculation shows that critical parameter is 0.05[4]. As a result, the efficiency of the above method is proved.

5 Conclusions

1) Sensitivity analysis based on stability margins is much faster than numerical calculation in a trial-and-error way in determining bifurcation value. It makes stability study of rotor systems upgrade from qualitative analysis to quantitative one.

2) Based on trajectories and a characteristic of this method, it has a wide application in high-dimensional nonlinear rotor systems and a prospect future in engineering.

References:

[1] Xue Yusheng. The stability-preserving trajectory-reduction methodology for analyzing nonlinear dynamics[A]. *Keynote in International Conferences on Info-tech & Info-net*[C]. IEEE Catalog Number: 01EX479, ICII2001, Beijing, 2001, 7 – 19.

[2] Xue Yusheng. *Quantitative Study of General Motion Stability and an Example on Power System Stability*[M]. Jiangsu Press of Science and Technology, Nanjing, 1999, 1 – 65 (in Chinese).

[3] Zheng Huiping, Xue Yusheng, Chen Yushu. Estimation of stability margin of parameter of nonlinear rotor system based on trajectory[J]. *Journal of Mechanical Strength*, 2002, **24**(2): 185 – 189 (in Chinese).

[4] Zhong Yi'e, He Yanzong. *Rotor Dynamics*[M]. Press of Tsinghua University, Beijing, 1987, 41 – 63 (in Chinese).

[5] Zheng Huiping. *Study on Nonlinear Dynamic Stability and Stability Margin of Unbalanced Rotor Systems with Lubricated Bearings*[D]. Ph D Dissertation, Tianjin University, Tianjin, 2000, 31 – 46, 74 – 84 (in Chinese).