GKZ-Generalized Hypergeometric Systems in Mirror Symmetry of Calabi–Yau Hypersurfaces

S. Hosono[1], B.H. Lian[2], S.-T. Yau[3]

[1] Department of Mathematics, Toyama University, Toyama 930, Japan.
E-mail: hosono@sci.toyama-u.ac.jp
[2] Department of Mathematics, Brandeis University, Waltham, MA 02154, USA.
E-mail: lian@max.math.brandeis.edu
[3] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA.
E-mail: yau@math.harvard.edu

Abstract: We present a detailed study of the generalized hypergeometric system introduced by Gel'fand, Kapranov and Zelevinski (GKZ-hypergeometric system) in the context of toric geometry. GKZ systems arise naturally in the moduli theory of Calabi–Yau toric varieties, and play an important role in applications of the mirror symmetry. We find that the Gröbner basis for the so-called toric ideal determines a finite set of differential operators for the local solutions of the GKZ system. At the special point called the large radius limit, we find a close relationship between the principal parts of the operators in the GKZ system and the intersection ring of a toric variety. As applications, we analyze general three dimensional hypersurfaces of Fermat and non-Fermat types with Hodge numbers up to $h^{1,1} = 3$. We also find and analyze several non-Landau–Ginzburg models which are related to singular models.

1. Introduction

Recent studies on nonperturbative aspects of string theory have made remarkable progress in understanding the structure of moduli spaces in string theory. Applications of mirror symmetry, for example, in type II string compactification to studying the geometry of moduli spaces is one of the most successful developments. Starting from the pioneering work by Candelas et al. [1], and subsequently by others, the quantum geometry of the moduli spaces for many Calabi–Yau models [2–11] have now been well understood via mirror symmetry. At the same time, there is parallel progress in studying the axiomatic framework of quantum geometry and its application to enumerative geometry [12]. Also in explicit constructions of the geometry of concrete Calabi–Yau models, it is now understood that for a large class of Calabi–Yau varieties, the mirror maps have remarkable modular and integrality properties [13–15]. These models present strong and even beautiful evidence for the recent proposal for the so-called type II-heterotic string duality [16]. These Calabi–Yau models continue to provide fruitful testing ground for string duality [17].

   Mirror symmetry was first recognized in the local operator algebra of the $N=2$ string theory [18]. Soon after the introduction of the framework of toric geometry into the study of Calabi–Yau models [19, 20], mirror symmetry has since been widely checked for many Calabi–Yau hypersurfaces and complete intersections in toric varieties. Mirror symmetry relates two moduli spaces with apparently very different properties – one moduli space is described by purely classical geometry, while the other is described by quantum geometry which receives nonperturbative corrections from the worldsheet instanton [21]. Mirror symmetry thus gives us a powerful means for studying quantum geometry of one moduli space via classical means such as the theory of variation of Hodge structures.

   Variation of Hodge structures allows us to study the period integrals for Calabi–Yau varieties. It is known that the period integrals satisfy differential equations with regular singularities, known as Picard–Fuchs differential equations. A general technique for constructing Picard–Fuchs equations is the reduction method of Dwork–Griffiths–Katz. For Calabi–Yau toric varieties, it was remarked in [22] that the period integrals satisfy a generalized hypergeometric system introduced by Gel'fand–Kapranov–Zelevinski [23]. It has been observed [8] in solving several examples that the GKZ system is not generic and is reducible. Moreover there is an irreducible part in which the period integrals live. In this paper we study the GKZ hypergeometric system for general Calabi–Yau hypersurfaces, and discuss the previous observations in a different light but with much greater generality. As applications, we determine the Picard–Fuchs differential equations for all hypersurfaces with Hodge numbers $h^{1,1} \leq 3$ in weighted projected spaces.

   In Sect. 1, we review the toric description of mirror symmetry, due to Batyrev. We introduce period integrals in the language of toric geometry, and introduce a GKZ system which we call $\varDelta^*$-hypergeometric system. The system is extended by incorporating the symmetry coming from the automorphism group of the ambient space [8]. We classify according to the toric data [8] Calabi–Yau hypersurfaces into three classes: types I, II and III.

   In Sect. 2, we analyze local solutions to the $\varDelta^*$-hypergeometric system. We construct a finite set of differential operators for local solutions by relating the system to an algebro-combinatorial object, known as a toric ideal. We find that the local properties near the so-called large radius limit are determined completely by the intersection ring of the ambient space. In the case of type I and type II models, we prove in general the existence of the large radius limit, hence establish the existence of the point of maximally unipotent monodromy. We give a natural explanation for the reducibility of our $\varDelta^*$-hypergeometric system in terms of certain aspects of the intersection ring of the ambient space. We also extend our arguments to type III models.

   In Sect. 3, we will apply our general framework to three dimensional Calabi–Yau hypersurfaces with $h^{1,1} \leq 3$. Detailed analyses are given for a few typical models. For others, we will append a list of the Picard–Fuchs equations to the source file of this article [24] for interested readers.

   In the final section we will discuss some relationships among different Calabi–Yau manifolds which come from the inclusion relations among reflexive polyhedra.

2. Toric Geometry and Generalized Hypergeometric Differential Equation

In this section we analyze the differential equations, known as Picard–Fuchs equations, satisfied by the periods of a toric variety. Applications of toric geometry to the description of the Picard–Fuchs equation was first initiated in [22] and further developed in [8]. Here we summarize some of the analyses in [8] and extract some combinatorial aspects of the Picard–Fuchs equations.

2.1. A construction of mirror manifolds. In order to fix some notations, we review Batyrev's construction of the mirror manifolds, which is applicable to the list of 7,555 hypersurfaces of [25, 26] as well as complete intersections [27] in a product of (weighted) projective spaces. In the following we restrict our attention to hypersurfaces, although generalization to complete intersections [28, 29] can be done.

Let us consider a weighted projective space $\mathbf{P}^n(w)$ and a hypersurface $X_d(w)$ with (weighted) homogeneous degree $d = w_1 + \cdots + w_{n+1}$. Without loss of generality, we may assume that the weight w is normalized [30], i.e., $gcd(w_1, \ldots, \hat{w}_i, \ldots, w_{n+1}) = 1, (i = 1, \ldots, n + 1)$. (See also [31].) For $n = 4$, the list of [25, 26] exhausts all hypersurfaces $X_d(w)$ defined by weighted homogeneous polynomials satisfying the transversality condition. Now let

$$W(z) = \sum_{(w, m)=d} a_m z^m = \sum_{(w, m)=d} a_{m_1,\ldots,m_{n+1}} z_1^{m_1} \cdots z_{n+1}^{m_{n+1}} . \qquad (2.1)$$

For generic a_m, the zero locus $\{W(z) = 0\}$ defines a hypersurface $X_d(w)$ in general position. Its intersection with singular locus of the ambient space $\mathbf{P}(w)$ gives the singular locus of the hypersurface. We denote the Newton polyhedron of $W(z)$ as $\Delta(w)$. It is the convex hull of the exponents of (2.1) m in $\mathbf{R}^{n+1}$, shifted by $(-1, \ldots, -1)$. If we take into account the condition $d = w_1 + \cdots + w_{n+1}$, it is easy to deduce that the shifted polyhedron can be written as

$$\Delta(w) = \mathrm{Conv.}(\{x \in \mathbf{Z}^{n+1} \mid (w, x) = 0, \ x_i \geqq -1 (i = 1, \ldots, n + 1)\}) . \qquad (2.2)$$

An n-dimensional polyhedron Δ in $\mathbf{R}^n$ is called integral if all its vertices are integral (with respect to the lattice $\mathbf{Z}^n$). A *reflexive* polyhedron is an integral polyhedron with exactly one integral interior point, the origin. The polar dual of Δ,

$$\Delta^* := \{y \in \mathbf{R}^n \mid (y, x) \geqq -1 \ (\forall x \in \Delta)\} \qquad (2.3)$$

is again integral and reflexive. If we consider the set of cones over the faces of a polyhedron, we will obtain a complete fan which covers $\mathbf{R}^n$. Thus to each pair of reflexive polyhedra (Δ, Δ^*), we can associate a pair of complete fans $(\Sigma(\Delta), \Sigma(\Delta^*))$ and in turn a pair of the n dimensional toric varieties $(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{P}_{\Sigma(\Delta)})$. In each of the toric varieties, there is a family of Calabi–Yau hypersurfaces given by the zero loci of certain sections of the anticanonical bundle. The toric variety $\mathbf{P}_{\Sigma(\Delta)}$ contains a canonical Zariski open torus $(\mathbf{C}^*)^n$ whose coordinates we denote as $X = (X_1, \ldots, X_n)$. In these coordinates, the sections are

$$f_{\Delta^*}(X, a) = \sum_{v_i^* \in \Delta^* \cap \mathbf{Z}^n} a_i X^{v_i^*} . \qquad (2.4)$$

For generic values of the a_i's in (2.4), the X_{Δ^*} in $\mathbf{P}_{\Sigma(\Delta)}$ admits a minimal resolution to a Calabi–Yau manifold (which we also denote X_{Δ^*}). Similarly there is a corresponding family of hypersurfaces X_Δ in $\mathbf{P}_{\Sigma(\Delta^*)}$. Batyrev showed that a pair of the Calabi–Yau manifolds (X_Δ, X_{Δ^*}) is mirror symmetric to each other in the sense

that we have the following relations for their Hodge numbers ($n \geq 4$);

$$h^{1,1}(X_\Delta) = h^{n-2,1}(X_{\Delta^*})$$

$$= l(\Delta^*) - (n+1) - \sum_{\text{codim } S^*=1} l'(S^*) + \sum_{\text{codim } S^*=2} l'(S^*)l'(S), \qquad (2.5)$$

$$h^{n-2,1}(X_\Delta) = h^{1,1}(X_{\Delta^*}) = l(\Delta) - (n+1) - \sum_{\text{codim } S=1} l'(S) + \sum_{\text{codim } S=2} l'(S)l'(S^*),$$

where the S are faces of Δ, S^* the polar dual face of S. The functions l and l' count the numbers of integral points in a face and in the interior of a face respectively.

When $W(z)$ is Fermat, the toric variety $\mathbf{P}_{\Sigma(\Delta^*)}$ is isomorphic to the weighted projective space $\mathbf{P}^n(w)$, with X_Δ isomorphic to some $X_d(w)$. Then the mirror hypersurface X_{Δ^*} can be understood [19] as an orbifold of the X_Δ in $\mathbf{P}_{\Sigma(\Delta^*)}$, giving the orbifold construction of Greene and Plesser [32] based on conformal field theory. For general hypersurfaces of non-Fermat type, $\mathbf{P}(w)$ and $\mathbf{P}_{\Sigma(\Delta^*)}$ are only birational. In fact the fan $\Sigma(\Delta^*)$ is a refinement of the fan of $\mathbf{P}(w)$. The hypersurfaces $X_d(w)$ and X_Δ are related by flop operations on the ambient spaces. It has been shown [33], in this way, that Batyrev's constructions applies to all 7,555 hypersurfaces and reproduces the generalized mirror constructions known to [34]. In addition, there are several mirror pairs (X_Δ, X_{Δ^*}) which do not come from hypersurfaces in weighted projective spaces.

The quantity most relevant to the applications of the mirror symmetry to the quantum geometry of X_Δ are the period integrals for its mirror X_{Δ^*}. For example,

$$\Pi(a) = \frac{1}{(2\pi i)^n} \int_{C_0} \frac{1}{f_{\Delta^*}(X,a)} \prod_{i=1}^n \frac{dX_i}{X_i}, \qquad (2.6)$$

is the period integral over the torus cycle $C_0 = \{|X_1| = |X_2| = \cdots = |X_n| = 1\}$ in $(\mathbf{C}^*)^n$. For other periods, we will analyze the differential equation satisfied by (2.6).

2.2. $\mathscr{A}$-hypergeometric system for the periods. In [28], it is remarked that the period integral (2.6) satisfies an $\mathscr{A}$-hypergeometric system introduced by Gel'fand, Kapranov and Zelevinski [23]. In [8], it is found that the hypergeometric system is not generic but reducible, and the period integrals can be extracted from the system as the irreducible part of its solution space. Furthermore, for most of the hypersurface models, it is noted that the hypergeometric system must be generalized in order to extract the irreducible part of the solutions. We reproduce here an extension which is called an extended Δ^*-hypergeometric system, from purely combinatorial data of the polyhedron. We note that for type I models (see below), the extended Δ^*-hypergeometric system coincides with the GKZ system. For type II or III, the extended Δ^*-hypergeometric system incorporates additional differential operators associated with the action of an automorphism group.

An $\mathscr{A}$-hypergeometric system is described by a finite set $\mathscr{A}$ in a lattice $\{1\} \times \mathbf{Z}^n$ with the property that $\mathscr{A}$ linearly spans $\mathbf{R}^{n+1}$. In our case of the Δ^*-hypergeometric system, the finite set is given by the set of all integral points in the polyhedron Δ^*. Namely we have $\mathscr{A} = \{\bar{v}_0^*, \bar{v}_1^*, \ldots, \bar{v}_p^* \mid \bar{v}_i^* = (1, v_i^*), \ v_i^* \in \Delta^* \cap \mathbf{Z}^n\}$. Here we let $\bar{v}_0^* = (1, v_0^*)$ for the origin v_0^* in Δ^*. We consider a lattice L of affine dependencies on $\mathscr{A}$:

$$L = \{(l_0, l_1, \ldots, l_p) \in \mathbf{Z}^{p+1} \mid l_0\bar{v}_0^* + l_1\bar{v}_1^* + \cdots + l_p\bar{v}_p^* = 0\}. \qquad (2.7)$$

Then it is found in [28] that the period integral (2.6) satisfies the following set of differential equations, ($\mathscr{A}$-hypergeometric system with exponents $\beta = (-1, 0, \ldots, 0) \in \mathbf{R}^{n+1}$),

$$\mathscr{D}_l \Pi(a) = 0 \quad (l \in L), \qquad \mathscr{Z}_j \Pi(a) = 0 \quad (j = 0, 1, \ldots, n), \qquad (2.8)$$

where the differential operators $\mathscr{D}_l$ and $\mathscr{Z}_j$ are defined to be

$$\mathscr{D}_l = \prod_{l_i > 0} \left(\frac{\partial}{\partial a_i} \right)^{l_i} - \prod_{l_j < 0} \left(\frac{\partial}{\partial a_j} \right)^{-l_j} \quad (l \in L),$$

$$\mathscr{Z}_j = \sum_{i=0}^{p} \bar{v}_{i,j}^* \theta_{a_i} - \beta_j \quad (j = 0, 1, \ldots, n). \qquad (2.9)$$

The solution space of (2.8) is typically too large – it contains more than the period integrals of the Calabi–Yau manifolds X_{Δ^*}. It turns out that the period integrals satisfy additional differential equations.

2.3. Automorphism of $\mathbf{P}_{\Sigma(\Delta)}$. It is easy to recognize the origin of the linear differential operators $\mathscr{Z}_j (j = 1, \ldots, n)$ as the invariance of the period integral (2.6) under the canonical torus action on a toric variety, $X_i \to \lambda_i X_i (\lambda_i \in \mathbf{C}^*)$. Since the algebraic torus acts by a subgroup of the automorphism group of the toric variety $\mathbf{P}_{\Sigma(\Delta)}$, it is natural to incorporate into the PDE system the invariance under infinitesimal action of the full automorphism group. To describe this action in full generality, we will introduce the root system for a toric variety.

Let us consider a compact nonsingular toric variety $\mathbf{P}_\Sigma$ based on a regular fan Σ in the scalar extension $N_\mathbf{R}$ of a lattice $N(\cong \mathbf{Z}^r)$ of rank r. Let $M(\cong \mathbf{Z}^r)$ be the lattice dual to N. We choose a basis $\{n_1, \ldots, n_r\}$ for N and a dual basis $\{m_1, \ldots, m_r\}$ for M. There is a canonical algebraic torus $T_N := \mathrm{Hom}_\mathbf{Z}(M, \mathbf{C}^*) = (\mathbf{C}^*)^r$ in $\mathbf{P}_\Sigma$ whose coordinate ring is $\mathbf{C}[M] = \bigoplus_{m \in M} \mathbf{C}e(m)$. We write it as $\mathbf{C}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ with $X_i = e(m_i)$. Define the derivations $\delta_n (n \in N)$ on $\mathbf{C}[M]$ by $\delta_n e(m) = \langle m, n \rangle e(m)$. These derivations describe the natural action of $\mathrm{Lie}(T_N)$ on T_N. We may write $\{\delta_{n_1}, \ldots, \delta_{n_r}\} = \{X_1 \frac{\partial}{\partial X_1}, \ldots, X_r \frac{\partial}{\partial X_r}\}$. The Lie algebra of the full automorphism group of $\mathbf{P}_\Sigma$ is described by the root system $R(\Sigma)$ in addition to the torus action. The root system $R(\Sigma)$ is determined by the data of the fan Σ as follows. We denote the subset of one dimensional cones in the fan as $\Sigma(1)$. In each one dimensional cone $\sigma^{(1)} \in \Sigma(1)$, there is a primitive element $n(\sigma^{(1)})$ in N. Let

$$R(\Sigma) = \{\alpha \in M \mid \exists \sigma_\alpha^{(1)} \in \Sigma(1) \text{ with } \langle \alpha, n(\sigma_\alpha^{(1)}) \rangle = -1$$

$$\text{and } \langle \alpha, n(\sigma^{(1)}) \rangle \geqq 0 \text{ for all } \sigma^{(1)} \neq \sigma_\alpha^{(1)}\}. \qquad (2.10)$$

In terms of the root system, the Lie algebra of the automorphism group can be expressed by (see Proposition 3.13 in [35] for details)

$$\mathrm{Lie}(\mathrm{Auto}(\mathbf{P}_\Sigma)) = \mathrm{Lie}(T_N) \oplus \left(\bigoplus_{\alpha \in R(\Sigma)} \mathbf{C}e(\alpha) \delta_{n(\sigma_\alpha^{(1)})} \right). \qquad (2.11)$$

The linear differential operators $\mathscr{Z}_1, \ldots, \mathscr{Z}_n$ in (2.8) express the invariance of the period integral $\Pi(a)$ under the action of $\mathrm{Lie}(T_N)$. In fact it is easy to check that

$$\mathscr{Z}_i \Pi(a) = \int_{C_0} \delta_{n_i} \left(\frac{1}{f_{\Delta^*}(X, a)} \right) \prod_{k=1}^{n} \frac{dX_k}{X_k} \quad (i = 1, \ldots, n). \qquad (2.12)$$

The operator $\mathscr{Z}_0$ represents the change of the period under the overall scaling of the Laurent polynomial $f_{\Delta^*}(X,a) \to \lambda f_{\Delta^*}(X,a)$. We can now clearly extend the formula (2.12) to define $\mathscr{Z}_Y \Pi(a)$ for every $Y \in \mathrm{Lie}(\mathrm{Auto}(\mathbf{P}_{\Sigma(\Delta)}))$ by replacing δ_{n_i} by Y. We thus arrive at the definition of the extended Δ^*-hypergeometric system

$$\mathscr{D}_l \Pi(a) = 0 \quad (l \in L), \qquad \mathscr{Z}_Y \Pi(a) = 0 \quad (Y \in \mathrm{Lie}(\mathrm{Auto}(\mathbf{P}_{\Sigma(\Delta)}))) \,. \tag{2.13}$$

This extended system was first introduced in [8] and was used successfully to determine the complete set of the period integrals.

Because of the special value of the exponent $\beta = (-1, 0, \ldots, 0) \in \mathbf{R}^{n+1}$, the following gauge for the period

$$\tilde{\Pi}(a) = a_0 \Pi(a) \,, \tag{2.14}$$

will be useful. We will denote the hypergeometric system in this gauge as $\tilde{\mathscr{D}}_l \tilde{\Pi}(a) = 0, \tilde{\mathscr{Z}}_i \tilde{\Pi}(a) = 0$. Especially the first order differential operators $\tilde{\mathscr{Z}}_0, \tilde{\mathscr{Z}}_1, \ldots, \tilde{\mathscr{Z}}_n$ may be written concisely as

$$\tilde{\mathscr{Z}}_u = \sum \langle u, \bar{v}_i^* \rangle \theta_{a_i} \quad (u \in \mathbf{R}^{n+1}) \,. \tag{2.15}$$

In ref. [8], several Calabi–Yau hypersurfaces with $h^{1,1} = 2$ and 3 have been studied. There hypersurfaces in a weighted projective space have been classified into three types depending on the properties of the fan $\Sigma(\Delta^*)$. Type I models are those which do not have any integral points in the interior of codimension-one faces of $\Delta^*(w)$ and for which we have a regular-fan $\Sigma(\Delta^*)$ after taking into account subdivisions of the cones resulting from the integral points on the lower dimensional faces. Type II models are those which have integral points in the interior of codimension-one faces of $\Delta^*(w)$ but for which we still have a regular fan $\Sigma(\Delta^*)$ after subdivisions of the cones resulting from the integral points on the faces. Type III models are those for which we do not have a regular fan $\Sigma(\Delta^*)$ even if we subdivide the cones by incorporating all the integral points on the faces. In this sense type III models may be called "singular." According to this classification, we reproduce here the models analyzed in [8]

Type I : $X_8(2, 2, 2, 1, 1)$

Type II : $X_{12}(6, 2, 2, 1, 1)$, $X_{14}(7, 2, 2, 2, 1)$, $X_{18}(9, 6, 1, 1, 1)$, $X_{12}(6, 3, 1, 1, 1)$,

$\quad\quad X_{24}(12, 8, 2, 1, 1)$,

Type III : $X_{12}(4, 3, 2, 2, 1)$, $X_{12}(3, 3, 3, 2, 1)$, $X_{15}(5, 3, 3, 3, 1)$, $X_{18}(9, 3, 3, 2, 1)$.

$$\tag{2.16}$$

It was found that for a model of type I or II, the extended Δ^*-hypergeometric system is sufficient to determine the complete set of the period integrals. Whereas for models of type III, one needs to consider additional (non-toric) differential operator(s) whose form can be determined from the Jacobian ring of the hypersurface. If we supplement these additional operators to the extended Δ^*-hypergeometric system, we can derive the Picard–Fuchs differential equations. Thus for type III models, the combinatorial data of the polyhedron Δ^* alone do not seem sufficient for the explicit construction of the full system of differential operators. Nevertheless we will find in the next section that the local solutions are determined purely by the combinatorial data of

the polyhedron, and this property is shared by all three types of the Calabi–Yau hypersurfaces.

Example. $X_{14}(7,2,2,2,1)$. This is a typical model with non-trivial automorphism group. The polyhedron $\Delta(w) = \{x \in \mathbf{R}^5 \mid w_1 x_1 + \cdots + w_5 x_5 = 0, \ x_i \geq -1 \ (i = 1, \ldots, 5)\}$ is simplicial and is given by the convex hull of the vertices

$$v_1 = (1, -1, -1, -1), \qquad v_2 = (-1, 6, -1, -1), \qquad v_3 = (-1, -1, 6, -1),$$
$$v_4 = (-1, -1, -1, 6), \qquad v_5 = (-1, -1, -1, -1), \tag{2.17}$$

where we fix a basis $\{\Lambda_1, \ldots, \Lambda_4\}$ for the lattice $H(w) = \{x \in \mathbf{Z}^5 \mid w_1 x_1 + \cdots + w_5 x_5 = 0\}$, with $\Lambda_1 = (1, 0, 0, 0, -w_1)$, $\Lambda_2 = (0, 1, 0, 0, -w_2)$, $\Lambda_3 = (0, 0, 1, 0, -w_3)$ and $\Lambda_4 = (0, 0, 0, 1, -w_4)$. The integral points in the dual polyhedron $\Delta^*(w)$ are

$$v_0^* = (0, 0, 0, 0), \qquad v_1^* = (1, 0, 0, 0), \qquad v_2^* = (0, 1, 0, 0),$$

$$v_3^* = (0, 0, 1, 0), \qquad v_4^* = (0, 0, 0, 1), \qquad v_5^* = (-7, -2, -2, -2), \tag{2.18}$$

$$v_6^* = (-3, -1, -1, -1), \qquad v_7^* = (-4, -1, -1, -1), \qquad v_8^* = (-1, 0, 0, 0).$$

The points $v_1^*, \ldots, v_5^*$ are the vertices of the simplicial polyhedron $\Delta^*(w)$ and all other points (except the origin) appear on some faces of the polyhedron. The point $v_6^* = \frac{1}{2}(v_1^* + v_5^*)$ appears on the edge (one dimensional face) and corresponds to an exceptional divisor in X_Δ. The point $v_7^* = \frac{1}{7}(v_2^* + v_3^* + v_4^* + 7v_5^*)$ and $v_8^* = \frac{1}{7}(2v_2^* + 2v_3^* + 2v_4^* + v_5^*)$ are both in the interior of the codimension-one face dual to the corner v_1 of $\Delta(w)$. Hence they describe the automorphism of $\mathbf{P}_{\Sigma(\Delta)}$ and of the family of hypersurfaces X_{Δ^*}. In fact the two points describe the root system for the fan $\Sigma(\Delta)$ and generate the nontrivial part of the automorphism,

$$\mathbf{C}\xi_1 \oplus \mathbf{C}\xi_2 := \mathbf{C}e(v_7^*)\delta_{v_1} \oplus \mathbf{C}e(v_8^*)\delta_{v_1}. \tag{2.19}$$

These infinitesimal actions on the coordinate ring can be expressed in terms of the natural basis for $N = \mathbf{Z}^4$ and $M = \mathbf{Z}^4$ as $\xi_i = X^{v_{6+i}^*}(\delta_{X_1} - \delta_{X_2} - \delta_{X_3} - \delta_{X_4}) \ (i = 1, 2)$ and have the expressions

$$\xi_1 = \frac{1}{X_1^4 X_2 X_3 X_4} \left(X_1 \frac{\partial}{\partial X_1} - X_2 \frac{\partial}{\partial X_2} - X_3 \frac{\partial}{\partial X_3} - X_4 \frac{\partial}{\partial X_4} \right),$$

$$\xi_2 = \frac{1}{X_1} \left(X_1 \frac{\partial}{\partial X_1} - X_2 \frac{\partial}{\partial X_2} - X_3 \frac{\partial}{\partial X_3} - X_4 \frac{\partial}{\partial X_4} \right). \tag{2.20}$$

We may verify the algebra $[\xi_1, \xi_2] = 0$. The linear differential operators $\mathscr{Z}_{\xi_1}$ and $\mathscr{Z}_{\xi_2}$, which follows from (2.12), turns out to be

$$\mathscr{Z}_{\xi_1} = a_0 \frac{\partial}{\partial a_7} + 2a_1 \frac{\partial}{\partial a_6} + a_6 \frac{\partial}{\partial a_5},$$

$$\mathscr{Z}_{\xi_2} = a_0 \frac{\partial}{\partial a_8} + 2a_1 \frac{\partial}{\partial a_0} + a_6 \frac{\partial}{\partial a_7}. \tag{2.21}$$

These linear operators together with $\mathscr{Z}_0, \ldots, \mathscr{Z}_5$ and the higher order operator $\mathscr{D}_l \ (l \in L)$ constitute the full extended Δ^*-hypergeometric system.

3. Secondary Fan, Gröbner Fan and Local Solutions

In this section, we analyze the local solutions of the Δ^*-hypergeometric system. We find that the local properties of the Δ^*-hypergeometric system are determined purely by an algebro-combinatoric object, known as a toric ideal. At a special point, called "large radius limit," the toric ideal is related to an ideal which determines the cohomology ring of the toric variety $\mathbf{P}_{\Sigma(\Delta^*)}$.

3.1. Convergent series solutions for $\mathscr{A}$-hypergeometric system. Here we will summarize, with some modification, the general results in [23] about the convergent series solutions of the $\mathscr{A}$-hypergeometric system. We set $\mathscr{A} = (1, \Delta^*) \cap \mathbf{Z}^{p+1} = \{\bar{v}_0^*, \ldots, \bar{v}_p^*\}$ for our case of the Δ^*-hypergeometric system. The description here is brief and is meant to fix notations and to prepare for later discussions. We refer the reader to the original paper [23] for details.

From the definition of the $\mathscr{A}$-hypergeometric system (2.8), (2.9), it is easy to check that a formal solution to the $\mathscr{A}$-hypergeometric system with exponent $\beta \in \mathbf{R}^{n+1}$ is given by

$$\Pi(a, \gamma) = \sum_{l \in L} \frac{1}{\prod_{0 \leq i \leq p} \Gamma(l_i + \gamma_i + 1)} a^{l+\gamma}, \qquad (3.1)$$

where $\beta = \sum_i \gamma_i \bar{v}_i^*$. Evidently the formal solution is invariant under $\gamma \to \gamma + v$ ($v \in L$). Define the affine subspace $\Phi(\beta) := \{\gamma \in \mathbf{R}^{p+1} \mid \beta = \sum \gamma_i \bar{v}_i^*\}$. If we choose a basis $l^{(1)}, \ldots, l^{(p-n)}$ for L, the formal series (3.1) takes the form $\Pi(a, \gamma) = a^\gamma \sum_{m_1, \ldots, m_{p-n} \in \mathbf{Z}} c_m x^m$, where $x_k = a^{l^{(k)}}$. The relevant solutions are those with $c_m(\gamma) = 0$ unless $m_i \geq 0$. One must therefore restrict the choices of the basis and of γ.

A subset $I \in \{0, 1, \ldots, p\}$ is a base if $\{\bar{v}_i^* \mid i \in I\}$ form a basis of $\mathbf{R}^{n+1}$. Given a base I and γ_j ($j \notin I$), we can solve for γ_j ($j \in I$) using the linear relation $\sum_{j \in I} \gamma_j \bar{v}_j^* = \beta - \sum_{j \notin I} \gamma_j \bar{v}_j^*$. Consider $\Phi_{\mathbf{Z}}(\beta, I) := \{\gamma \in \Phi(\beta) \mid \gamma_j \in \mathbf{Z} \ (j \notin I)\}$, and $\Phi_{\mathbf{Z}}^A(\beta, I) := \{\gamma \in \Phi_{\mathbf{Z}}(\beta, I) \mid \gamma_j = \sum_{k=1}^{p-n} \lambda_k l_j^{(k)} \ (0 \leq \lambda_k < 1, \ j \notin I)\}$. It is clear that $\Phi_{\mathbf{Z}}^A(\beta, I)$ is a set of representatives of $\Phi_{\mathbf{Z}}(\beta, I)/L$. Consider the cone $\mathscr{K}(\mathscr{A}, I) = \{l \in L_{\mathbf{R}} \mid l_i \geq 0 \ (i \notin I)\}$ where $L_{\mathbf{R}} = L \otimes \mathbf{R}$. A $\mathbf{Z}$-basis $A \subset L$ is said to be compatible with the base I if the cone generated by the basis A contains the cone $\mathscr{K}(\mathscr{A}, I)$.

If $A = \{l^{(1)}, \ldots, l^{(p-n)}\}$ is compatible with the base I, then the formal series (3.1) takes the form $\Pi(a, \gamma) = a^\gamma \sum_{m_1, \ldots, m_{p-n} \geq 0} c_m x^m$ for each $\gamma \in \Phi_{\mathbf{Z}}^A(\beta, I)$ with

$x_k = a^{l^{(k)}}$, and this power series converges for sufficiently small $|x_k|$. (3.2)

By definition we may write the formal series (3.1) as above with $c_m = c_m(\gamma) := \prod_{i=0}^p 1/\Gamma(\sum m_k l_i^{(k)} + \gamma_i + 1)$. For $\gamma \in \Phi_{\mathbf{Z}}(\beta, I)$, we have $\sum m_k l_j^{(k)} + \gamma_j + 1 \in \mathbf{Z}$ for $j \notin I$. It follows that if $c_m \neq 0$, then $\sum m_k l_j^{(k)} + \gamma_j = \sum (m_k + \lambda_k) l_j^{(k)} \geq 0$ ($j \notin I$), where we use $\gamma_j = \sum \lambda_k l_j^{(k)}$ ($0 \leq \lambda_k < 1$, $j \notin I$). Since the basis A is compatible with the base I, we have $m_k + \lambda_k \geq 0$ for all k, implying $m_k \geq 0$. Thus given a basis A compatible with the base I, if for every $\gamma \in \Phi_{\mathbf{Z}}^A(\beta, I)$ there is $c_m(\gamma) \neq 0$ for some m, then we have $|\Phi_{\mathbf{Z}}(\beta, I)/L| = |\det(\bar{v}_{j,i}^*)_{1 \leq i \leq n+1, j \in I}|$ linearly independent power series solutions [23].

However it can happen that $c_m(\gamma) = 0$ for all m, i.e., the series solution becomes trivial $\Pi(a, \gamma) \equiv 0$ when $\sum m_k l_i^{(k)} + \gamma_i + 1 \in \mathbf{Z}_{\leq 0}$ ($m_k \geq 0$) for some $i \in I$.

In this case, we multiply c_m by a constant infinite renormalization $\Gamma(\gamma_i + 1)$. More precisely, we assume that γ is such that the following limit exists:

$$\frac{\Gamma(\gamma_i + 1)}{\Gamma(l_i + \gamma_i + 1)} := \lim_{\varepsilon \to 0} \frac{\Gamma(\gamma_i + 1 + \varepsilon)}{\Gamma(l_i + \gamma_i + 1 + \varepsilon)} \tag{3.3}$$

for all $l \in L$.

All linearly independent power series solutions are constructed from a set of bases $\{I\}$ which form a triangulation of the polyhedron $P := \text{Conv.}(\{0, \bar{v}_0^*, \bar{v}_1^*, \ldots, \bar{v}_p^*\})$, where 0 is the origin in $\mathbf{R}^{n+1}$. We call a collection of bases $T = \{I\}$ a triangulation of P if $\bigcup_{I \in T} \langle \bar{v}_I^* \rangle = P$ and $\langle \bar{v}_{I_1}^* \rangle \cap \langle \bar{v}_{I_2}^* \rangle$ $(I_1, I_2 \in T)$ is a lower dimensional common face. Here $\langle \bar{v}_I^* \rangle$ a $n + 1$ dimensional simplex with vertices $\bar{v}_i^*$ $(i \in I)$ and the origin. Because the $n + 1$-simplices in P are in 1–1 correspondence with the n-simplex in Δ^*, there is a notion of a triangulation of Δ^* (or $\mathcal{A}$). We use the two notions interchangeably. A triangulation T is called maximal if T gives the maximum number of n-simplices in Δ^* and $0 \in I$ for all $I \in T$. A **Z**-basis A of L is called compatible with a triangulation T if A is compatible with every $I \in T$.

For a base I and a point $\eta \in \mathbf{R}^{p+1}$, we consider a linear function $h_{I,\eta}$ on $\mathbf{R}^{p+1}$ such that $h_{I,\eta}(\bar{v}_i^*) = \eta_i$ $(i \in I)$. We define a cone $\mathscr{C}(\mathcal{A}, I)$ by $\{\eta \in \mathbf{R}^{p+1} \mid h_{I,\eta}(\bar{v}_i^*) \leq \eta_i \ (i \notin I)\}$. For a triangulation T, we define the cone $\mathscr{C}(\mathcal{A}, T) := \bigcap_{I \in T} \mathscr{C}(\mathcal{A}, I)$. We may associate with $\eta \in \mathbf{R}^{p+1}$ and a triangulation T, a piecewise linear continuous function $h_{T,\eta}$ on the polyhedron P defined by 1) $h_{T,\eta}(\bar{v}_i^*) = \eta_i$ for each vertex $\bar{v}_i^*$ of the triangulation T, 2) the restriction $h_{T,\eta}|_{\langle \bar{v}_I^* \rangle}$ $(I \in T)$ is a linear function. Then the cone $\mathscr{C}(\mathcal{A}, T)$ consists of $\eta \in \mathbf{R}^{p+1}$ for which the function $h_{T,\eta}$ is convex and $h_{T,\eta}(\bar{v}_i^*) \leq \eta_i$ for $\bar{v}_i^*$ not a vertex of T [23]. A regular triangulation is a triangulation for which we have interior points in the cone $\mathscr{C}(\mathcal{A}, T)$. For every regular triangulation T, there are infinitely many **Z**-basis of L compatible with T. We set $\Phi_{\mathbf{Z}}^A(\beta, T) := \bigcup_{I \in T} \Phi_{\mathbf{Z}}^A(\beta, I)$. Now we may state the result (Theorem 3) in [23]:

> For a regular triangulation T of the polyhedron P, and a **Z**-basis $A = \{l^{(1)}, \ldots, l^{(p-n)}\}$ of L compatible with T, we have integral power series in the variables $x_k = a^{l^{(k)}}$ for $a^{-\gamma} \Pi(a, \gamma)$ $(\gamma \in \Phi_{\mathbf{Z}}^A(\beta, T))$, which converge for sufficiently small $|x_k|$. If the exponent β is T-nonresonant, the series $\Pi(a, \gamma)$ $(\gamma \in \Phi_{\mathbf{Z}}^A(\beta, T))$ constitute vol(P) linearly independent solutions for (2.8) . (3.4)

In the above theorem, the exponent β is called T-nonresonant if the sets $\Phi_{\mathbf{Z}}(\beta, I)$ $(I \in T)$ are pairwise disjoint. It turns out that in our Δ^*-hypergeometric system there are many regular triangulations for which the exponent $\beta = (-1, 0, \ldots, 0)$ is T-resonant. In particular, if T is a maximal triangulation and the polyhedron Δ^* is of type I or II, then β is "maximally T-resonant," i.e., $\Phi_{\mathbf{Z}}(\beta, I)$ consists of a unique element $\gamma = (-1, 0, \ldots, 0)$ modulo L for all $I \in T$. (Note that each simplex $I \in T$ has volume $|\det(\bar{v}_{j,i}^*)_{1 \leq i \leq n+1, j \in I}| = 1$.) In this case, we will obtain only one power series solution (3.1), and all other solutions contain logarithms, whose forms we will determine by the Frobenius method.

Given a regular triangulation T, a compatible **Z**-basis $A = \{l^{(1)}, \ldots, l^{(p-n)}\}$ and $\gamma \in \Phi_{\mathbf{Z}}^A(\beta, T)$, we define a power series $w_0(x, \rho) = a_0 \Pi(a, \gamma)$, where $\rho = (\rho_1, \ldots, \rho_{p-n})$ is defined by $\gamma = \sum \rho_k l^{(k)} + (-1, 0, \ldots, 0)$ and $x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}}$. Explicitly

we have

$$w_0(x, \rho) = \sum_{m_1, \ldots, m_k \geq 0} \frac{\Gamma(-\sum((m_k + \rho_k)l_0^{(k)} + 1)}{\prod_{1 \leq i \leq p} \Gamma(\sum(m_k + \rho_k)l_i^{(k)} + 1)} x^{m+\rho} . \tag{3.5}$$

The ρ can also be determined by the indicial equations of the hypergeometric system.

Given a regular triangulation T, we shall now construct a compatible **Z**-basis A with the criterion that the cone generated by A in $L_{\mathbf{R}}$ contains the cone $\mathcal{K}(\mathcal{A}, T) := \bigcup_{I \in T} \mathcal{K}(\mathcal{A}, I)$. First we introduce the Gale transformation. Consider the exact sequence

$$0 \to \mathbf{R}^{n+1} \underset{\mathbf{A}}{\longrightarrow} \mathbf{R}^{p+1} \underset{\mathbf{B}}{\longrightarrow} \mathbf{R}^{p-n} \to 0 , \tag{3.6}$$

where we let $\mathbf{R}^{n+1}$ be the span of the integral points $\bar{v}_i^*$'s and $\mathbf{R}^{p+1}$ in the middle is the span of a basis $\{e_{\bar{v}_0^*}, e_{\bar{v}_1^*}, \ldots, e_{\bar{v}_p^*}\}$ labeled by the points. The linear map $\mathbf{A}$ sends $v \in \mathbf{R}^{n+1}$ to $\sum_i \langle v, \bar{v}_i^* \rangle e_{\bar{v}_i^*}$ and $\mathbf{B}$ is the natural map onto $\mathbf{R}^{p+1}/\mathbf{R}^{n+1} = \mathbf{R}^{p-n}$. The Gale transform of a point configuration $\mathcal{A}$ in $\mathbf{R}^{n+1}$, which we denote $\{\mathcal{A}, \mathbf{R}^{n+1}\}$, is defined to be a point configuration $\{\mathcal{B}, \mathbf{R}^{p-n}\}$ with $\mathcal{B} := \{\mathbf{B}(e_{\bar{v}_0^*}), \ldots, \mathbf{B}(e_{\bar{v}_p^*})\}$. Now we consider a cone in $\mathbf{R}^{p-n}$,

$$\mathscr{C}'(\mathcal{A}, T) = \bigcap_{I \in T} \left(\sum_{i \notin I} \mathbf{R}_{\geq 0} \mathbf{B}(e_{\bar{v}_i^*}) \right) . \tag{3.7}$$

Then it is shown in [36, 37] that the cone $\mathscr{C}(\mathcal{A}, T)$ decomposes into $\mathbf{R}^{n+1} \oplus \mathscr{C}'(\mathcal{A}, T)$. The secondary fan $\mathscr{F}(\mathcal{A})$ is defined as

$$\mathscr{F}(\mathcal{A}) = \{\mathscr{C}'(\mathcal{A}, T) \mid T: \text{regular triangulation}\} . \tag{3.8}$$

It is known that the secondary fan is complete and strongly polytopal polyhedral fan [36, 37].

In our point configuration $\{\mathcal{A}, \mathbf{R}^{n+1}\}$, the set $\mathcal{A}$ consists of integral points. Therefore the sequence (3.6) can be considered with an integral structure: $0 \to \mathbf{Z}^{n+1} \underset{\mathbf{A}}{\to} \mathbf{Z}^{p+1} \underset{\mathbf{B}}{\to} \mathbf{Z}^{p+1}/\mathbf{A}(\mathbf{Z}^{n+1}) \to 0$. The dual of this sequence is $0 \leftarrow \text{Coker}(\mathbf{A}^*) \leftarrow (\mathbf{Z}^{n+1})^* \underset{\mathbf{A}^*}{\leftarrow} (\mathbf{Z}^{p+1})^* \leftarrow L \leftarrow 0$, where $\mathbf{A}^*$ maps $s \in (\mathbf{Z}^{p+1})^*$ to $\sum_i \langle e_{\bar{v}_i^*}, s \rangle \bar{v}_i^*$, and so $L(= \text{Ker}(\mathbf{A}^*))$ is the lattice of the affine relations among $\mathcal{A}$. The cone dual to $\mathscr{C}'(\mathcal{A}, T) \subset \mathbf{R}^{p-n}$ is the cone $\mathcal{K}(\mathcal{A}, T) \subset L_{\mathbf{R}}$. In general $\mathscr{C}'(\mathcal{A}, T)$ is strongly convex but not necessarily regular. There is a canonical refinement of the secondary fan known as the Gröbner fan (see next subsection). However even a cone in this refinement is not necessarily regular. By suitably subdividing the cone, we obtain a regular subcone and hence a **Z**-basis of this subcone. The dual basis $A = \{l^{(1)}, \ldots, l^{(p-n)}\}$ thus generates a cone containing $\mathcal{K}(\mathcal{A}, T)$. This gives us a **Z**-basis of L compatible with T. Note that when $\mathscr{C}'(\mathcal{A}, T)$ is already regular, the basis A is uniquely determined by T.

Suppose now the polyhedron Δ^* is of type I or II. Then endowed with a maximal subdivision, it defines a regular fan $\Sigma(\Delta^*)$ and $\mathbf{P}_{\Sigma(\Delta^*)}$ is smooth. It is known that the Gale transform $\{\mathcal{B}, \mathbf{Z}^{p-n}\}$ generates the Picard group [35, 38]. Now associated with $\Sigma(\Delta^*)$ is a maximal triangulation T. In this case, $\mathscr{C}'(\mathcal{A}, T)$ is the Kähler cone, and $\mathcal{K}(\mathcal{A}, T)$ is the Mori cone of $\mathbf{P}_{\Sigma(\Delta^*)}$. In particular $\mathscr{C}'(\mathcal{A}, T)$ is a maximal cone (hence has interior points), implying that T is regular.

3.2. Toric ideal and universal Gröbner basis. Here we will focus on the differential operators $\mathscr{D}_l$ ($l \in L$) in (2.9). Although there are infinitely many operators, we can describe the system by a finite set of the operators. The problem is how to construct such a finite set. We will see that the so-called toric ideal in the theory of Gröbner basis gives us a powerful tool for this purpose.

Let $\mathscr{A}$ be the finite set in the previous subsection and L be the lattice representing the integral relations among the vertices in $\mathscr{A}$. We may decompose any element $l \in L$ uniquely into $l_+ - l_-$ with two nonnegative vectors l_+, l_- having disjoint support, where the support $m \in L$ is defined by supp$(m) := \{i \mid m_i \neq 0\}$. Toric ideal $\mathscr{I}_{\mathscr{A}}$ is defined as the ideal in $\mathbf{C}[y_0, \ldots, y_p]$ which is generated by $y^{l_+} - y^{l_-}$, i.e.,

$$\mathscr{I}_{\mathscr{A}} = \langle y^{l_+} - y^{l_-} \mid l \in L \rangle . \tag{3.9}$$

Let ω be a term order in $\mathbf{C}[y_0, \ldots, y_p]$. It is a vector $\omega = (\omega_0, \ldots, \omega_p) \in \mathbf{R}^{p+1}$ which defines an monomial ordering by the weights: the weight of $y_0^{\alpha_0} \cdots y_p^{\alpha_p}$ being $\omega_0\alpha_0 + \cdots + \omega_p\alpha_p$. With respect to this term order, we consider an ideal $LT_\omega(\mathscr{I}_{\mathscr{A}}) = \langle LT_\omega(f) \mid f \in \mathscr{I}_{\mathscr{A}} \rangle$ of the leading terms of $\mathscr{I}_{\mathscr{A}}$. Two different weights ω and ω' may give the same ideal. The equivalence class in $\mathbf{R}^{p+1}$

$$\mathscr{C}(\mathscr{I}_{\mathscr{A}}, \omega) := \{\omega' \in \mathbf{R}^{p+1} \mid LT_\omega(\mathscr{I}_{\mathscr{A}}) = LT_{\omega'}(\mathscr{I}_{\mathscr{A}})\} , \tag{3.10}$$

is an open convex polyhedral cone. The collection of cones $\{\mathscr{C}(\mathscr{I}_{\mathscr{A}}, \omega)\}$ is known to be finite and defines a polyhedral fan called the Gröbner fan $\mathscr{F}(\mathscr{I}_{\mathscr{A}})$ of $\mathscr{I}_{\mathscr{A}}$.

The Gröbner basis of $\mathscr{I}_{\mathscr{A}}$ with respect to a term order ω is a finite generating set $\mathscr{B}_\omega$ of $\mathscr{I}_{\mathscr{A}}$, with the property that the ideal $\langle LT_\omega(g) \mid g \in \mathscr{B}_\omega \rangle$ is equal to $LT_\omega(\mathscr{I}_{\mathscr{A}})$. By Hilbert's basis theorem, $\mathscr{I}_{\mathscr{A}}$ is generated by a finite set of binomials $y^{l_+} - y^{l_-}$ with $l \in L$. Starting from such a finite set, the (reduced) Gröbner basis $\mathscr{B}_\omega$ obtained by Buchberger's algorithm [39] is also a set of binomials. This is because the algorithm consists of forming the S-polymonials for the generators and the reductions of the minimal Gröbner basis and both processes close in the set of binomials. Moreover the elements of the reduced Gröbner basis take the form $y^{l_+} - y^{l_-}$ ($l \in L$) of binomials.

Next given a term order ω, we shall obtain a regular triangulation T_ω and hence a compatible $\mathbf{Z}$-basis A of L (last section). The elements of the toric ideal $\mathscr{I}_{\mathscr{A}}$ may be identified as differential operators which annihilate the formal series $\Pi(a, \gamma)$ with $\gamma \in \Phi_\mathbf{Z}^A(\beta, T_\omega)$. The ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$ is then a set of of "leading" terms of the operators which determine the indices for the series $w_0(x, \rho)$. Therefore the Gröbner basis $\mathscr{B}_\omega$ with respect to ω gives a finite set of the differential operators $\{\mathscr{D}_l\}$ which suffices to describe the local solutions. A finite set which contains the Gröbner basis $\mathscr{B}_\omega$ for all term orders is known as a universal Gröbner basis $\mathscr{U}_{\mathscr{A}}$. This basis is useful to describe the global property of the system.

A nonzero integral relation $l \in L$ is called elementary if 1) l is primitive, i.e., gcd$(l_0, l_1, \ldots, l_p) = 1$, 2) supp$(l)$ is minimal with respect to inclusion. It is known that the set $\{l^{(1)}, l^{(2)}, \ldots, l^{(m)}\}$ of all elementary integral relations generates a $(p+1)$-dimensional zonotope $\mathscr{P}_{\mathscr{A}} := \langle 0, l^{(1)} \rangle + \langle 0, l^{(2)} \rangle + \cdots + \langle 0, l^{(m)} \rangle$, where $\langle 0, l^{(k)} \rangle$ represents a one-dimensional simplex and the sum means the Minkowski sum. The universal Gröbner basis is then given by [40]

$$\mathscr{U}_{\mathscr{A}} = \{y^{l_+} - y^{l_-} \mid l \in \mathscr{P}_{\mathscr{A}} \cap \mathbf{Z}^{p+1}\} . \tag{3.11}$$

Given a term order ω the notion of a regular triangulation can in fact be recast as follows. Consider the polytope $P_\omega := \mathrm{Conv}.\{(\omega_0, v_0^*), \ldots, (\omega_p, v_p^*)\}$ in $\mathbf{R}^{n+1}$, which is a lifting of $\mathscr{A}$ by assigning the weights ω_i as height to each point v_i^*. For sufficiently generic ω, the lower envelope of P_ω naturally induces a triangulation T_ω of $\mathscr{A}$.

It turns out that a triangulation T of $\mathscr{A}$ is regular if and only if $T = T_\omega$ for some generic weight ω. Also the interior points of the cone $\mathscr{C}(\mathscr{A}, T)$ consists of all weights $\omega \in \mathbf{R}^{p+1}$ such that $T_\omega = T$ [40]. The Stanley–Reisner ideal SR_T for a triangulation T of $\mathscr{A}$ is the ideal in $\mathbf{C}[y_0, \ldots, y_p]$ generated by all monomials $y_{i_1} y_{i_2} \cdots y_{i_k}$, where $\{i_1, i_2, \ldots, i_k\} \notin T$. Then the following is shown in [40]:

> *If a weight vector ω defines a term order for the toric ideal $\mathscr{I}_{\mathscr{A}}$, then the corresponding subdivision T_ω is a regular triangulation. The Stanley–Reisner ideal SR_{T_ω} is equal to the radical of the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$.* (3.12)

As an immediate corollary to (3.12), the Gröbner fan $\mathscr{F}(\mathscr{I}_{\mathscr{A}})$ is a refinement of the fan $\{\mathscr{C}(\mathscr{A}, T)\}$. Since each cone $\mathscr{C}(\mathscr{A}, T)$ has a decomposition $\mathbf{R}^{n+1} \oplus \mathscr{C}'(\mathscr{A}, T)$, we have a similar decomposition $\mathscr{C}(\mathscr{I}_{\mathscr{A}}, \omega) = \mathbf{R}^{n+1} \oplus \mathscr{C}'(\mathscr{I}_{\mathscr{A}}, \omega)$. We will call the collection $\{\mathscr{C}'(\mathscr{I}_{\mathscr{A}}, \omega)\}$ the Gröbner fan which we also denote by $\mathscr{F}(\mathscr{I}_{\mathscr{A}})$.

Example. $\mathbf{P}(2, 2, 2, 1, 1)$. This is a simple example of a toric variety in which we can define a Calabi–Yau hypersurface with $h^{1,1}(X_\Delta) = 2$. The polyhedron $\Delta(w)$ is given by the convex hull of the following integral points,

$$v_1 = (3, -1, -1, -1), \qquad v_2 = (-1, 3, -1, -1), \qquad v_3 = (-1, -1, 3, -1),$$
$$v_4 = (-1, -1, -1, 7), \qquad v_5 = (-1, -1, -1, -1),$$
(3.13)

where the vector components are those with respect to a fixed basis $\Lambda_1 = (1, 0, 0, 0, -w_1), \ldots, \Lambda_4 = (0, 0, 0, 1, -w_4)$ for the lattice $H(w)$ (see the example in the previous section). The integral points in the dual $\Delta^*(w)$ are

$$v_1^* = (1, 0, 0, 0), \qquad v_2^* = (0, 1, 0, 0), \qquad v_3^* = (0, 0, 1, 0),$$
$$v_4^* = (0, 0, 0, 1), \qquad v_5^* = (-2, -2, -2, -1), \qquad v_6^* = (-1, -1, -1, 0),$$
(3.14)

The point $v_6^* = \frac{1}{2}(v_4^* + v_5^*)$ in a codimension 3 face of Δ^* corresponds to a A_1-type Du Val singularity in the affine subvariety determined by the cone $\mathbf{R}_{\geq 0} v_4^* + \mathbf{R}_{\geq 0} v_5^*$ in the fan $\Sigma(\Delta^*)$. We can find three elementary relations (up to sign) in $\mathscr{A} = (1, \Delta^*(w)) \cap \mathbf{Z}^5$ expressed by

$$l^{(1)} = (-4, 1, 1, 1, 0, 0, 1), \qquad l^{(2)} = (0, 0, 0, 0, 1, 1, -2), \qquad l^{(3)} = (-8, 2, 2, 2, 1, 1, 0).$$
(3.15)

Then the zonotope $\mathscr{P}_{\mathscr{A}}$ determines the universal Gröbner basis

$$\mathscr{U}_{\mathscr{A}} = \{y_1 y_2 y_3 y_6 - y_0^4, y_4 y_5 - y_6^2, y_1^2 y_2^2 y_3^2 y_4 y_5 - y_0^8, y_1 y_2 y_3 y_4 y_5 - y_0^4 y_6\}.$$
(3.16)

It is straightforward to find all possible regular triangulations of the set $\mathscr{A}$, or equivalently the polyhedron $\Delta^*(w)$. We find the following four regular triangulations:

$$T_0 = \{\langle 0, 2, 3, 5, 6 \rangle, \langle 0, 1, 3, 5, 6 \rangle, \langle 0, 1, 2, 5, 6 \rangle, \langle 0, 2, 3, 4, 6 \rangle,$$
$$\langle 0, 1, 3, 4, 6 \rangle, \langle 0, 1, 2, 4, 6 \rangle, \langle 0, 1, 2, 3, 5 \rangle, \langle 0, 1, 2, 3, 4 \rangle\},$$

$$T_1 = \{\langle 1, 2, 3, 4, 5 \rangle\}, \qquad T_2 = \{\langle 1, 2, 3, 4, 6 \rangle, \langle 1, 2, 3, 5, 6 \rangle\},$$
(3.17)

$$T_3 = \{\langle 0, 2, 3, 4, 5 \rangle, \langle 0, 1, 3, 4, 5 \rangle, \langle 0, 1, 2, 4, 5 \rangle, \langle 0, 1, 2, 3, 5 \rangle, \langle 0, 1, 2, 3, 4 \rangle\},$$

where, for example, $\langle 0, 2, 3, 4, 5 \rangle$ represents a simplex with vertices $\bar{v}_0^*, \bar{v}_2^*, \ldots, \bar{v}_5^*$.

Table 1. Gröbner cones with typical weights. Each cone determines the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$ and its radical. The radical coincides with the Stanley–Reisner ideal SR_{T_ω} according to (3.12)

cone	weight ω	$LT_\omega(\mathscr{I}_{\mathscr{A}})$	$\mathrm{rad}(LT_\omega(\mathscr{I}_{\mathscr{A}}))$
τ_1	$(0,1,1,1,1,1,0)$	$\langle y_1 y_2 y_3 y_6, y_4 y_5 \rangle$	$\langle y_1 y_2 y_3 y_6, y_4 y_5 \rangle$
τ_2	$(1,0,0,0,1,1,0)$	$\langle y_0^4, y_4 y_5 \rangle$	$\langle y_0, y_4 y_5 \rangle$
τ_3	$(1,0,0,0,0,0,1)$	$\langle y_0^4, y_6^2 \rangle$	$\langle y_0, y_6 \rangle$
τ_4	$(1,0,0,0,0,0,5)$	$\langle y_1 y_2 y_3 y_6, y_6^2, y_0^8, y_0^4 y_6 \rangle$	$\langle y_0, y_6 \rangle$
τ_5	$(0,1,1,1,0,0,4)$	$\langle y_1 y_2 y_3 y_6, y_6^2, y_1^2 y_2^2 y_3^2 y_4 y_5, y_0^4 y_6 \rangle$	$\langle y_6, y_1 y_2 y_3 y_4 y_5 \rangle$
τ_6	$(0,1,1,1,0,0,1)$	$\langle y_1 y_2 y_3 y_6, y_6^2, y_1 y_2 y_3 y_4 y_5 \rangle$	$\langle y_6, y_1 y_2 y_3 y_4 y_5 \rangle$

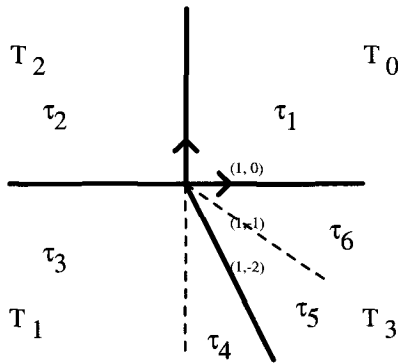**Fig. 1.** The secondary fan and the Gröbner fan for $P(2,2,2,1,1)$. The secondary fan consists of the polyhedral cones parametrized by the regular triangulations $T_0, \ldots, T_3$ in the text. The Gröbner fan provides a refinement consisting of $\tau_1, \ldots, \tau_6$ represented by the typical weights in Table 1

The Gröbner fan consists of six two-dimensional cones, together with lower dimensional cones as their faces. We list the typical weight with the corresponding ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$ and its radical in Table 1. We draw, in Fig. 1, the secondary fan and the Gröbner fan as its refinement using a **Z**-basis $\{\tilde{l}^{(1)}, \tilde{l}^{(2)}\}$ which is dual to a **Z**-basis $\{l^{(1)}, l^{(2)}\}$ in (3.15) of the lattice L.

3.3. Cohomology ring of $\mathbf{P}_{\Sigma(\Delta^*)}$ and the local solutions – when $\Sigma(\Delta^*)$ is regular. In this subsection we will study the local solutions of the Δ^*-hypergeometric system. Since the Gröbner fan is a refinement of the secondary fan, each cone of the Gröbner fan naturally defines a convergent series for (3.1). Namely we consider a cone τ with typical weight ω. If τ is simplicial and regular we consider a **Z**-basis $\{\tilde{l}_\tau^{(1)}, \ldots, \tilde{l}_\tau^{(p-n)}\}$ of τ, and if not we subdivide τ into simplicial and regular cones and take a **Z**-basis for one of these cones. Then the dual basis $\{l_\tau^{(1)}, \ldots, l_\tau^{(p-n)}\}$ gives us a **Z**-basis compatible with the regular triangulation T_ω and the series (3.5). Even though the choice of **Z**-basis of L is not unique, once a choice is made we refer to it as a **Z**-basis of L for the cone τ with typical weight ω.

Since the exponent β is T-resonant for some regular triangulations, we do not have $\mathrm{vol}(\Delta^*(w))$ linearly independent power series solutions, and so we need to

search for the logarithmic solutions. The type of logarithmic solutions which arises for a given triangulation depends on the type of degenerations of the hypersurface, hence on the monodromy of its period integrals. In general, the differential equations satisfied by the period integrals have regular singularities [41]. Therefore we can determine the local solutions from the data of the leading terms of the differential equations – the so-called indicial equations. We expect that among the singularities, the quotient singularities can be resolved by the toric method via a refinement – such as the Gröbner fan – of the secondary fan ([22] Conjecture 13.2). Thus near these singularities, we should recover the data for our local solutions from the structure of the cones in the fan.

Let us consider a power series solution (3.5) determined by a $\mathbf{Z}$-basis $\{l_\tau^{(1)}, \ldots, l_\tau^{(p-n)}\}$ of L for a cone τ with typical weight $\omega \in \tau$. We identify the toric ideal $\mathscr{I}_{\mathscr{A}}$ in $\mathbf{C}[y_1, \ldots, y_p]$ with the ideal generated by $\{\mathscr{D}_l\}$ in $\mathbf{C}[\frac{\partial}{\partial a_1}, \ldots, \frac{\partial}{\partial a_p}]$. While we will consider a multiplication of $\mathscr{D}_l$ by the rational functions of a_k's extending the coefficient, we need to be careful with the noncommutativity resulting from this extension. Now let us consider an operator in the Gröbner basis $\mathscr{B}_\omega$. If $\mathscr{D}_l \in \mathscr{B}_\omega$ and $\omega \cdot l_+ - \omega \cdot l_- > 0$ (< 0), then $(\frac{\partial}{\partial a})^{l_+}((\frac{\partial}{\partial a})^{l_-})$ is one of the generators for the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$. For the case $\omega \cdot (l_+ - l_-) > 0$, we multiply $\mathscr{D}_l$ by a^{l_+} to obtain

$$a^{l_+}\mathscr{D}_l = a^{l_+}\left(\frac{\partial}{\partial a}\right)^{l_+} - a^{l_+ - l_-}a^{l_-}\left(\frac{\partial}{\partial a}\right)^{l_-}. \qquad (3.18)$$

Since $\omega \cdot (l_+ - l_-) > 0$ we have $l_+ - l_- \in \tau^\vee$. Since $\tau^\vee$ is generated by $\{l^{(1)}, \ldots, l^{(p-n)}\}$, it follows that $a^{l_+ - l_-}$ in the second term can be expressed by a monomial of $\{x_\tau^{(k)}\}$ which vanishes when $x_\tau^{(k)} \to 0$. Other parts in (3.18) are "homogeneous" and can be rewritten in terms of the log derivatives $\theta_{a_i} = a_i \frac{\partial}{\partial a_i}$. The same argument applies to the case $\omega \cdot (l_+ - l_-) < 0$. Therefore the principal part of $\mathscr{D}_l$ which determines the local properties about $x_\tau^{(k)} = 0$ are given, through the generators of $\mathscr{B}_\omega$, by

$$\left(\frac{\partial}{\partial a}\right)^{l_+} - \left(\frac{\partial}{\partial a}\right)^{l_-} \mapsto a^{l_\pm}\left(\frac{\partial}{\partial a}\right)^{l_\pm}, \qquad (3.19)$$

where l_+ and l_- in the right-hand side for $\omega \cdot (l_+ - l_-) > 0$ and $\omega \cdot (l_+ - l_-) < 0$, respectively. Clearly we can express the principal part (3.19) as a polynomial $I_l(\theta_{a_0}, \ldots, \theta_{a_p})$. In the gauge $\tilde{\Pi} = \tilde{\Pi}(x_\tau^{(1)}, \ldots, x_\tau^{(p-n)})$ (2.14), using $x_k = a^{l_\tau^{(k)}}$ we can write

$$I_l(\theta_{a_0}, \ldots, \theta_{a_p}) = J_l(\theta_{x_\tau^{(1)}}, \ldots, \theta_{x_\tau^{(p-n)}}), \qquad (3.20)$$

where the right-hand side is a polynomial in the $\theta_{x_\tau^{(k)}}$. Note also that J_l is homogeneous if the entries of $l_\pm$ are 0 or 1. Due to the property of the Gröbner basis, the principal parts (3.20) for the elements in $\mathscr{B}_\omega$ give us a complete set of the indicial equations for ρ. Summarizing our results:

Consider a local solution $w_0(x_\tau, \rho)$ (3.5) with a $\mathbf{Z}$-basis of L for a cone τ with typical weight ω. Then the indices ρ are determined from the finite set of indicial equations $J_l(\rho_\tau^{(1)}, \ldots, \rho_\tau^{(p-n)}) = 0$, with $y^{l_+} - y^{l_-} \in \mathscr{B}_\omega$. $\qquad (3.21)$

This result combined with the Frobenius method enables us to construct missing solutions in the general theorem (3.4) for the case of T-resonant.

Now let us turn to the description of the intersection ring $A^*(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{Z})$, which is isomorphic to the cohomology ring, $H^{2*}(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{Z})$ of the nonsingular projective toric variety $\mathbf{P}_{\Sigma(\Delta^*)}$. In the following we assume $\mathbf{P}_{\Sigma(\Delta^*)}$ is nonsingular, which means that we take one of the finest subdivisions of the fan $\Sigma(\Delta^*)$. Note that for the models of type I and II, such finest subdivision comes from a maximal triangulation T_0 of the polyhedron Δ^*. We have seen that T_0 is also regular. In the next section, we will find that our results apply also to the singular models of type III with some modifications.

In toric geometry, each integral point v_i^* ($i = 1, \ldots, p$) in $\Delta^* \cap \mathbf{Z}^n$ corresponds to an irreducible T-invariant divisor D_i. It is known that if $v_{i_1}^*, \ldots, v_{i_k}^*$ generate a cone in $\Sigma(\Delta^*)$, the divisors $D_{i_1} \cdots D_{i_k}$ intersect transversally with the subvariety determined by the cone. Also there are linear relations among the toric divisors since we are working modulo the divisors of rational functions on $\mathbf{P}_{\Sigma(\Delta^*)}$. It is then known that [35]:

For a compact nonsingular toric variety $\mathbf{P}_{\Sigma(\Delta^*)}$, the intersection ring $A^*(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{Z})$ is described by $\mathbf{Z}[D_1, \ldots, D_p]/\mathscr{I}$, where $\mathscr{I}$ is the ideal generated by

> (i) $D_{i_1} \cdot \cdots \cdot D_{i_k}$ *for* $v_{i_1}^*, \ldots, v_{i_k}^*$ *not in a cone of* $\Sigma(\Delta^*)$,
>
> (ii) $\sum_{i=1}^{p} \langle u, v_i^* \rangle D_i$ *for* $u \in \mathbf{Z}^n$. $\qquad\qquad$ (3.22)

We can fix the normalization of the "volume form" in the ring by the property that the Euler number of $\mathbf{P}_{\Sigma(\Delta^*)}$ is equal to the number of the n-dimensional cones in the fan $\Sigma(\Delta^*)$. This is the number of the n-simplices in the corresponding maximal triangulation T_0 of the polyhedron Δ^*.

To relate toric ideals to our previous discussion on the Δ^*-hypergeometric system, we introduce a formal variable D_0 and rewrite the intersection ring as $A^*(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{Z}) = \mathbf{Z}[D_0, D_1, \ldots, D_p]/\bar{\mathscr{I}}$, where we define $\bar{\mathscr{I}}$ as the ideal generated by

> (i)′ $D_{i_1} \cdots D_{i_k}$ *for* $\bar{v}_{i_1}^*, \ldots, \bar{v}_{i_k}^*$ *not in a cone of* $\Sigma((1, \Delta^*))$,
>
> (ii)′ $\sum_{i=0}^{p} \langle u, \bar{v}_i^* \rangle D_i$ *for* $u \in \mathbf{Z}^{n+1}$. $\qquad\qquad$ (3.23)

The fan $\Sigma((1, \Delta^*))$ in (i)′ is defined to be a set of cones over the simplices of the triangulations T_0 of $(1, \Delta^*)$. If the fan $\Sigma(\Delta^*)$ is regular, so is the fan $\Sigma((1, \Delta^*))$, although the latter fan is not complete.

Now note that the set of the generators (i)′ is the same as the generators of the Stanley–Reisner ideal for the maximal triangulation T_0 of Δ^*. Note also the similarity of the linear relations (ii)′ and the first order relations (2.15) in the hypergeometric system. By (3.12) the Stanley–Reisner ideal SR_{T_0} is the radical of $LT_\omega(\mathscr{I}_{\mathscr{A}})$, where ω is a weight with $T_\omega = T_0$. In the following, we will show that the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$ is radical. This allows us to determine the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$, or equivalently the principal parts (3.20) of the Δ^*-hypergeometric system that governs the local solutions, via a purely combinatorial object – the Stanley–Reisner ideal. As an immediate consequence we show that for the maximal triangulation, $\rho = (0, \ldots, 0)$ is the unique solution to the indicial equations. Furthermore we observe that the local solutions for the maximal triangulation T_0 can be described by the intersection numbers. The latter

are computable from the intersections ring (3.22). We will also see that the Stanley–Reisner ideal above can be easily computed in terms of the so-called primitive collections.

To discuss the combinatorial description of the Stanley–Reisner ideal, we introduce the notions of *a primitive collection* and *a primitive relation* [42]. A primitive collection of a complete fan $\Sigma(\varDelta^*)$ is a set of integral vectors $\mathscr{P} = \{v_{i_1}^*, v_{i_2}^*, \ldots, v_{i_a}^*\}$ such that if we remove any one of $v_{i_s}^*$ from $\mathscr{P}$, then the integral vectors in $\mathscr{P} \backslash \{v_{i_s}^*\}$ generate a cone in $\Sigma(\varDelta^*)$ while $\mathscr{P}$ itself does not generate any cone in $\Sigma(\varDelta^*)$. It is easy to prove that i) in (3.22) can be replaced by the monomials $D_{i_1} \cdots D_{i_a}$ corresponding to the primitive collections of $\Sigma(\varDelta^*)$. Once we fix a triangulation T_0 which underlies the $\Sigma(\varDelta^*)$, it is straightforward to read off all primitive collections. So far we don't need the regularity of $\Sigma(\varDelta^*)$. But to discuss primitive relations, we must assume that $\Sigma(\varDelta^*)$ is regular. A primitive relation will be a certain element of L attached to each primitive collection.

Consider a primitive collection $\mathscr{P} = \{v_{i_1}^*, v_{i_2}^*, \ldots, v_{i_a}^*\}$. There is a unique cone $\mathscr{C} \in \Sigma(\varDelta^*)$ of minimum dimension such that the integral point $v_{i_1}^* + v_{i_2}^* + \cdots + v_{i_a}^*$ is in the interior of $\mathscr{C}$. By regularity, there is a set of generators $\{v_{j_1}^*, \ldots, v_{j_s}^*\}$ of the cone such that for some positive integers c_k, we have

$$v_{i_1}^* + v_{i_2}^* + \cdots + v_{i_a}^* = \sum_{k \geq 1} c_k v_{j_k}^* . \qquad (3.24)$$

It is easy to translate the above statement about the fan $\Sigma(\varDelta^*)$ into a statement about the fan $\Sigma((1, \varDelta^*))$, which is not complete but regular. We get

$$\bar{v}_{i_1}^* + \bar{v}_{i_2}^* + \cdots + \bar{v}_{i_a}^* = \sum_{k \geq 0} c_k \bar{v}_{j_k}^* , \qquad (3.25)$$

where $\bar{v}_{j_0}^* = \bar{v}_0^*$ and $c_0 = a - \sum_{k \geq 1} c_k \geq 0$. Equation (3.25) defines a primitive relation $l(\mathscr{P}) \in L$. It is easy to deduce from the defining property of a primitive collection that the index sets $\{i_1, \ldots, i_a\}$ and $\{j_0, \ldots, j_s\}$ are disjoint.

Let ω be a weight vector such that T_ω is the maximal triangulation T_0. Recall that the convex polytope P_ω is defined by the convex hull of the points $\tilde{v}_k^* := (\omega_k, v_k^*)$ $(k = 0, \ldots, p)$. Then we can show the following "height" inequality $(\tilde{v}_{i_1}^* + \cdots + \tilde{v}_{i_a}^*)_0 > (\Sigma c_k \tilde{v}_{j_k}^*)_0$, i.e.,

$$\omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_a} > \sum c_k \omega_{j_k} . \qquad (3.26)$$

This means that $LT_\omega(y^{l(\mathscr{P})+} - y^{l(\mathscr{P})-}) = y_{i_1} y_{i_2} \cdots y_{i_a}$. Since the Stanley–Reisner ideal SR_{T_ω} is generated by those $y^{i_1} y^{i_2} \cdots y^{i_a}$ with $\{v_{i_1}^*, v_{i_2}^*, \ldots, v_{i_a}^*\}$ primitive, it follows that $SR_{T_\omega} \subset LT_\omega(\mathscr{I}_\mathscr{A})$. Combining this with the property (3.12), we see that $LT_\omega(\mathscr{I}_\mathscr{A})$ is radical. Moreover we also have $SR_{T_\omega} = \langle y^{l(\mathscr{P})+} \, | \, \mathscr{P}$ is primitive$\rangle$.

In the example $\mathbf{P}(2, 2, 2, 1, 1)$ discussed in the last subsection, we find two primitive collections $\{v_1^*, v_2^*, v_3^*, v_6^*\}, \{v_4^*, v_5^*\}$ for the maximal triangulation T_0 and the corresponding primitive relations turn out to be $l^{(1)}$ and $l^{(2)}$ in (3.15), respectively. As is evident in Table 1, these primitive collections give the generators $LT_\omega(y^{l^{(i)}_+} - y^{l^{(i)}_-})$, $(i = 1, 2)$ of the ideal $LT_\omega(\mathscr{I}_\mathscr{A})$ for the cone τ_1.

Note that the above generators of SR_{T_0} are nothing but the leading symbols of a generating set of operators $\mathscr{D}_l$ of the $\varDelta^*$-hypergeometric system. Combining with the operators $\mathscr{Z}_i$, we have a correspondence between the symbols of the full $\varDelta^*$-hypergeometric system and the ideal $\tilde{\mathscr{I}}$ (3.23) for the intersection ring. This

motivates the following map m from $\mathbf{Z}[\theta_{a_0}, \ldots, \theta_{a_p}]$ to the intersection ring $m : \theta_{a_i} \mapsto D_i$. Define the following intersection coupling,

$$C_{i_1 i_2 \cdots i_n} = \langle m(\theta_{x_\tau^{(i_1)}}) m(\theta_{x_\tau^{(i_2)}}) \cdots m(\theta_{x_\tau^{(i_n)}}) \rangle , \qquad (3.27)$$

where the bracket means taking the coefficient of the "volume element" in the ring $A^*(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{Z})$. Then we *observe* the following:

> *If τ is the cone in which T_ω ($\omega \in \tau$) is a maximal triangulation, then all the indices at the point $x_\tau^{(i)} = 0$ ($i = 1, \ldots, p - n$) of the hypergeometric system are identically zero. And the local solutions near this point are given by*

$$w_0(x_\tau, \rho)|_{\rho=0}, \qquad \partial_{\rho_i} w_0(x_\tau, \rho)|_{\rho=0}, \qquad \sum_{i_1, i_2} C_{i_1 i_2 \cdots i_n} \partial_{\rho_{i_1}} \partial_{\rho_{i_2}} w_0(x_\tau, \rho)|_{\rho=0} ,$$

$$\cdots \qquad (3.28)$$

$$\sum_{i_1, i_2, \cdots i_n} C_{i_1 i_2 \cdots i_n} \partial_{\rho_{i_1}} \partial_{\rho_{i_2}} \cdots \partial_{\rho_{i_n}} w_0(x_\tau, \rho)|_{\rho=0} .$$

Recall that $\mathrm{rad}(LT_\omega(\mathscr{I}_\mathscr{A})) = LT_\omega(\mathscr{I}_\mathscr{A})$ for the weight ω such that $T_\omega = T_0$, and that $LT_\omega(\mathscr{I}_\mathscr{A})$ is generated by $LT_\omega(\mathscr{B}_\omega)$. Since an element of the Gröbner basis has the form $y^{l+} - y^{l-}$, it follows that the entries of either vector $l_\pm$ are 0 or 1. In either case, we see that the corresponding indicial equation $J_l(\rho_1, \ldots, \rho_{p-n}) = 0$ is homogeneous. But the finiteness of the solution set implies that zero is the only solution.

Example. $\mathbf{P}(2, 2, 2, 1, 1)$. As we have seen, there is one maximal triangulation T_0 in (3.17) for the polyhedron Δ^*. The corresponding cone is τ_1 in Table 1, and thus the Gröbner basis $\mathscr{B}_\omega$ consists of $y^{l_+^{(1)}} - y^{l_-^{(1)}} = y_1 y_2 y_3 y_6 - y_0^4$ and $y^{l_+^{(2)}} - y^{l_-^{(2)}} = y_4 y_5 - y_6^2$. From this we obtain the leading term operators for (3.20);

$$J_{l^{(1)}} = \theta_{a_1} \theta_{a_2} \theta_{a_3} \theta_{a_6} = \theta_{x_{\tau_1}}^3 (\theta_{x_{\tau_1}} - 2\theta_{y_{\tau_1}}) ,$$

$$J_{l^{(2)}} = \theta_{a_4} \theta_{a_5} = \theta_{y_{\tau_1}}^2 , \qquad (3.29)$$

with the corresponding linear operators

$$\mathscr{L}_1 = \theta_{x_{\tau_1}}^3 (\theta_{x_{\tau_1}} - 2\theta_{y_{\tau_1}}) - x_{\tau_1}(4\theta_{x_{\tau_1}} + 4)(4\theta_{x_{\tau_1}} + 3)(4\theta_{x_{\tau_1}} + 2)(4\theta_{x_{\tau_1}} + 1) ,$$

$$\mathscr{L}_2 = \theta_{y_{\tau_1}}^2 - y_{\tau_1}(\theta_{x_{\tau_1}} - 2\theta_{y_{\tau_1}} - 1)(\theta_{x_{\tau_1}} - 2\theta_{y_{\tau_1}}) . \qquad (3.30)$$

Since the generators of the Stanley–Reisner ideal is given by the primitive collections, it is easy to determine the intersection ring. The results for the intersection couplings are

$$C_{xxxx} = 2, \qquad C_{xxxy} = 1, \qquad C_{xxyy} = C_{xyyy} = C_{yyyy} = 0 , \qquad (3.31)$$

where $C_{xxxx} = \langle m(\theta_{x_{\tau_1}}) \cdots m(\theta_{x_{\tau_1}}) \rangle$ for example. From the indicial equations $J_{l^{(1)}}(\rho) = J_{l^{(2)}}(\rho) = 0$, we see that all indices at the point $x_{\tau_1} = y_{\tau_1} = 0$ are zero. In fact we

find the following 8 solutions with only one power series solution;

$$w_0(x,0); \qquad \partial_{\rho_x} w_0(x,0), \partial_{\rho_y} w_0(x,0); \qquad (2\partial_{\rho_x}^2 + 2\partial_{\rho_x}\partial_{\rho_y})w_0(x,0), \partial_{\rho_x}^2 w_0(x,0);$$

$$(2\partial_{\rho_x}^3 + 3\partial_{\rho_x}^2\partial_{\rho_y})w_0(x,0), \partial_{\rho_x}^3 w_0(x,0); \qquad (2\partial_{\rho_x}^4 + 4\partial_{\rho_x}^3\partial_{\rho_y})w_0(x,0) , \qquad (3.32)$$

where

$$w_0(x,\rho) = \sum \frac{\Gamma(4(n + \rho_x) + 1)}{\Gamma(n + \rho_x + 1)^3 \Gamma(m + \rho_y + 1)^2 \Gamma(n - 2m + \rho_x - 2\rho_y + 1)} x_{\tau_1}^{n+\rho_x} y_{\tau_1}^{m+\rho_y} .$$

$$(3.33)$$

Because we have $\mathscr{L}_i w_0(x,\rho) = J_{l^{(i)}}(\rho)x^\rho + $ (power series in x and y) $(i = 1,2)$, we can verify that (3.32) solve the hypergeometric system by inspecting $(2\partial_{\rho_x}^4 + 4\partial_{\rho_x}^3\partial_{\rho_y})(J_{l^{(i)}}(\rho)x^\rho)|_{\rho=0} = 0$ $(i = 1,2)$, for example (see Eqs. (4.4) and (4.5) in [8] for detailed arguments about the remaining terms).

Similarly we obtain for τ_2 the principal parts (3.20) in the $\tilde{\Pi}(a)$ gauge

$$(4\theta_{x_{\tau_2}} - 4)(4\theta_{x_{\tau_2}} - 3)(4\theta_{x_{\tau_2}} - 2)(4\theta_{x_{\tau_2}} - 1), \qquad \theta_{y_{\tau_2}}^2 , \qquad (3.34)$$

with $x_{\tau_2} = a^{-l^{(1)}} = 1/x_{\tau_1}$, $y_{\tau_2} = a^{l^{(2)}}$. From these principal parts, we see that not all solutions to the indicial equations at $x_{\tau_2} = y_{\tau_2} = 0$ are zero. Thus the local properties of τ_1 and τ_2 are quite different. The fact that $LT_\omega(\mathscr{I}_{\mathscr{A}})$ is radical in τ_1 but not in τ_2 is responsible for this difference.

3.4. Cohomology ring of X_Δ and the local solutions. We consider the restriction map $H^*(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{Z}) \to H^*(X_\Delta, \mathbf{Z})$ induced by the inclusion $X_\Delta \to \mathbf{P}_{\Sigma(\Delta^*)}$, and denote the image by $H^*_{\text{toric}}(X_\Delta, \mathbf{Z})$. The restriction map can be realized by considering the intersection of the elements of $H^*(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{Z})$ with the divisor X_Δ. By construction of the Calabi–Yau hypersurface X_Δ, the divisor class $[X_\Delta]$ coincides with the anti-canonical class of the ambient space $\mathbf{P}_{\Sigma(\Delta^*)}$, namely

$$[X_\Delta] = D_1 + D_2 + \cdots + D_p , \qquad (3.35)$$

in the intersection ring. The toric part of the cohomology $H^*_{\text{toric}}(X_\Delta, \mathbf{Z})$ can then be written as $A^*_{\text{toric}}(X_\Delta, \mathbf{Z}) = A^*(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{Z})/\text{Ann}(D_1 + \cdots + D_p)$ (where $\text{Ann}(x)$ in a ring R is $\text{Ann}(x) := \{y \in R \mid yx = 0\}$) or equivalently

$$A^*_{\text{toric}}(X_\Delta, \mathbf{Z}) = \mathbf{Z}[D_1, D_2, \ldots, D_p]/\mathscr{I}_{\text{quot}} , \qquad (3.36)$$

where $\mathscr{I}_{\text{quot}}$ is the ideal quotient $\mathscr{I}_{\text{quot}} = \mathscr{I} : (D_1 + \cdots + D_p)$. (Here $(I : x) = \{y \in R \mid yx \in I\}$.)

Now recall the close relationship between the ideal $\tilde{\mathscr{I}}$ in (3.23) and the ideal of the symbols for the Δ^*-hypergeometric system with respect to the cone τ of maximal triangulations T_ω. First we have $\mathscr{I}_{\text{quot}} = \tilde{\mathscr{I}} : (D_1 + D_2 + \cdots + D_p) = \tilde{\mathscr{I}} : (-D_0) = \tilde{\mathscr{I}} : D_0$. In fact we observe more: suitable linear combinations of the differential operators $a^{l\pm}\mathscr{D}_l$ factorize from the left by the operator θ_{a_0}, implying that the hypergeometric system is a reducible system. Factorization by the operator θ_{a_0} should be understood as corresponding to the restriction to the hypersurface X_Δ. As we shall see, the solutions to the factorized system can be obtained from (3.28) by a similar restriction of the intersection couplings (cf. $m(-\theta_{a_0}) = -D_0 = [X_\Delta]$).

In cases of type II models, we observed that the quotient (3.36) results in setting to zero the divisors D_i for which the integral points v_i^* are on a codimension one face of Δ^*. This can be understood as follows: the above divisors come from the desingularizations of point singularities of the ambient space; a hypersurface X_Δ in general position will not meet these singularities. In accordance with this "decoupling" of the divisors it is natural to consider the lattice $L' = \{l \in L \mid l_i = 0,\ v_i^*$ is on a codimension one face of $\Delta^*\}$. We define a $\mathbf{Z}$-basis $\{l_\tau^{(1)}, \ldots, l_\tau^{(p-n')}\}$ ($n' = \dim \mathrm{Auto}(\mathbf{P}_{\Sigma(\Delta^*)})$) of L' of the reduced cone $(\mathbf{R}_{\geq 0} l_\tau^{(1)} + \cdots + \mathbf{R}_{\geq 0} l_\tau^{(p-n)}) \cap L'_\mathbf{R}$ as follows. We make subdivisions of the reduced cone if it is not regular, in which case the $\mathbf{Z}$-basis is not uniquely determined. However our observations in the following do not depend on this. We will call L' the *reduced lattice*, or the reduction of L.

The decoupling of some divisor D_i in the intersection ring implies that we can turn off the monomial deformation via a_i (which corresponds to the divisor D_i under the monomial-divisor map [43]). In fact we observe that these variables can be eliminated in the extended Δ^*-hypergeometric system which is originally defined to act on functions on $\mathbf{C}^{\mathscr{A}}$ as follow. Recall that the GKZ Δ^*-hypergeometric system is enlarged by adjoining $n' - n$ additional linear differential operators $\mathscr{Z}_i$ ($i = n+1, \ldots, \dim \mathrm{Auto}(\mathbf{P}_{\Sigma(\Delta)}) = n'$ in (2.21)). This creates just enough equations to eliminate those operators $\frac{\partial}{\partial a_i}$ corresponding to points v_i^* on the codimension one faces, from the operators $\mathscr{D}_l$ ($l \in L$). We may then set $a_i = 0$ after the elimination. We denote by $\mathscr{D}'_l$ the resulting new operators which act on functions on $\mathbf{C}^{\mathscr{A}'}$, where the set $\mathscr{A}'$ consists of all integral points on the faces with codimension greater than one. Note that the set $\{\mathscr{D}'_l \mid l \in L\}$ is in general larger than the set $\{\mathscr{D}_{l'} \mid l' \in L'\}$.

We now define the intersection couplings on $A^*_{\mathrm{toric}}(X_\Delta, \mathbf{Z})$ by

$$K^{cl}_{i_1 i_2 \cdots i_{n-1}} = \langle m(\theta_{x_\tau^{(i_1)}}) m(\theta_{x_\tau^{(i_2)}}) \cdots m(\theta_{x_\tau^{(i_{n-1})}}) \cdot m(-\theta_{a_0}) \rangle, \qquad (3.37)$$

then we may state the observation given in [10] as follows:

For a cone τ with typical weight ω, some of the operators $a^{l\pm} \mathscr{D}'_l (l \in L)$ or their linear combinations factorize by the operator θ_{a_0} from the left, indicating that the Δ^*-hypergeometric system is reducible. If T_ω is a maximal triangulation, the local solutions about the point $x_\tau^{(i)} = 0$ ($i = 1, \ldots, p - n'$) for the reduced system are given by

$$w_0(x_\tau, \rho)|_{\rho=0}, \quad \partial_{\rho_i} w_0(x_\tau, \rho)|_{\rho=0}, \quad \sum_{i_1, i_2} K^{cl}_{i_1 i_2 \cdots i_{n-1}} \partial_{\rho_{i_1}} \partial_{\rho_{i_2}} w_0(x_\tau, \rho)|_{\rho=0},$$

$$\cdots \qquad \qquad (3.38)$$

$$\sum_{i_1, i_2, \cdots i_{n-1}} K^{cl}_{i_1 i_2 \cdots i_{n-1}} \partial_{\rho_{i_1}} \partial_{\rho_{i_2}} \cdots \partial_{\rho_{i_{n-1}}} w_0(x_\tau, \rho)|_{\rho=0}.$$

We also observe that in the case of Fermat hypersurfaces, the operators $a^{l\pm} \mathscr{D}'_l$ which factorize whose leading term generate the ideal $\bar{\mathscr{I}}_{\mathrm{quot}}$ can be constructed from operators $\mathscr{D}_l$ in the Gröbner basis $\mathscr{B}_\omega$. However for general models of non-Fermat type we need to consider operators $a^{l\pm} \mathscr{D}'_l$ outside the basis $\mathscr{B}_\omega$ as well as their linear combinations with coefficients in the ring generated by the θ_{a_i} (see examples in Sect. 4).

Examples. 1) $\mathbf{P}(2,2,2,1,1)$. We have seen a unique maximal triangulation T_0 in (3.17) and have constructed local solutions for the corresponding cone in the

Gröbner fan. Now we note that $\theta_{a_0} = -(\theta_{a_1} + \cdots + \theta_{a_6}) = -4\theta_{x_{\tau_1}}$. It is easy to observe that the operator $a^{l_+^{(1)}}\mathscr{D}_{l^{(1)}}$ expressed in the $\tilde{\Pi}(a)$ gauge factorizes from the left by $\theta_{a_0} = -4\theta_{x_{\tau_1}}$, i.e., $\mathscr{L}_1 = \theta_{x_{\tau_1}}\mathcal{O}$ in (3.30) for some third order operator $\mathcal{O}$. If we write the divisor $m(\theta_{x_{\tau_1}})$ as J_x and similarly for J_y, the topological data for the solutions are summarized as follows:

$$K_{xxx}^{cl} = 8, \qquad K_{xxy}^{cl} = 4, \qquad K_{xyy}^{cl} = K_{yyy}^{cl} = 0 ,$$
$$c_2 \cdot J_x = 56, \qquad c_2 \cdot J_y = 24 , \tag{3.39}$$

where the invariants $c_2 \cdot J$'s are listed for later use. For their calculation we use the adjunction formula [44]; $c(X_\Delta) = \prod_{i=1}^{P}(1 + D_i)/(1 + [X_\Delta])$.

2) $\mathbf{P}(7, 2, 2, 2, 1)$. The toric data of this model have been summarized in the end of the previous section. Although this model has the same moduli as the above model, two integral points on the codimension one face make the combinatorics of this model much more complicated. It turns out that there are 14 elementary relations which generate the zonotope $\mathscr{P}_{\mathscr{A}}$, and there are more than 2,000 elements for the universal Gröbner basis. The secondary fan has 32 four dimensional cones, most of which are singular.

It seems to be a formidable task to determine the Gröbner fan, however it is easy to find the maximal triangulation of Δ^* and the corresponding Stanley–Reisner ideal. As proved in the previous subsection, for a weight ω such that T_ω is the maximal triangulation, we have $LT_\omega(\mathscr{I}_{\mathscr{A}}) = \mathrm{rad}(LT_\omega(\mathscr{I}_{\mathscr{A}})) = SR_{T_\omega}$ and we can determine the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$ by the Stanley–Reisner ideal which is simply described by the primitive collections. In this case, it turns out that the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}}) = SR_{T_\omega}$ is generated by

$$\begin{array}{cccccc} y_1 y_5, & y_1 y_7, & y_1 y_8, & y_5 y_8, & y_6 y_7, & y_6 y_8 , \\ & y_2 y_3 y_4 y_5, & y_2 y_3 y_4 y_6, & y_2 y_3 y_4 y_7 . & & \end{array} \tag{3.40}$$

These generators may be translated into the generators (i)$'$ in (3.23). Then together with the linear relations (ii)$'$ in (3.23), they define the intersection ring of the ambient space. We find that the ideal $\bar{\mathscr{I}}_{\mathrm{quot}} = \mathscr{I} : (D_1 + \cdots + D_8)$ as defined earlier is generated by the monomials

$$D_1 D_5, \qquad D_3 D_4 D_6, \qquad D_7, \qquad D_8 , \tag{3.41}$$

with the linear relations (ii)$'$. The divisors D_7 and D_8 in (3.41) being among the generators show that these divisors decouple from the intersection ring.

We find that a $\mathbf{Z}$-basis of L for a cone τ with the typical weight ω is $\{l_\tau^{(1)}, l_\tau^{(2)}, l_\tau^{(3)}, l_\tau^{(4)}\}$ with

$$l_\tau^{(1)} = (-1, 0, 0, 0, 0, -1, 1, 1, 0), \qquad l_\tau^{(2)} = (0, 1, 0, 0, 0, 1, -2, 0, 0) ,$$
$$l_\tau^{(3)} = (0, 0, 1, 1, 1, 0, 0, 1, -4), \qquad l_\tau^{(4)} = (0, 0, 0, 0, 0, 1, 0, -2, 1) . \tag{3.42}$$

The intersection of the cone generated by (3.42) with $L_\mathbf{R}'$ is a two dimensional regular cone generated by $l^{(1)} = 7l_\tau^{(1)} + l_\tau^{(3)} + 4l_\tau^{(4)} = (-7, 0, 1, 1, 1, -3, 7, 0, 0)$ and $l^{(2)} = l_\tau^{(2)}$.

The form of our generators of $\bar{\mathscr{I}}_{\text{quot}}$ suggests that we should try to factorize the following differential operators:

$$\mathscr{D}_{l_{\{1,5\}}} = \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_5} - \left(\frac{\partial}{\partial a_6}\right)^2,$$

$$\mathscr{D}_{l_{\{2,3,4,6\}}} = \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_6} - \frac{\partial}{\partial a_0} \left(\frac{\partial}{\partial a_8}\right)^3,$$

(3.43)

where $l_{\{1,5\}}$ and $l_{\{2,3,4,6\}}$ are the primitive relations corresponding respectively to the primitive collections $\{v_1^*, v_5^*\}$ and $\{v_2^*, v_3^*, v_4^*, v_6^*\}$. Although the second operator contains a derivative with respect to a_8, we can eliminate it using the order one operators $\mathscr{Z}_{\xi_1}$ and $\mathscr{Z}_{\xi_2}$ corresponding to the automorphisms (2.21). Defining the local variables $x = -a^{l^{(1)}}, y = a^{l^{(2)}}$, we observe the factorization of the operator θ_{a_0} in $a_0 a_2 a_3 a_4 a_6 \mathscr{D}_{l_{\{2,3,4,6\}}} \frac{1}{a_0}$, and find a complete set of differential equations for the period integrals:

$$\mathscr{D}_1 = (\theta_y - 3\theta_x)\theta_y - y(7\theta_x - 2\theta_y - 1)(7\theta_x - 2\theta_y),$$

$$\mathscr{D}_2 = \theta_x^2(7\theta_x - 2\theta_y) - 7x(y(28\theta_x - 4\theta_y + 18) + \theta_y - 3\theta_x - 2))$$

(3.44)

$$\times (y(28\theta_x - 4\theta_y + 10) + \theta_y - 3\theta_x - 1)(y(28\theta_x - 4\theta_y + 2) + \theta_y - 3\theta_x),$$

in the $\tilde{\Pi}(a)$ gauge (2.14). The local solutions of this system are given by (3.38) with the following topological data:

$$K_{xxx}^{cl} = 2, \qquad K_{xxy}^{cl} = 7, \qquad K_{xyy}^{cl} = 21, \qquad K_{yyy}^{cl} = 63,$$

$$c_2 \cdot J_x = 44, \qquad c_2 \cdot J_y = 126.$$

(3.45)

3.5. Singular models of type III. In the previous subsections, we have considered the non-singular models, i.e., models of type I and II in our classification (2.16). However in actual applications, singular models dominate the others. We will see, nevertheless, that several properties observed in the previous subsections apply with some modifications even to the singular cases.

Since a complete analysis of the secondary (Gröbner) fan for Δ^* is formidable in general (cf. the example $P(7,2,2,2,1)$), we focus only on the Calabi–Yau phase(s) which corresponds to maximal triangulation(s). For the nonsingular models of type I and II, we have seen that the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$ for a maximal triangulation T_ω coincides with the Stanley–Reisner ideal. For the singular models of type III however, the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$ differs from its radical and from the Stanley–Reisner ideal because $\Sigma(\Delta^*)$ is no longer regular.

For a singular model, the fan $\Sigma(\Delta^*)$ is singular even relative to the maximum subdivision incorporating all integral points in Δ^*. To obtain a regular fan, which we denote as $\Sigma(\Delta^*)_{\text{reg}}$, we subdivide further the singular cones taking into account integral points *outside* the polyhedron Δ^*. Since the polyhedron Δ^* is reflexive, the integral points which generate an n-dimensional cone in $\Sigma(\Delta^*)$ are on a hyperplane with integral distance one from the origin [19]. Moving this hyperplane in a parallel way to the integral points outside Δ^*, we can speak of the integral distance of these points. For the hypersurfaces $X_d(w)$ in (2.1), a point with the integral distance $k > 0$ corresponds to a monomial of the homogeneous degree kd.

Let us denote all the integral points generating the one dimensional cones of $\Sigma(\Delta^*)_{\text{reg}}$ as $\{v_1^*, \ldots, v_p^*, v_{p+1}^*, \ldots, v_q^*\}$, where $v_{p+1}^*, \ldots, v_q^*$ are those new points introduced by the subdivision. (Note that even though the new points have distance greater than 1, they are still primitive vectors of the lattice.) Since we have a non-singular toric variety $\mathbf{P}_{\Sigma(\Delta^*)_{\text{reg}}}$, we can describe its intersection ring according to (3.22) with additional divisors $D_{p+1}, \ldots, D_q$. It turns out that the divisor class of the Calabi–Yau hypersurface X_Δ in this fully resolved ambient space is given by

$$[X_\Delta] = D_1 + \cdots + D_p + d_{p+1}D_{p+1} + \cdots + d_q D_q , \qquad (3.46)$$

where d_k is the integral distance of the point v_k^* described above. We should note that the regular fan $\Sigma(\Delta^*)_{\text{reg}}$ need not be the fan associated with a triangulation of the polyhedron $\Delta'^* := \text{Conv.}(\{v_1^*, \ldots, v_p^*, v_{p+1}^*, \ldots, v_q^*\})$. Therefore in general, we do not have a description (3.22) of the intersection ring via the Stanley–Reisner ideal in terms of a triangulation of Δ'^*. However in many cases, it happens that the convex hull Δ'^* is itself a reflexive polyhedron. In such a case we have another family of Calabi–Yau manifolds $X_{\Delta'}$ in the ambient space $\mathbf{P}_{\Sigma(\Delta'^*)}$. This ambient space is in general different from $\mathbf{P}_{\Sigma(\Delta^*)_{\text{reg}}}$. However if we have the relation $\Sigma(\Delta'^*) = \Sigma(\Delta^*)_{\text{reg}}$, then we will have two different families of Calabi–Yau hypersurfaces in the same ambient space $\mathbf{P}_{\Sigma(\Delta'^*)} = \mathbf{P}_{\Sigma(\Delta^*)_{\text{reg}}}$. One hypersurface X_Δ represents the divisor class (3.46) and the other hypersurface $X_{\Delta'}$ represents

$$[X_{\Delta'}] = D_1 + \cdots + D_p + D_{p+1} + \cdots + D_q . \qquad (3.47)$$

We will see an example of this type in Sect. 4.

Now let us see the detailed analysis in a typical example $X_{12}(4, 3, 2, 2, 1)$ which was analyzed in [8]. The polyhedron $\Delta(w)$ for this model has vertices

$$v_1 = (2, -1, -1, -1), \quad v_2 = (-1, 3, -1, -1), \quad v_3 = (-1, -1, 5, -1),$$
$$v_4 = (-1, -1, -1, 5), \quad v_2 = (-1, -1, -1, -1) , \qquad (3.48)$$

with respect to the basis $\Lambda_1, \ldots, \Lambda_4$ for the lattice $H(w)$ as in (2.17). The integral points in the dual polyhedron $\Delta^*(w)$ are as

$$v_1^* = (1, 0, 0, 0), \quad v_2^* = (0, 1, 0, 0), \quad v_3^* = (0, 0, 1, 0),$$
$$v_4^* = (0, 0, 0, 1), \quad v_5^* = (-4, -3, -2, -2), \quad v_6^* = (-2, -1, -1, -1) , \qquad (3.49)$$

together with the origin $v_0^* = (0, 0, 0, 0)$. The maximal triangulation of the polyhedron $\Delta^*(w)$ is unique and is given by

$$\begin{aligned}
T_0 = \{ &\langle 0, 3, 4, 5, 6 \rangle, \langle 0, 1, 3, 4, 5 \rangle, \langle 0, 2, 3, 4, 6 \rangle, \langle 0, 1, 4, 5, 6 \rangle, \\
&\langle 0, 1, 3, 5, 6 \rangle, \langle 0, 1, 2, 4, 6 \rangle, \langle 0, 1, 2, 3, 6 \rangle, \langle 0, 1, 2, 3, 4 \rangle \} .
\end{aligned} \qquad (3.50)$$

It is easy to see that the corresponding fan $\Sigma(\Delta^*)$ is not regular because the first three simplices in T_0 respectively have volumes $2, 3$, and 2. We subdivide the first cone by introducing a point $v_7^* = \frac{v_3^* + v_4^* + v_5^* + v_6^*}{2}$. Similarly by introducing $v_8^* = \frac{2v_1^* + v_4^*}{3} + \frac{v_3^* + 2v_5^*}{3}$ and $v_9^* = \frac{v_1^* + 2v_4^*}{3} + \frac{2v_3^* + v_5^*}{3}$ for the second cone and $v_{10}^* = \frac{v_2^* + v_3^* + v_4^* + v_6^*}{2}$ for the third cone, we finally obtain the regular fan $\Sigma(\Delta^*)_{\text{reg}}$. All these additional points $v_7^*, \ldots, v_{10}^*$ have integral distance two and correspond to the charge two monomials

$z_2 z_3^3 z_4^3 z_5^9$, $z_1^2 z_3^2 z_4^2 z_5^8$, $z_1 z_3^4 z_4^4 z_5^4$ and $z_2^3 z_3^3 z_4^3 z_5^3$, respectively. The generators (i) in (3.22) are determined by the primitive collections for the fan $\Sigma(\Delta^*)_{\text{reg}}$, and there are 20 such generators. Together with the linear generators (ii) in (3.22), these determine the defining ideal $\mathcal{I}$ for the intersection ring $A^*(\mathbf{P}_{\Sigma(\Delta^*)_{\text{reg}}})$. The ideal quotient by $[X_\Delta] = D_1 + \cdots + D_6 + 2(D_7 + \cdots + D_{10})$ determines $A^*_{\text{toric}}(X_\Delta)$. It turns out that $\mathcal{I}_{\text{quot}}$ is generated by

$$D_2 D_5, \qquad D_1 D_3 D_6, \qquad D_7, \qquad D_8, \qquad D_9, \qquad D_{10}, \qquad (3.51)$$

together with the linear relations (ii). The generators D_7, D_8, D_9 and D_{10} indicate that these divisors decouple from the intersection ring. This can be understood as follows: the additional points $v_7^*, \ldots, v_{10}^*$ represent point singularities in the ambient space and the divisors $D_7, \ldots, D_{10}$ resulting from the desingularization of these points do not intersect with the hypersurface X_Δ in the general position.

Now let us turn to the set of the convex piecewise linear functions over the fan $\Sigma(\Delta^*)_{\text{reg}}$, i.e., the Kähler cone of $\mathbf{P}_{\Sigma(\Delta^*)_{\text{reg}}}$ (see [37]). Since the regular fan $\Sigma(\Delta^*)_{\text{reg}}$ does not come from any triangulation of the polyhedron Δ'^* (in fact we verify $\Sigma(\Delta^*)_{\text{reg}}$ has 21 four dimensional regular cones whereas $\text{vol}(\Delta'^*) = 24$), the Kähler cone so obtained cannot be interpreted as a cone of the secondary fan for Δ'^*. It is straightforward to find a $\mathbf{Z}$-basis for the dual cone of Kähler cone τ and we have

$$l_\tau^{(1)} = (-1, 0, 0, 1, 1, 0, 0, 0, 1, -2, 0), \qquad l_\tau^{(2)} = (-1, 1, 0, 0, 0, 1, 0, 0, -2, 1, 0),$$

$$l_\tau^{(3)} = (-2, 0, 0, 1, 1, 1, 1, -2, 0, 0, 0), \qquad l_\tau^{(4)} = (1, 0, 0, -1, -1, -1, 0, 1, 0, 1, 0),$$

$$l_\tau^{(5)} = (-2, 0, 1, 1, 1, 0, 1, 0, 0, 0, -2), \qquad l_\tau^{(6)} = (2, 0, 0, -1, -1, 0, -2, 1, 0, 0, 1).$$

$$(3.52)$$

The decoupling of the divisors $D_7, \ldots, D_{10}$ in (3.51) corresponds to reducing from L to the lattice L' generated by $l^{(1)} = 4l_\tau^{(1)} + 2l_\tau^{(2)} + 3l_\tau^{(3)} + 3l_\tau^{(4)}$ and $l^{(2)} = l_\tau^{(3)} + l_\tau^{(5)} + 2l_\tau^{(6)}$ with

$$l^{(1)} = (-6, 2, 0, 1, 1, -1, 3, 0, 0, 0, 0), \qquad l^{(2)} = (0, 0, 1, 0, 0, 1, -2, 0, 0, 0, 0). \quad (3.53)$$

We verify that the above basis for the reduced lattice generates the cone $\mathcal{K}(\mathcal{A}, T_0)$ dual to $\mathcal{C}'(\mathcal{A}, T_0)$ for the maximal triangulation T_0 of Δ^*. However this is not a general phenomenon as we will see in the example $X_{14}(7, 3, 2, 1, 1)$ presented in Sect. 4.

The operators $\mathcal{D}_l$ we deduce from the first two of (3.51) are

$$\mathcal{D}_{l_{\{2,5\}}} = \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_5} - \left(\frac{\partial}{\partial a_6} \right)^2,$$

$$\mathcal{D}_{l_{\{1,3,4,6\}}} = \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_6} - \left(\frac{\partial}{\partial a_0} \right)^4 \frac{\partial}{\partial a_9},$$

$$(3.54)$$

where $l_{\{2,5\}} = l_\tau^{(3)} + l_\tau^{(5)} + 2l_\tau^{(6)}$ and $l_{\{1,3,4,6\}} = 2l_\tau^{(1)} + l_\tau^{(2)} + l_\tau^{(3)} + 2l_\tau^{(4)}$ are primitive relations for $\Sigma(\Delta^*)_{\text{reg}}$. These two operators are the analogues of (3.43) of the nonsingular model, but with one crucial difference. In this singular case, we do not have an order one differential operator in the extended Δ^*-hypergeometric system

to eliminate $\frac{\partial}{\partial a_9}$. In order to eliminate this we must study the Jacobian ring of the hypersurface in detail. In [8], a second order operator was found which has the form

$$\frac{\partial}{\partial a_0}\frac{\partial}{\partial a_9} = \frac{12a_1a_2}{a_0^2}\left(\frac{\partial}{\partial a_0}\right)^2 - \frac{24a_1a_2a_6}{a_0^3}\frac{\partial}{\partial a_0}\frac{\partial}{\partial a_6} - \frac{12a_1a_6^2}{a_0^3}\frac{\partial}{\partial a_0}\frac{\partial}{\partial a_5}, \qquad (3.55)$$

when acting on the period $\Pi(a)$, see Eq. (3.39) in [8]. Using this relation and the definition $x = a^{l^{(1)}}$ and $y = a^{l^{(2)}}$, a third order differential operator is derived from $\mathscr{D}_{l_{\{1,3,4,6\}}}$ after a factorization θ_x from the left. As is evident, the linear differential operators represent relations among the monomial with the homogeneous degree d or the charge one in the Jacobian ring. In contrast, the differential operator (3.55) represents a relation among the monomials of charge two. While the order one differential operators have been related to the symmetry of the period under automorphisms and thus to the combinatorial data of the polyhedron Δ^*, the form of the operators for the charge two monomials above do not have a clear description in terms of the combinatorial data. This is a typical feature we encounter in the analysis of the singular models. We observe that despite having to use charge two operators to factorize $\mathscr{D}_l$, the principal part of the factorized operators still coincide with those monomial generators of the defining ideal $\mathscr{I}_{\text{quot}}$ for $A^*_{\text{toric}}(X_\Delta)$ – just as in the case of type I, II models. This means that the structure of the local solutions is not affected by the usage of the charge two operators. That is, the same properties in (3.38) hold for type III models as they do for type I and II models. In our example here, the local solutions are described by the following topological data:

$$K^{cl}_{xxx} = 2, \qquad K^{cl}_{xxy} = 3, \qquad K^{cl}_{xyy} = 3, \qquad K^{cl}_{yyy} = 3,$$
$$c_2 \cdot J_x = 32, \qquad c_2 \cdot J_y = 42. \qquad (3.56)$$

We verify that the convex hull of the points $\{v_1^*, \ldots, v_{10}^*\}$ is again reflexive and defines a family of Calabi–Yau manifolds $X_{\Delta'}$ with Hodge numbers (2.5) $h^{1,1}(X_{\Delta'}) = 6$ and $h^{2,1}(X_{\Delta'}) = 71$. Thus this is a case in which a polyhedron Δ^* results in topologically distinct Calabi–Yau manifolds $X_{\Delta'}$ and X_Δ sitting inside two distinct ambient spaces (because $\Sigma(\Delta^*)_{\text{reg}} \neq \Sigma(\Delta'^*)$ as we have seen). In fact $X_{\Delta'}$ is not even in the list of 7,555 Laudau–Ginzburg models of [25].

Finally let us calculate the Stanley–Reisner ideal for the triangulation T_0 in (3.50). It is straightforward to see that the ideal is generated by

$$D_2D_5, \qquad D_1D_3D_4D_6. \qquad (3.57)$$

Since the model (or the fan $\Sigma(\Delta^*)$) is only simplicial but not regular, the odd homology groups of the singular toric variety can have torsion. Thus we consider the homology groups over $\mathbf{Q}$. Then the groups are given by the intersection ring (3.22) $\mathscr{A}^*(\mathbf{P}_{\Sigma(\Delta^*)}, \mathbf{Q})$ over $\mathbf{Q}$. Thus it is $\mathbf{Q}[D_1, \ldots, D_6]/\mathscr{I}$ with the ideal $\mathscr{I}$ generated by (3.57) and the linear relations among the vertices $\{v_1^*, \ldots, v_6^*\}$ as in (ii) of (3.22) [38]. The normalization of the "volume form" of this ring becomes less clear because the Euler number of the singular $\mathbf{P}_{\Sigma(\Delta^*)}$ is not given simply by the number of the cones with maximal dimensions in $\Sigma(\Delta^*)$. However we know that the hypersurface X_Δ in the general position does not meet the point

singularities of the $\mathbf{P}_{\Sigma(\varDelta^*)}$, and the hypersurface divisor class is given by $[X_\varDelta] = D_1 + \cdots + D_6$. Therefore we naturally introduce a normalization of $\mathscr{A}^*(\mathbf{P}_{\Sigma(\varDelta)}, \mathbf{Q})$ using the Euler number of the hypersurface, rather than that of the singular ambient space:

$$\frac{\prod_{i=1}^p (1 + D_i)}{(1 + [X_\varDelta])}[X_\varDelta]\big|_{\text{top}} = 2(h^{1,1}(X_\varDelta) - h^{2,1}(X_\varDelta)). \tag{3.58}$$

Here we evaluate the component of the top degree on the left-hand side and we use the Hodge numbers $h^{1,1}(X_\varDelta)$ and $h^{2,1}(X_\varDelta)$ in (2.5). In the left-hand side, we adopt the expression $\prod(1 + D_i)$ for the total Chern class [38] which is justified for the nonsingular $\mathbf{P}_{\Sigma(\varDelta^*)}$, but naively extended to our singular case. We have verified experimentally that the normalization (3.58) indeed results in the right topological couplings and the linear form $c_2 \cdot J$'s. We may summarize our observation in general,

> For a smooth Calabi–Yau models $X_\varDelta$ in a singular toric variety $\mathbf{P}_{\Sigma(\varDelta^*)}$, the intersection ring $\mathscr{A}^*_{\text{toric}}(X_\varDelta, \mathbf{Q})$ is given by $\mathscr{A}^*(\mathbf{P}_{\Sigma(\varDelta^*)}, \mathbf{Q})/ \text{Ann}(D_1 + \cdots + D_p)$ with the normalization determined by (3.58). (3.59)

The effect of taking the quotient by $\text{Ann}(D_1 + \cdots + D_p)$ may be replaced by the ideal quotient $\mathscr{I}_{\text{quot}} = \mathscr{I} : (D_1 + \cdots + D_p)$ as in the nonsingular case. In our example, it is easy to derive the first two of (3.51) from (3.57) via the ideal quotient.

Finally we note that all notions in the theory of toric ideals apply to the singular cases as well. Therefore it would be helpful to compare the Gröbner fan of a singular model with that of a nonsingular model. By an analysis similar to (3.15), we obtain the following elementary relations for the model $\mathbf{P}(4, 3, 2, 2, 1)$:

$$l^{(1)} = (-6, 2, 0, 1, 1, -1, 3), \qquad l^{(2)} = (0, 0, 1, 0, 0, 1, -2),$$
$$l^{(3)} = (-6, 2, 1, 1, 1, 0, 1), \qquad l^{(4)} = (-12, 4, 3, 2, 2, 1, 0). \tag{3.60}$$

The universal Gröbner basis are determined from the zonotope $\mathscr{P}_{\mathscr{A}}$ as

$$\mathscr{U}_{\mathscr{A}} = \{y_1^8 y_2^5 y_3^4 y_4^4 y_5 y_6^2 - y_0^{24}, y_1^6 y_2^4 y_3^3 y_4^3 y_5 y_6 - y_0^{18}, y_1^4 y_2 y_3^2 y_4^2 y_6^4 - y_5 y_0^{12},$$
$$y_1^6 y_2^5 y_3^3 y_4^3 y_5^2 - y_0^{18} y_6, y_1^4 y_2^3 y_3^2 y_4^2 y_5 - y_0^{12} y_6, y_1^2 y_3 y_4 y_6^2 - y_0^6 y_5,$$
$$y_1^2 y_2 y_3 y_4 y_6 - y_0^6, y_1^2 y_2^2 y_3 y_4 y_5 - y_0^6 y_6, y_2 y_5 - y_6^2\}. \tag{3.61}$$

In Table 2, we present the cones in the Gröbner fan with the ideals of the leading terms. There the cone τ_1 corresponds to the maximal triangulation T_0 (3.50) and should be compared with τ_1 in Table 1. The difference we should note is that the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$ is not radical and does not coincide with the Stanley–Reisner ideal ST_ω. To see the consequence of this fact, recall that the generators of the ideal $LT_\omega(\mathscr{I}_{\mathscr{A}})$ may be mapped to the symbol of the differential operators $\mathscr{D}_l$ by (3.19). As we see in the Table 2 explicitly, we simply obtain higher order differential operators rather than (3.54).

Table 2. Gröbner cones with typical weights for $\mathbf{P}(4,3,2,2,1)$. The first cone τ_1 corresponds to the maximal triangulation T_0 in the text

cone	weight ω	$LT_\omega(\mathcal{I}_{\mathscr{A}})$	$\mathrm{rad}(LT_\omega(\mathcal{I}_{\mathscr{A}}))$
τ_1	$(0,1,1,0,0,0,0)$	$\langle y_1^2 y_2 y_3 y_4 y_6,\ y_1^2 y_3 y_4 y_6^3,\ y_2 y_5 \rangle$	$\langle y_1 y_3 y_4 y_6,\ y_2 y_5 \rangle$
τ_2	$(0,0,2,0,0,1,0)$	$\langle y_1^2 y_2 y_3 y_4 y_6,\ y_0^6 y_5,\ y_2 y_5 \rangle$	$\langle y_1 y_2 y_3 y_4 y_6,\ y_0 y_5,\ y_2 y_5 \rangle$
τ_3	$(1,0,0,0,0,1,0)$	$\langle y_2 y_5,\ y_0^6 \rangle$	$\langle y_2 y_5,\ y_0 \rangle$
τ_4	$(1,0,0,0,0,0,1)$	$\langle y_0^2,\ y_6^2 \rangle$	$\langle y_0,\ y_6 \rangle$
τ_5	$(1,0,0,0,0,0,7)$	$\langle y_1^2 y_2 y_3 y_4 y_6,\ y_6^2,\ y_0^6 y_6,\ y_0^{12} \rangle$	$\langle y_0,\ y_6 \rangle$
τ_6	$(0,0,0,0,0,1,3)$	$\langle y_6^2,\ y_0^6 y_6,\ y_1^2 y_2 y_3 y_4 y_6,\ y_1^4 y_2^3 y_3^2 y_4^2 y_5 \rangle$	$\langle y_6,\ y_1 y_2 y_3 y_4 y_5 \rangle$
τ_7	$(0,1,0,0,0,1,2)$	$\langle y_6^2,\ y_1^2 y_2 y_3 y_4 y_6,\ y_1^2 y_2^2 y_3 y_4 y_5 \rangle$	$\langle y_6,\ y_1 y_2 y_3 y_4 y_5 \rangle$

4. Applications to Mirror Symmetry and Mirror Map

In the application to mirror symmetry, the secondary fan can be regarded as a collection of different phases of a type II string theory compactified on a Calabi–Yau manifold (see for example [45, 46]). The triangulations of Δ^* which induce different subdivisions of the fan $\Sigma(\Delta^*)$, and their corresponding cones in the secondary fan are known to have a clear physical meaning in terms of orbifold as well as the smooth Calabi–Yau manifold. Among them, the maximal triangulations of Δ^* or the finest refinements of the fan $\Sigma(\Delta^*)$ constitute the Calabi–Yau phase. In this phase we have the large radius limit of the smooth Calabi–Yau manifold where the non-perturbative instanton corrections are suppressed exponentially. The structure observed in (3.38) is consistent with the quantum cohomology ring near the large radius limit.

In this section, we use several models to show how our general framework applies.

4.1. Quantum cohomology ring. Quantum cohomology ring is one of the nontrivial consequences of the local operator algebra of the type II string theory compactified on a Calabi–Yau threefold. In $N = 2$ string theory, two different kinds of the local topological operator algebras, called (a, c)- and (c, c)-ring, correspond respectively to the $H^{1,1}$-type cohomology and the $H^{2,1}$-type cohomology in the topological σ-model [47, 48]. On physical ground, the (a, c)-ring receives quantum corrections from σ-model instantons whereas the (c, c)-ring does not [21]. Mirror symmetry which exchanges the two provides a powerful hypothesis to determine the quantum cohomology ring in terms of the (c, c)-ring:

$$\bigoplus_{i=0}^{3} H_q^{i,i}(X_\Delta) \cong \bigoplus_{i=0}^{3} H^{3-i,i}(X_{\Delta^*,a}),\qquad (4.1)$$

where q in the left-hand side represents the quantum deformation and a in the right-hand side represents the classical deformation of the mirror hypersurface in (2.4). More precisely, we may regard the right-hand side as the Jacobian ring of the mirror hypersurface and we can use the theory of variation of Hodge structures to study this side. The isomorphism can then be realized in terms of the flat coordinates on moduli spaces. This map is called the mirror map. It is known that the mirror

map has many remarkable properties such as modular property, integrality in the q-expansion, etc. [13].

In the classical limit, the instanton corrections in the quantum cohomology ring are exponentially suppressed. The monodromy of the periods near the limit is maximally unipotent [2]. This is the property we established in general in (3.38) for any maximal triangulation of the polyhedron Δ^* of type I or II. It is found in [8, 10] that if we define the local variables via the basis $\{l^{(k)}\}$ of the Mori cone by

$$x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}} , \tag{4.2}$$

then we may express the mirror map as

$$t_j = \frac{1}{2\pi i} \frac{\partial_{\rho_j} w_0(x,0)}{w_0(x,0)} , \tag{4.3}$$

where $q_j = e^{2\pi i t_j}$. The inverse map is denoted as $x_k = x_k(q)$. The quantum couplings are related to the geometrical couplings, $K_{ijk}(x) := \int \Omega(x) \wedge \partial_i \partial_j \partial_k \Omega(x) - \Omega(x)$ being the holomorphic threeform – by

$$K_{ijk}(q) = \frac{1}{(2\pi i)^3} \left(\frac{1}{w_0(x)}\right)^2 \sum_{l,m,n} K_{lmn}(x) \frac{dx_l}{dt_i} \frac{dx_m}{dt_j} \frac{dx_n}{dt_k} \bigg|_{x_k=x_k(q)} . \tag{4.4}$$

Special geometry in the $H^{2,1}$-moduli space enables us to express the same couplings using the so-called prepotential $F(t)$ [49]:

$$K_{ijk}(q) = \frac{1}{(2\pi i)^3} \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \frac{\partial}{\partial t_k} F(t) . \tag{4.5}$$

For the prepotential, there is a concise formula given in [10] based on the local structure (3.38):

$$F(t) = \frac{1}{2} \left(\frac{1}{w_0(x)}\right)^2 \left\{ w_0(x)D^{(3)}w_0(x) + \sum_l D_l^{(1)}w_0(x)D_l^{(2)}w_0(x) \right\} \bigg|_{x_k=x_k(q)} , \tag{4.6}$$

where we define

$$D_l^{(1)} = \partial_{\rho_l}, \qquad D_l^{(2)} = \frac{1}{2} \sum_{m,n} K_{lmn}^{cl} \partial_{\rho_m} \partial_{\rho_n}, \qquad D^{(3)} = -\frac{1}{6} \sum_{l,m,n} K_{lmn}^{cl} \partial_{\rho_l} \partial_{\rho_m} \partial_{\rho_n} . \tag{4.7}$$

It is also observed that the prepotential defined above has the following asymptotic form with topological data in the leading terms, i.e.,

$$F(t) = \frac{1}{6} \sum_{i,j,k} K_{ijk}^{cl} t_i t_j t_k - \sum_k \frac{(c_2 \cdot J_k)}{24} t_k - i \frac{\zeta(3)}{16\pi^3} \chi + \mathcal{O}(q) , \tag{4.8}$$

where χ is the Euler number of X_Δ and the $\mathcal{O}(q)$- terms represent the quantum corrections. The first example understood was the case of the quintic in $\mathbf{P}^4$ studied by Candelas et al [1]. We denote by $N^r(d)$ the predicted number of σ-model instantons with multi degree $(d_1, \ldots, d_{h^{1,1}})$. The genus one (string 1 loop) topological

amplitude [50] F_1^{top} has the form

$$F_1^{\text{top}} = \log\left\{\left(\frac{1}{w_0}\right)^{3+h^{1,1}-\frac{x}{12}} \frac{\partial(x_1,\ldots,x_{h^{1,1}})}{\partial(t_1,\ldots,t_{h^{1,1}})} \prod_j \text{dis}_j^{r_j} \prod_i x_i^{s_i}\right\} + \text{const.}, \qquad (4.9)$$

where the dis_j are irreducible parts of the discriminant of the hypersurface X_Δ and r_j and s_i are some parameters to be fixed by the asymptotic form of the topological amplitude. It is known that the amplitude has an expansion of the form

$$F_1^{\text{top}} = \text{const.} - \frac{2\pi i}{12}\sum_k (c_2 \cdot J_k)t_k$$

$$-\sum_d \left\{2N^e(d)\log(\eta(q^d)) + \frac{1}{6}N^r(d)\log(1-q^d)\right\}, \qquad (4.10)$$

where $q^d = q_1^{d_1}\cdots q_{h^{1,1}}^{d_{h^{1,1}}}$ and the number $N^e(d)$ is the prediction for the number of 1 loop instantons, i.e., elliptic curves in the Calabi–Yau manifold X_Δ with multi degree n.

   In the following, based on our general observation (3.38), we analyze the large radius limit. In this paper, we will be mainly concerned with the determination of the Picard–Fuchs operators from which we can determine the quantum corrections in a straightforward way. For example we can calculate the quantum corrected yukawa couplings (4.4) using the Mathematica program INSTANTON appended to [10]. The required input data come from the Picard–Fuchs operators and the classical couplings given here in Appendix C. For the interested reader, a complete list of the Picard–Fuchs operators for the Calabi–Yau hypersurfaces with $h^{1,1} \leq 3$ is appended in the source file of this text [24]. The determination of the numbers $N^e(d)$ is a little involved because we need to know the form of the discriminants of the hypersurfaces and need to fix unknown parameters r_i and s_i in (4.9). We will list, in the appendix to the source file, the form of the discriminants for some of our models. However the detailed analysis, together with the analysis of the conifold singularities where one Calabi–Yau model may be connected to another (cf. [51, 52]) will be presented elsewhere.

4.2. Selected Examples.

$X_9(3,2,2,1,1)^2_{-168}$. This is a singular model of type III. The polyhedron $\Delta(w)$ for this model has the vertices

$$v_1 = (2,-1,-1,-1), \qquad v_2 = (-1,3,-1,-1), \qquad v_3 = (-1,3,-1,0),$$

$$v_4 = (-1,-1,3,-1), \qquad v_5 = (-1,-1,3,0), \qquad v_6 = (-1,-1,-1,8), \qquad (4.11)$$

$$v_7 = (-1,-1,-1,-1), \qquad v_8 = (0,2,-1,-1), \qquad v_9 = (0,-1,2,-1),$$

with respect to the basis $\{\Lambda_1,\ldots,\Lambda_4\}$ for the lattice $H(w)$ defined after (2.17). Then the vertices of the dual polyhedron $\Delta^*(w)$ are given by

$$v_1^* = (1,0,0,0), \qquad v_2^* = (0,1,0,0), \qquad v_3^* = (0,0,1,0),$$

$$v_4^* = (0,0,0,1), \qquad v_5^* = (-3,-2,-2,-1), \qquad v_6^* = (-1,-1,-1,0). \qquad (4.12)$$

There are no integral points inside the polyhedron except the origin. For the maximal triangulation of $\Delta^*(w)$, we obtain

$$
\begin{aligned}
T_0 = \{ &\langle 0,3,4,5,6 \rangle, \langle 0,2,4,5,6 \rangle, \langle 0,1,3,5,6 \rangle, \langle 0,1,2,5,6 \rangle, \langle 0,1,3,4,6 \rangle, \\
&\langle 0,1,2,4,6 \rangle, \langle 0,2,3,4,5 \rangle, \langle 0,1,2,3,5 \rangle, \langle 0,1,2,3,4 \rangle \} \,.
\end{aligned}
\tag{4.13}
$$

This triangulation induce the fan $\Sigma(\Delta^*)$, however the resulting fan is singular because the simplex $\langle 0,2,3,4,5 \rangle$ has volume three. In fact we find two integral points $v_7^* = \frac{(v_2^*+v_3^*)+2(v_4^*+v_5^*)}{3} = (-2,-1,-1,0)$, $v_8^* = \frac{2(v_2^*+v_3^*)+(v_4^*+v_5^*)}{3} = (-1,0,0,0)$ which are inside the cone spanned by v_2^*, v_3^*, v_4^* and v_5^* but outside the polyhedron, indicating that this model is of type III. As described in the previous section, we subdivide the fan $\Sigma(\Delta^*)$ by v_7^* and v_8^* to obtain a regular fan $\Sigma(\Delta^*)_{\mathrm{reg}}$. The intersection ring $A^*(\mathbf{P}_{\Sigma(\Delta^*)_{\mathrm{reg}}}, \mathbf{Z})$ is described by the ideal (i) in (3.22) with generators

$$
\begin{aligned}
&D_1 D_7, \quad D_1 D_8, \quad D_6 D_7, \quad D_6 D_8, \quad D_1 D_4 D_5, \quad D_2 D_3 D_6 \,, \\
&D_2 D_3 D_7, \quad D_4 D_5 D_8, \quad D_2 D_3 D_4 D_5 \,,
\end{aligned}
\tag{4.14}
$$

and the linear relations (ii) among the integral points $v_1^*, \ldots, v_8^*$. The hypersurface divisor $[X_\Delta] = D_1 + \cdots + D_6 + 2D_7 + 2D_8$ determines the ideal quotient $\mathscr{I}_{\mathrm{quot}}$. It is generated by

$$
D_4 D_5 - D_4 D_6 + 4 D_3 D_6, \quad D_1 D_4 D_5, \quad D_7, \quad D_8 \,,
\tag{4.15}
$$

together with the linear relations. Starting from those operators $\mathscr{D}_l$ whose leading terms match (4.14) (under the correspondence $\theta_{a_i} \leftrightarrow D_i (i = 1, \ldots, p)$), we can derive the Picard–Fuchs operators via some nontrivial factorizations.

We first note that the generator $D_1 D_4 D_5$ induces $l_{\{1,4,5\}} = (-1,1,0,0,1,1,-2,0,0)$ in L. From this we immediately see that the operator

$$
\mathscr{D}_{l_{\{1,4,5\}}} = \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_5} - \frac{\partial}{\partial a_0} \left(\frac{\partial}{\partial a_6} \right)^2
\tag{4.16}
$$

is one of the Picard–Fuchs operators. To find the other, we need to derive the following relations from the analysis of the Jacobian ring of the hypersurface:

$$
\frac{\partial}{\partial a_0} \frac{\partial}{\partial a_7} = -\frac{3a_1}{a_0} \left(\frac{\partial}{\partial a_0} \right)^2 - \frac{a_0}{a_6} \frac{\partial}{\partial a_0} \frac{\partial}{\partial a_8} \,,
$$

$$
\begin{aligned}
\frac{\partial}{\partial a_0} \frac{\partial}{\partial a_8} = {}&-\frac{3a_1 a_4 a_5}{16 a_2 a_3 a_6} \left(\frac{\partial}{\partial a_6} \right)^2 - \frac{a_0 a_4 a_5}{16 a_2 a_3 a_6} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_5} \\
&+ \frac{a_0 a_4}{16 a_2 a_3} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_6} - \frac{a_0}{4 a_2} \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_6} \,.
\end{aligned}
\tag{4.17}
$$

The derivation of the above relations may be done most efficiently by representing the hypersurface in terms of the homogeneous coordinate of $\mathbf{P}(3,2,2,1,1)$:

$$
W = z_1^3 + z_2^4 z_4 + z_3^4 z_5 + z_4^9 + z_5^9 \,.
\tag{4.18}
$$

The mirror of this hypersurface, whose period we are analyzing, can be constructed by the transposition argument of Berglund and Hübsch [34]. We consider the

orbifold $\hat{W}/\mathbf{Z}_4 \times \mathbf{Z}_9$ with the transposed potential $\hat{W}$. Relating to our toric description, we may write the transposed potential

$$\hat{W} = a_1 z_1^3 + a_2 z_2^4 + a_3 z_3^4 + a_4 z_2 z_4^9 + a_5 z_3 z_5^9 + a_0 z_1 z_2 z_3 z_4 z_5 + a_6 z_1 z_4^4 z_5^4 , \quad (4.19)$$

which is regarded as the degree 12 hypersurface in $\mathbf{P}(4,3,3,1,1)$. Then the integral points v_7^*, v_8^* are mapped, respectively, to degree 24 (charge two) monomials $z_2^2 z_3^2 z_4^6 z_5^6$ and $z_2^3 z_3^3 z_4^3 z_5^3$ under the monomial-divisor map [43]. The equations in (4.17) represent the relations among the charge two monomials in the Jacobian ring $\mathbf{C}[z_1, \ldots, z_5]/(\partial \hat{W})$.

We now focus on the generators $D_1 D_8$ and $D_6 D_7$ which correspond to the primitive collections whose primitive relations are $l_{\{1,8\}} = (-2,1,0,0,0,0,0,0,1)$ and $l_{\{6,7\}} = (0,0,0,0,-1,-1,1,1,0)$ in L. We find that the operator

$$\mathcal{O} = 3 \frac{a_1 a_2 a_3 a_6}{a_0} \frac{\partial}{\partial a_0} \mathscr{D}_{l_{\{1,8\}}} - \frac{a_2 a_3 a_6^3}{a_0^2} \frac{\partial}{\partial a_0} \mathscr{D}_{l_{\{6,7\}}} \qquad (4.20)$$

has the property that $a_0 \mathcal{O} \Pi(a) = (\theta_{a_0} - 2) \mathscr{D}_2 \tilde{\Pi}(a)$. Using this we obtain a complete set of the Picard–Fuchs operators in the $\tilde{\Pi}(a)$ gauge,

$$\mathscr{D}_1 = \theta_y (\theta_y - \theta_x)^2 - y(3\theta_x + \theta_y + 1)(3\theta_x - 2\theta_y - 1)(3\theta_x - 2\theta_y) ,$$

$$\mathscr{D}_2 = (\theta_x - \theta_y)^2 + (\theta_x - \theta_y)(3\theta_x - 2\theta_y) + 4\theta_x (3\theta_x - 2\theta_y)$$
$$- 48xy(3\theta_x - 2\theta_y - 1)(3\theta_x + \theta_y + 1) \qquad (4.21)$$
$$- 3y(3\theta_x - 2\theta_y - 1)(3\theta_x - 2\theta_y)$$
$$- 48xy(3\theta_x + \theta_y + 3)(3\theta_x + \theta_y + 1) - 16x(\theta_x - \theta_y)^2 ,$$

where x and y are defined by (4.2) using the basis $l^{(1)} = (-3,0,1,1,-1,-1,3,0,0)$ and $l^{(2)} = (-1,1,0,0,1,1,-2,0,0)$ generating the Mori cone in the reduced lattice $\{l \in L \mid l_7 = l_8 = 0\}$.

Using the hypersurface divisor $[X_\varDelta] = D_1 + \cdots + D_6 + 2D_7 + 2D_8$ we determine the following topological data:

$$K_{xxx}^{cl} = 6, \qquad K_{xxy}^{cl} = 9, \qquad K_{xyy}^{cl} = 13, \qquad K_{yyy}^{cl} = 17 ,$$
$$c_2 \cdot J_x = 48, \qquad c_2 \cdot J_y = 74 . \qquad (4.22)$$

According to the general form (3.38), these topological data determine the local solutions of the Picard–Fuchs equations (4.21) near $x = y = 0$. We notice that this model has the same Hodge numbers as the model $X_8(2,2,2,1,1)_{-168}^2$. However there is no linear transformation which relates the topological data: the cubic and linear forms of the two manifolds. By Wall's theorem [53] we see that the two manifolds are topologically distinct.

Non-LG model related to $X_9(3,2,2,1,1)_{-168}^2$. For the model analyzed in the last subsection, we can verify that the polyhedron $\varDelta'^* = \text{Conv.}(\{v_1^*, \ldots, v_8^*\})$ is reflexive and the complete fan $\Sigma(\varDelta'^*)$ for a triangulation of $\varDelta'^*$ (i.e., the triangulation $T_\varDelta$ below) coincides with $\Sigma(\varDelta^*)_{\text{reg}}$. Therefore we have another family of Calabi–Yau hypersurfaces $X_{\varDelta'}$ in the same ambient space $\mathbf{P}_{\Sigma(\varDelta'^*)} = \mathbf{P}_{\Sigma(\varDelta^*)_{\text{reg}}}$. The hypersurface

represents the divisor class

$$[X_{\Delta'}] = D_1 + \cdots + D_6 + D_7 + D_8 . \tag{4.23}$$

The dual polyhedron Δ'^* is the convex hull of the points $v_1^*, \ldots, v_8^*$. The polyhedron Δ' has vertices $v_2, v_3, \ldots, v_9$ (the corner v_1 is deleted) and

$$v_{10} = (1, -1, -1, 2), \quad v_{11} = (1, 0, -1, 0), \quad v_{12} = (1, 0, -1, -1),$$

$$v_{13} = (1, -1, -1, -1), \quad v_{14} = (1, -1, 0, 0), \quad v_{15} = (1, -1, 0, -1) . \tag{4.24}$$

By the formula (2.5), we know that Hodge numbers of $X_{\Delta'}$ are $h^{1,1} = 4$ and $h^{2,1} = 85$. It turns out that this model is not in the list of [25]. Also this model gives an example of a topology change due to flop operations [43].

There are 37 triangulations for the polyhedron Δ'^* and among them two triangulations give us different resolutions of the ambient space. The first one is the triangulation corresponding to the subdivision $\Sigma(\Delta^*)_{\text{reg}}$:

$$T_A = \{\langle 0, 3, 5, 7, 8 \rangle, \langle 0, 2, 5, 7, 8 \rangle, \langle 0, 3, 4, 7, 8 \rangle, \langle 0, 2, 3, 5, 8 \rangle, \langle 0, 2, 3, 4, 8 \rangle,$$

$$\langle 0, 1, 3, 5, 6 \rangle, \langle 0, 1, 2, 5, 6 \rangle, \langle 0, 1, 3, 4, 6 \rangle, \langle 0, 1, 2, 4, 6 \rangle, \langle 0, 1, 2, 3, 5 \rangle, \langle 0, 2, 4, 7, 8 \rangle,$$

$$\langle 0, 1, 2, 3, 4 \rangle, \langle 0, 3, 4, 5, 7 \rangle, \langle 0, 2, 4, 5, 7 \rangle, \langle 0, 3, 4, 5, 6 \rangle, \langle 0, 2, 4, 5, 6 \rangle\} . \tag{4.25}$$

The second triangulation T_B is T_A but with the last four simplices replaced by

$$\langle 0, 3, 4, 6, 7 \rangle, \quad \langle 0, 2, 5, 6, 7 \rangle, \quad \langle 0, 3, 4, 6, 7 \rangle, \quad \langle 0, 2, 4, 6, 7 \rangle . \tag{4.26}$$

We verify that the difference in the two triangulations is due to two different triangulations of the two dimensional face (square) $\langle v_4^*, v_5^*, v_5^*, v_6^* \rangle$. They are $\{\langle v_4^*, v_5^*, v_6^* \rangle, \langle v_4^*, v_5^*, v_7^* \rangle\}$ for T_A, and $\{\langle v_4^*, v_6^*, v_7^* \rangle, \langle v_5^*, v_6^*, v_7^* \rangle\}$ for T_B.

For the triangulation T_A, we have in (4.14) the generators of the Stanley–Reisner ideal. Each generator $D_{i_1} D_{i_2} \cdots D_{i_k}$ determines uniquely the element $l_{\{i_1, i_2, \ldots, l_k\}}$ in the lattice L, and in turn the operator $\mathscr{D}_{l_{\{i_1, i_2, \ldots, i_k\}}}$. We observe that some combinations of the operators factorize to give a complete set of Picard–Fuchs operators. The principal parts of these operators generate the ideal $\mathscr{I}_{\text{quot}}$ as in (4.15). In Appendix A, we list the resulting Picard–Fuchs operators in terms of the local coordinates x, y, z and w defined by $l_A^{(1)} = (-1, 1, 0, 0, 1, 1, -2, 0, 0)$, $l_A^{(2)} = (-1, 0, 0, 0, 1, 1, 0, -2, 0)$, $l_A^{(3)} = (-1, 0, 1, 1, 0, 0, 0, 1, -2)$ and $l_A^{(4)} = (0, 0, 0, 0, -1, -1, 1, 1, 0)$, respectively. The intersection ring (3.36) determines the topological data as follows;

$$K_{xxx}^{A,cl} = 17, \quad K_{xxy}^{A,cl} = 26, \quad K_{xyy}^{A,cl} = 36, \quad K_{yyy}^{A,cl} = 46, \quad K_{xxz}^{A,cl} = 13 ,$$

$$K_{xyz}^{A,cl} = 18, \quad K_{yyz}^{A,cl} = 23, \quad K_{xzz}^{A,cl} = 9, \quad K_{yzz}^{A,cl} = 11, \quad K_{zzz}^{A,cl} = 4 ,$$

$$K_{xxw}^{A,cl} = 39, \quad K_{xyw}^{A,cl} = 54, \quad K_{yyw}^{A,cl} = 72, \quad K_{xzw}^{A,cl} = 27, \quad K_{yzw}^{A,cl} = 36 ,$$

$$K_{zzw}^{A,cl} = 18, \quad K_{xww}^{A,cl} = 81, \quad K_{yww}^{A,cl} = 108, \quad K_{zww}^{A,cl} = 54, \quad K_{www}^{A,cl} = 162 ,$$

$$c_2 \cdot J_x^A = 74, \quad c_2 \cdot J_y^A = 100, \quad c_2 \cdot J_z^A = 52, \quad c_2 \cdot J_w^A = 144 . \tag{4.27}$$

For the triangulation T_B, we find the generators for theStanley–Reisner ideal

$$D_1 D_7, \qquad D_1 D_8, \qquad D_4 D_5, \qquad D_6 D_8, \qquad D_2 D_3 D_6, \qquad D_2 D_3 D_7 \,. \qquad (4.28)$$

We observe again some (less trivial) factorizations among the operators $\{\mathscr{D}_{l_{\{i_1, i_2, \dots, i_k\}}}\}$ and their combinations in order to obtain the Picard–Fuchs operators listed in Appendix A. The local coordinates x, y, z and w for this triangulation are defined by $l_B^{(1)} = l_A^{(1)} + l_A^{(4)}$, $l_B^{(2)} = l_A^{(2)} + l_A^{(4)}$, $l_B^{(3)} = l_A^{(3)} + l_A^{(4)}$ and $l_B^{(4)} = -l_A^{(4)}$, respectively. Then the topological data turns out to be

$$K_{xxw}^{B,cl} = 4, \qquad K_{xyw}^{B,cl} = 8, \qquad K_{yyw}^{B,cl} = 10 \,,$$

$$K_{xzw}^{B,cl} = 4, \qquad K_{yzw}^{B,cl} = 5, \qquad K_{zzw}^{B,cl} = 2 \,, \qquad (4.29)$$

$$c_2 \cdot J_x^B = 74, \qquad c_2 \cdot J_y^B = 100, \qquad c_2 \cdot J_z^B = 52, \qquad c_2 \cdot J_w^B = 24 \,,$$

where the cubic couplings among J_x, J_y, J_z are the same as in (4.27) and $K_{www}^{B,cl} = K_{*ww}^{B,cl} = 0$ ($* = x, y, z$). As we observe in (4.29), the topological data for the phase B indicate that the Calabi–Yau hypersurface has the property of a $K3$ fibration [14]. In fact, we verify that $\Delta_{K3}^* := \text{Conv.}(\{v_1^*, v_2^*, v_3^*, v_6^*, v_7^*, v_8^*\})$ is a three dimensional reflexive polyhedron. We observe that $c_2 \cdot J_i = 24$ for some i (cf. (4.29)) is necessary for the Calabi–Yau hypersurface to contain a $K3$. We also remark that the existence of a three dimensional reflexive polyhedron Δ_{K3}^* in Δ^* does not always yield the above topological condition. We will return to this point later in the final section.

$X_{14}(7,3,2,1,1)_{-260}^2$. This model provides us an example in which we have two different resolutions of point singularities in the ambient space, however the difference of the two resolutions does not affect the topology of the Calabi–Yau hypersurface. This model has also been solved in [11].

Let us summarize the toric data for this model. The reflexive polyhedron we consider is given by the convex hull of the following integral points:

$$v_1 = (1, -1, -1, -1), \qquad v_2 = (-1, 3, 0, -1), \qquad v_3 = (-1, 3, -1, -1) \,,$$

$$v_4 = (-1, 3, -1, 1), \qquad v_5 = (-1, -1, 6, -1) \,, \qquad (4.30)$$

$$v_6 = (-1, -1, -1, 13), \qquad v_7 = (-1, -1, -1, -1) \,,$$

with respect to the basis $\{\Lambda_1, \dots, \Lambda_4\}$ of $H(w)$ given after (2.17). Then the vertices of the dual polyhedron $\Delta^*(w)$ are

$$v_1^* = (1, 0, 0, 0), \qquad v_2^* = (0, 1, 0, 0), \qquad v_3^* = (0, 0, 1, 0) \,,$$

$$v_4^* = (0, 0, 0, 1), \qquad v_5^* = (-7, -3, -2, -1), \qquad v_6^* = (-2, -1, 0, 0) \,. \qquad (4.31)$$

We will find one point $v_7^* = (-1, 0, 0, 0)$ on a codimension one face $\langle v_2^*, v_3^*, v_4^*, v_5^*, v_6^* \rangle$. If we triangulate the polyhedron $\Delta^*(w)$, we will find the following two different triangulations T_A and T_B which induce the complete fans $\Sigma(\Delta^*)^A$ and $\Sigma(\Delta^*)^B$
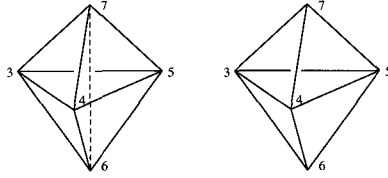
Fig. 2. Two different triangulations T_A (left) and T_B (right) for the models $X_7(7,3,2,1,1)$. In the left, we see three 3-simplices whereas in the right we see two 3-simplices. This results in different regular fans $\Sigma(\Delta^*)^A_{\text{reg}}$ and $\Sigma(\Delta^*)^B_{\text{reg}}$ for the different desingularizations of the ambient space. However the Calabi–Yau hypersurfaces in them have the same topology

respectively;

$$T_A = \{\langle 0,4,5,6,7\rangle, \langle 0,3,5,6,7\rangle, \langle 0,3,4,6,7\rangle, \langle 0,2,4,5,7\rangle,$$

$$\langle 0,2,3,5,7\rangle, \langle 0,2,3,4,7\rangle, \langle 0,1,4,5,6\rangle, \langle 0,1,3,5,6\rangle,$$

$$\langle 0,1,3,4,6\rangle, \langle 0,1,2,4,5\rangle, \langle 0,1,2,3,5\rangle, \langle 0,1,2,3,4\rangle\}. \qquad (4.32)$$

T_B can be obtained from T_A by replacing the first three simplices of T_A by $\langle 0,3,4,5,7\rangle$ and $\langle 0,3,4,5,6\rangle$. The difference between T_A and T_B are depicted in Fig. 2. Since it turns out that some of the cones in the fan $\Sigma(\Delta^*)$ are singular for both triangulations, we need to subdivide them. In the case of T_A, we find the following integral points make the cones regular:

$$v_8^* = \frac{1}{2}(v_4^* + v_5^* + v_6^* + v_7^*), \qquad v_9^* = \frac{1}{2}(v_2^* + v_4^* + v_5^* + v_7^*),$$

$$v_{10}^* = \frac{1}{2}(v_1^* + v_4^* + v_5^* + v_6^*), \qquad v_{11}^* = \frac{1}{2}(v_1^* + v_2^* + v_4^* + v_5^*), \qquad (4.33)$$

and for T_B we find

$$v_8^* = \frac{1}{3}(v_3^* + v_7^*) + \frac{2}{3}(v_4^* + v_5^*), \qquad v_9^* = \frac{1}{2}(v_2^* + v_4^* + v_5^* + v_7^*),$$

$$v_{10}^* = \frac{1}{2}(v_1^* + v_4^* + v_5^* + v_6^*), \qquad v_{11}^* = \frac{1}{2}(v_1^* + v_2^* + v_4^* + v_5^*), \qquad (4.34)$$

$$v_{12}^* = \frac{1}{3}(v_4^* + v_5^*) + \frac{2}{3}(v_3^* + v_7^*).$$

Subdividing $\Sigma(\Delta^*)^A$ and $\Sigma(\Delta^*)^B$ by these integral points results in the regular fans $\Sigma(\Delta^*)^A_{\text{reg}}$ and $\Sigma(\Delta^*)^B_{\text{reg}}$, respectively, both of which do not come from any triangulation of the polyhedron Δ'^* – the convex hull of all the integral points. Using each of the two regular fans, we determine the basis for the Kähler cone, and the Mori cone of the ambient space. We summarize in Appendix B the bases $\{\eta_A^1, \ldots, \eta_A^8\}$ and $\{\eta_B^1, \ldots, \eta_B^9\}$ of the Mori cones for $\Sigma(\Delta^*)^A_{\text{reg}}$ and $\Sigma(\Delta^*)^B_{\text{reg}}$ respectively. We observe that the Mori cones for both ambient spaces are not simplicial, implying that neither are the Kähler cones of the ambient spaces.
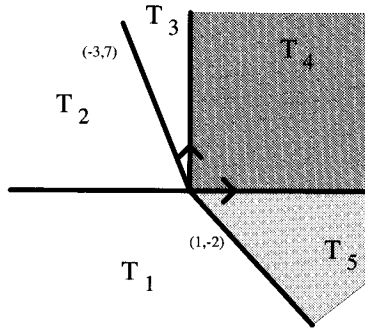
Fig. 3. The secondary fan for the polyhedron $\Delta^*(7,3,2,1,1)$. The Kähler cones of the smooth ambient spaces $\mathbf{P}_{\Sigma(\Delta^*)^A_{reg}}$ and $\mathbf{P}_{\Sigma(\Delta^*)^B_{reg}}$ have different restrictions to the secondary fan. The restricted Kähler cone for the former space is given by the union of the cones parametrized by T_4 and T_5, while for the latter it is given by the cone parametrized T_4

The divisor $[X_\Delta]$ of the form (3.46) determines the same intersection ring for the two hypersurfaces, and for both cases we find that the divisors D_i ($i \geq 7$) decouple. In fact the ideal $\mathscr{I}_{\text{quot}}$ is generated by

$$D_2 D_6, \qquad D_3 D_4 D_5, \qquad D_i \quad (i \geq 7), \qquad (4.35)$$

together with the linear relations (ii) in (3.22). We remark that in this model both $\{v_2^*, v_6^*\}$ and $\{v_3^*, v_4^*, v_5^*\}$ are the primitive collections of $\Sigma(\Delta)^A_{\text{reg}}$ and $\Sigma(\Delta)^B_{\text{reg}}$. The reduced lattices which we denote L'_A, L'_B in the two cases are generated by

$$l_A^{(1)} = 3\eta_A^2 + \eta_A^3 + 2\eta_A^4 + 2\eta_A^5 + 4\eta_A^6 = (-4, 2, 1, 0, 0, 0, 1, 0, \ldots, 0),$$
$$l_A^{(2)} = 2\eta_A^1 + \eta_A^2 + \eta_A^3 + \eta_A^7 + 2\eta_A^8 = (-2, 1, 0, 2, 1, 1, -3, 0, \ldots, 0), \qquad (4.36)$$

for L'_A and

$$l_B^{(1)} = 3\eta_B^2 + 6\eta_B^3 + 7\eta_B^4 + \eta_B^5 + 2\eta_B^6 = (0, 0, 1, -4, -2, -2, 7, 0, \ldots, 0),$$
$$l_B^{(2)} = 2\eta_B^1 + \eta_B^7 = (-2, 1, 0, 2, 1, 1, -3, 0, \ldots, 0), \qquad (4.37)$$

for L'_B. We remark that the Mori cone for the ambient spaces are not simplicial but their intersection with L'_R's are. We also note that two restricted cones for $\Sigma(\Delta^*)^A_{\text{reg}}$ and $\Sigma(\Delta^*)^B_{\text{reg}}$ have an intersection, in fact the former is included in the latter since $l_A^{(1)} = l_B^{(1)}$ and $l_A^{(2)} = l_B^{(2)} + 2l_B^{(1)}$. We draw in Fig. 3 the restricted Kähler cones in the secondary fan for the polyhedron Δ^*, more precisely in the secondary fan for the point configurations $v_0^*, v_1^*, \ldots, v_6^*$ (we delete the point v_7^* corresponding to automorphisms). Since $\mathscr{I}_{\text{quot}}$ is the same for both $\Sigma(\Delta^*)^A_{\text{reg}}$ and $\Sigma(\Delta^*)^B_{\text{reg}}$, we expect that the two triangulations define the same Calabi–Yau hypersurface in different ambient spaces, i.e., the only difference is in the topology of the ambient space which is irrelevant to the hypersurface.

Now we derive the Picard–Fuchs operators based on the triangulation T_A. We note that this model is of type III with non-trivial automorphisms. This is the most general situation. The point v_7^* on a codimension one face is a root vector in (2.10) for the fan $\Sigma(\Delta)$. According to (2.12), this results in the following linear operator annihilating the periods:

$$\mathscr{L}' = 2a_1 \frac{\partial}{\partial a_0} + a_0 \frac{\partial}{\partial a_7} . \tag{4.38}$$

Now we look at the operators which correspond to the first two generators in (4.35),

$$\mathscr{D}_{l_{\{2,6\}}} = \frac{\partial}{\partial a_2} \frac{\partial}{\partial a_6} - \left(\frac{\partial}{\partial a_7} \right)^2 ,$$

$$\mathscr{D}_{l_{\{3,4,5\}}} = \frac{\partial}{\partial a_3} \frac{\partial}{\partial a_4} \frac{\partial}{\partial a_5} - \frac{\partial}{\partial a_0} \frac{\partial}{\partial a_6} \frac{\partial}{\partial a_8} . \tag{4.39}$$

Starting with these operators, we derive the Picard–Fuchs operators for the period restricted to the sublattice (4.36). It is easy to see the first operator $\mathscr{D}_{l_{\{2,6\}}}$ combined with the linear operator (4.38) results in a second order differential operator. For the operator $\mathscr{D}_{l_{\{3,4,5\}}}$, we need to look into the structure of the Jacobian ring of the hypersurface. For this, as in the previous example, it would be most efficient to express the hypersurface in terms of the homogeneous coordinates:

$$W = z_1^2 + z_2^4 z_3 + z_3^7 + z_4^{14} + z_5^{14} , \tag{4.40}$$

in $\mathbf{P}(7,3,2,1,1)$. Then the transposition argument in [34] applied to this hypersurface indicates that the mirror is given by the orbifold $\hat{W}/\mathbf{Z}_2 \times \mathbf{Z}_{14}$ with

$$\hat{W} = a_1 z_1^2 + a_2 z_2^4 + a_3 z_2 z_3^7 + a_4 z_4^{14} + a_5 z_5^{14} + a_0 z_1 z_2 z_3 z_4 z_5$$
$$+ a_6 z_3^4 z_4^4 z_5^4 + a_7 z_2^2 z_3^2 z_4^2 z_5^2 , \tag{4.41}$$

in $\mathbf{P}(14,7,3,2,2)$. We note that, in this form, the automorphism used for (4.38) is identified with

$$z_1 \mapsto z_1 + \varepsilon\, z_2 z_3 z_4 z_5, \qquad z_i \mapsto z_i \ (i \geqq 2) , \tag{4.42}$$

in infinitesimal form. The deformation parameters $a_8, \dots, a_{11}$ corresponds to the degree 56 (charge two) monomials $z_2 z_3^3 z_4^{10} z_5^{10}$, $z_2^3 z_3 z_4^8 z_5^8$, $z_1 z_3^2 z_4^9 z_5^9$ and $z_1 z_2^2 z_4^7 z_5^7$, respectively. Since we can verify $(1 - \frac{64 a_1^2 a_2 a_6}{a_0^4}) z_2 z_3^3 z_4^{10} z_5^{10} = \frac{112 a_1^2 a_2 a_3}{a_0^4} z_2^2 z_3^6 z_4^6 z_5^6 + \frac{2 a_1 a_3}{a_0^2} z_3^8 z_4^8 z_5^8$ modulo terms in the Jacobian ring $(\partial \hat{W})$ which vanish inside the period integral, we have the relation

$$\left(1 - \frac{64 a_1^2 a_2 a_6}{a_0^4}\right) \frac{\partial}{\partial a_0} \frac{\partial}{\partial a_8} = -224 \frac{a_1^3 a_2 a_3}{a_0^5} \frac{\partial}{\partial a_6} \frac{\partial}{\partial a_0} + 2 \frac{a_1 a_3}{a_0^2} \left(\frac{\partial}{\partial a_6} \right)^2 , \tag{4.43}$$

where we use (4.38) in the derivation. If we combine (4.43) with the operator $\mathscr{D}_{l_{\{3,4,5\}}}$ in (4.39), we will obtain a third order differential operator. Thus we obtain the Picard–Fuchs operators which determine the local solutions with the

property (3.38);

$$\mathcal{D}_1 = \theta_x(\theta_x - 3\theta_y) - 4x(2\theta_y + 4\theta_x + 3)(2\theta_y + 4\theta_x + 1),$$

$$\mathcal{D}_2 = (1 - 64x)^2\theta_y^3 - 64\{112x^2y(\theta_x - 3\theta_y)(2\theta_y + 4\theta_x + 1)$$

$$+xy(\theta_x - 3\theta_y - 1)(\theta_x - 3\theta_y)\} - (1 - 64x)\{112xy(\theta_x - 3\theta_y - 1)$$

$$\times(\theta_x - 3\theta_y)(2\theta_y + 4\theta_x + 1) + y(\theta_x - 3\theta_y - 2)(\theta_x - 3\theta_y - 1)(\theta_x - 3\theta_y)\},$$

$$(4.44)$$

with $x = a^{l^{(1)}_A} = \frac{a_1 a_2 a_6}{a_0^4}$, $y = a^{l^{(2)}_A} = \frac{a_1 a_3^2 a_4 a_5}{a_0^2 a_6^3}$. The topological data for the local solutions about $x = y = 0$ are given by

$$K_{xxx}^{cl,A} = 9, \qquad K_{xxy}^{cl,A} = 3, \qquad K_{xyy}^{cl,A} = 1, \qquad K_{yyy}^{cl,A} = 0,$$

$$c_2 \cdot J_x^A = 66, \qquad c_2 \cdot J_y^A = 24.$$

$$(4.45)$$

The analysis for T_B is the same as the above and the Picard–Fuchs operators are given by (4.44) with the variables $(x_A, y_A) := (x, y)$ changed to (x_B, y_B) under the relations $x_A = x_B y_B^2$, $y_A = y_B$. The topological data are connected by the linear relations which results from these relations.

5. Conclusion and Discussions

We have analyzed the GKZ hypergeometric system – which we call Δ^*-hypergeometric – for a reflexive polyhedron. The characteristic feature of this system in mirror symmetry is that it is T-resonant in general. Especially, for a maximal triangulation T_0 of the polyhedron Δ^*, the monodromy of this system becomes maximally unipotent. We have found close relationships between the Stanley–Reisner ideal for the triangulation T_0 and the ring of the leading terms of the Δ^*-hypergeometric system at the maximally unipotent point. For the models of type I and II, we have proved these two ideals are actually equal, using the general theory of toric ideals. We have found a closed formula for the local solutions near the maximally unipotent point, in terms of the intersection form. As was observed in [8, 10], the Δ^*-hypergeometric system is reducible. If we extract the irreducible part of the system by factoring out the operator θ_{a_0}, the resulting system gives a sufficient set of differential operators to determine the quantum geometry of moduli space. We have verified our observations for the Calabi–Yau hypersurfaces in weighted projective spaces up to $h^{1,1} \leq 3$, including models of type III.

In the table of Appendix C, we have summarized the topological data for each models. There we can see several isomorphisms or relations between different models. For example we have $X_{14}^{I\!I}(7,2,2,2,1)_{-240}^2 \cong X_8^{I\!I}(3,1,1,1,1)_{-240}^2$, $X_{15}^{I\!I}(5,3,3,3,1)_{-144}^3 \cong X_{10}^{I\!I}(3,2,2,2,1)_{-144}^3$ and $X_{18}^{I}(9,4,2,2,1)_{-240}^3 \cong X_9^{I}(4,2,1,1,1)_{-240}^3$, all of which can be explained by a fractional change of the variables [54]. Also there can be a reflexive polyhedron $\Delta^*(w')$ in another reflexive polyhedron $\Delta^*(w)$.[1] For example, by listing all integral points in the polyhedra, we see $\Delta^*(2,2,2,1,1) \subset$

---

[1]This observation has also been made in Ref. [11].

$\Delta^*(3,3,3,2,1)$, $\Delta^*(6,2,2,1,1) \subset \Delta^*(9,3,3,2,1)$ and $\Delta^*(1^5), \Delta^*(2,1^4) \subset \Delta^*(3,2,2, 2,1) \subset \Delta^*(5,3,3,3,1)$. Since all integral points in $\Delta^*(w')$ are contained in $\Delta^*(w)$, the inclusion relation $\Delta^*(w') \subset \Delta^*(w)$ implies that the fan $\Sigma(\Delta^*(w))$ is a refinement of the fan $\Sigma(\Delta^*(w'))$. This reminds us of the cases we encountered in the singular models of type III, in which we found that topologically different Calabi–Yau hypersurfaces can sit in the same ambient space.

To see the details, let us consider the case $\Delta^*(6,2,2,1,1) \subset \Delta^*(9,3,3,2,1)$. The integral points in $\Delta^*(9,3,3,2,1)$ with respect to the basis given after (2.17) are $v_0^* = (0,0,0,0)$, $v_1^* = (1,0,0,0)$, $v_2^* = (0,1,0,0)$, $v_3^* = (0,0,1,0)$, $v_4^* = (0,0,0,1)$, $v_5^* = (-9,-3,-3,-2)$, $v_6^* = (-6,-2,-2,-1)$, $v_7^* = (-3,-1,-1,0)$ and $v_8^* = (-1,0, 0,0)$, where the last points v_8^* are on a codimension one face of the polyhedron. The polyhedron $\Delta^*(6,2,2,1,1)$ has integral points $v_0^*, v_1^*, v_2^*, v_3^*, v_4^*, v_6^*, v_7^*, v_8^*$, where the point v_8^* is also on a codimension one face. Therefore $\Sigma(\Delta^*(9,3,3,2,1))$ is a refinement of the fan $\Sigma(\Delta^*(6,2,2,1,1))$, and we will have two different Calabi–Yau hypersurfaces in the same ambient space $\mathbf{P}_{\Sigma(\Delta^*(9,3,3,2,1))}$.[2] According to (3.46), the divisor for the hypersurface is given by

$$[X_{\Delta(w)}] = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 \,, \qquad (5.1)$$

for the model $X_{18}(9,3,3,2,1)^3_{-186}$; and

$$[X_{\Delta(w')}] = D_1 + D_2 + D_3 + D_4 + 2D_5 + D_6 + D_7 + D_8 \,, \qquad (5.2)$$

for the model $X_{12}(6,2,2,1,1)^2_{-252}$. This can also be understood by the fractional transformation on the defining polynomial. The polynomial $W(z) = \hat{W}(z)$ for the mirror of $X_{18}(9,3,3,2,1)^3_{-186}$ is

$$\hat{W} = a_1 z_1^2 + a_2 z_2^6 + a_3 z_3^6 + a_4 z_4^9 + a_5 z_5^{18} + a_0 z_1 z_2 z_3 z_4 z_5 + a_6 z_4^3 z_5^{12} + a_7 z_4^6 z_5^6 \,, \quad (5.3)$$

in $\mathbf{P}(9,3,3,2,1)/(\mathbf{Z}_6)^2$, where the deformation by a_8, which corresponds to the divisor D_8, is eliminated using the automorphism. Now consider the transformation $\xi_i = z_i$ $(i = 1,2,3), \xi_4 = z_4^{3/4}, \xi_5 = z_4^{1/4} z_5$. Then the potential becomes, if we set $a_5 = 0$,

$$\hat{W}(\xi) = a_1 \xi_1^2 + a_2 \xi_2^6 + a_3 \xi_3^6 + a_4 \xi_4^{12} + a_0 \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 + a_6 \xi_5^{12} + a_7 \xi_4^6 \xi_5^6 \,, \qquad (5.4)$$

which can be regarded as a hypersurface in $\mathbf{P}(6,2,2,1,1)/(\mathbf{Z}_6^2 \times \mathbf{Z}_{12})$, the mirror of $X_{12}(6,2,2,1,1)^2_{-252}$. The additional quotient by $\mathbf{Z}_{12}$ comes from the identification $(\xi_4, \xi_5) \equiv (\alpha^4 \xi_4, \alpha \xi_5)$ with $\alpha^4 = 1$ (see [8] for the detailed form of the actions for $\mathbf{Z}_6^2$). The Mori cone of each model may be obtained by restricting the Mori cone of the ambient space to the sublattice L', namely $l \in L$ with $l_8 = 0$ for $\Delta^*(9,3,3,2,1)$ and $l_5 = l_8 = 0$ for $\Delta^*(6,2,2,1,1)$. Thus the inclusion of the dual polyhedron, $\Delta^*(w') \subset \Delta^*(w)$, implies an embedding of the (quantum) Kähler moduli of $X_{\Delta(w')}$ to that of $X_{\Delta(w)}$, or equivalently under mirror symmetry, the complex structure moduli for the mirror $X_{\Delta^*(w')}$ to that of $X_{\Delta^*(w)}$.

As a different kind of inclusion relation, we also observe that the dual polyhedron $\Delta^*_{K3}(w')$ for some $K3$ hypersurface [55] sits inside the polyhedron $\Delta^*(w)$ for a Calabi–Yau hypersurface. It has also been observed that if, in addition, we

---

[2]Since the ambient space is still singular, we need further subdivisions of some cones. However the following arguments are valid for the fully resolved ambient space.

have the following specific form of the topological data; $c_2 \cdot K = 24$, $J \cdot K \cdot K = K \cdot K \cdot K = 0$ for some divisor class K, then the following "CY-K3 correspondence" occurs: the Picard–Fuchs operators for the Calabi–Yau manifold specialize to those for a $K3$-model under a suitable limit of the variables. In our list, the following models shows these specific properties: $X_8(2,2,2,1,1)^2_{-168}$, $X_{12}(6,2,2,1,1)^2_{-252}$, $X_{12}(3,3,3,2,1)^3_{-132}$, $X_{18}(9,3,3,2,1)^3_{-192}$, $X_{24}(12,8,2,1,1)^3_{-480}$, $X_{10}(4,2,2,1,1)^3_{-192}$ and $X_{16}(8,3,3,1,1)^3_{-256}$. Also our non-Landau–Ginzburg model found in relation to $X_9(3,2,2,1,1)^2_{-168}$ shows this property as well. The K3 polyhedron Δ^*_{K3} contained in the reflexive polyhedron Δ'^* provides an example of non-Landau–Ginzburg K3 hypersurface. We have noticed that the specific form of the topological data depends on how we triangulate the polyhedron, namely in this example, the CY-K3 correspondence occurs only in the phase B (see (4.29)). Some of the models where the CY-K3 correspondence occurs has been studied extensively, and has provided strong evidence for the so-called heterotic-type II string duality [16, 14, 17]. We believe that our general framework outlined here will provide powerful techniques for studying questions in heterotic-type II duality.

Appendix A. Picard–Fuchs Equations for the Model in Section 3

This non-Landau–Ginzburg model is defined by the reflexive polyhedron Δ'^* which has the property $\Sigma(\Delta'^*) = \Sigma(\Delta^*)_{\text{reg}}$ for $\Delta^* = \Delta^*(3,2,2,1,1)$. There are two Calabi–Yau phases, phase A and phase B, which are connected by flop operations.

Phase A:

$$\mathcal{D}_1 = \theta_x(\theta_z - 2\theta_y + \theta_w) - xw(\theta_x + \theta_z + \theta_y + 11)(\theta_w - 2\theta_x),$$

$$\mathcal{D}_2 = \theta_x(\theta_y - 2\theta_z) - xyw^2(\theta_x + \theta_z + \theta_y + 2)(\theta_x + \theta_z + \theta_y + 1),$$

$$\mathcal{D}_3 = (\theta_w - 2\theta_x)(\theta_z - 2\theta_y + \theta_w) - w(\theta_x + \theta_y - \theta_w)(\theta_x + \theta_y - \theta_w),$$

$$\mathcal{D}_4 = (2\theta_x - \theta_w)(2\theta_z - \theta_y) - yw(\theta_x + \theta_z + \theta_y + 1)(\theta_z - 2\theta_y + \theta_w),$$

$$\mathcal{D}_5 = (\theta_x + \theta_y - \theta_w)^2\theta_x - x(\theta_x + \theta_z + \theta_y + 1)(\theta_z - 2\theta_y + \theta_w - 1)(\theta_z - 2\theta_y + \theta_w),$$

$$\mathcal{D}_6 = (\theta_x + \theta_y - \theta_w)^2(\theta_y - 2\theta_z)$$
$$\quad - y(\theta_x + \theta_z + \theta_y + 1)(\theta_z - 2\theta_y + \theta_w - 1)(\theta_z - 2\theta_y + \theta_w),$$

$$\mathcal{D}_7 = \theta_z(\theta_x + \theta_y - \theta_w)^2 + 3yz(\theta_x + \theta_z + \theta_y + 1)(\theta_z - \theta_y + \theta_w)(\theta_y - 2\theta_z)$$
$$\quad - y\theta_z(\theta_z - 2\theta_y + \theta_w - 1)(\theta_z - 2\theta_y + \theta_w) - x\theta_z(\theta_w - 2\theta_x)(\theta_w - 2\theta_x - 1),$$

$$\mathcal{D}_8 = 9\theta_x^2 - 18\theta_x\theta_w + 250\theta_x\theta_y - 410\theta_x\theta_z + 160\theta_z\theta_w$$
$$\quad - 48yzw(\theta_x + \theta_z + \theta_y + 1)(\theta_y - 2\theta_z) - 9x(\theta_w - 2\theta_x)(\theta_w - 2\theta_x - 1)$$
$$\quad + yw(\theta_x - 2\theta_z)(\theta_z - 2\theta_y + \theta_w) + 4xyw^2(\theta_y + 1)(\theta_x + \theta_z + \theta_y + 1)$$
$$\quad + xw(9\theta_z + 100\theta_y + 9\theta_w + 9)(\theta_w - 2\theta_x),$$

$$\mathscr{D}_9 = 3\theta_x\theta_y - 6\theta_x\theta_z - 6\theta_w\theta_y + 3\theta_w\theta_y + 3\theta_y^2 - 9\theta_y\theta_z + 13\theta_w\theta_z + 3\theta_z^2$$
$$-3y(\theta_z - 2\theta_y + \theta_w - 1)(\theta_z - 2\theta_y + \theta_w) - 3z(2\theta_z - \theta_y + 1)(2\theta_z - \theta_y)$$
$$-xw\theta_z(\theta_w - 2\theta_x) + yw(5\theta_z + 3\theta_w + 3)(\theta_z - 2\theta_y + \theta_w)$$
$$+xyw^2(\theta_x + \theta_z + \theta_y + 1)(8\theta_z + 6\theta_w + 12) . \tag{A.1}$$

Phase B:
$$\mathscr{D}_1 = \theta_x(\theta_x - \theta_y + \theta_z - \theta_w) - x(\theta_x + \theta_y + \theta_z + 1)(\theta_y - \theta_x - \theta_w) ,$$
$$\mathscr{D}_2 = \theta_x(\theta_y - 2\theta_z) - xy(\theta_x + \theta_y + \theta_z + 2)(\theta_x + \theta_y + \theta_z + 1) ,$$
$$\mathscr{D}_3 = \theta_w^2 - w(\theta_y - \theta_x - \theta_w)(\theta_x - \theta_y + \theta_z - \theta_w) ,$$
$$\mathscr{D}_4 = (\theta_y - \theta_x - \theta_w)(\theta_y - 2\theta_z) - y(\theta_x + \theta_y + \theta_z + 1)(\theta_x - \theta_y + \theta_z - \theta_w) ,$$
$$\mathscr{D}_5 = 9\theta_x\theta_w - 2\theta_x\theta_y - 16\theta_z\theta_w - 16\theta_x\theta_z + 16\theta_y\theta_z - 48yz(\theta_x + \theta_y + \theta_z + 1)$$
$$(\theta_y - 2\theta_z) - 16y\theta_z(\theta_x - \theta_y + \theta_z - \theta_w) - 9xw(\theta_y - \theta_x - \theta_w - 1)$$
$$(\theta_y - \theta_x - \theta_w) - 4xy(2\theta_x - 3\theta_y + 2\theta_w - 1)(\theta_x + \theta_y + \theta_z + 1)$$
$$-x(9\theta_w - 10\theta_y)(\theta_y - \theta_x - \theta_w) ,$$

$$\mathscr{D}_6 = 3\theta_y\theta_w + 8\theta_y\theta_w + 6\theta_x\theta_y - 24\theta_z\theta_w - 16\theta_x\theta_z + 8\theta_z^2$$
$$-8z(\theta_y - 2\theta_z - 1)(\theta_y - 2\theta_z) - 16yz(\theta_x + \theta_y + \theta_z + 1)(\theta_y - 2\theta_z)$$
$$-8yw(\theta_x - \theta_y + \theta_z - \theta_w - 1)(\theta_x - \theta_y + \theta_z - \theta_w)$$
$$-3xw(\theta_y - \theta_x - \theta_w - 1)(\theta_y - \theta_x - \theta_w) - 8y\theta_w(\theta_x - \theta_y + \theta_z - \theta_w)$$
$$-4xy(5\theta_w + 2\theta_x + \theta_y + 3)(\theta_x + \theta_y + \theta_z + 1) - x(3\theta_w - 2\theta_y)(\theta_y - \theta_x - \theta_w) . \tag{A.2}$$

Appendix B. Basis of the Mori Cone for $P(7, 3, 2, 1, 1)$

For this weighted projective space, we have two different desingularizations of the ambient space, $\mathbf{P}_{\Sigma(\Delta^*)_{\mathrm{reg}}^A}$ and $\mathbf{P}_{\Sigma(\Delta^*)_{\mathrm{reg}}^B}$ in the text. For each desingularization, we obtain the basis of the Mori cone following [37]. We see the Mori cone for $\Sigma(\Delta^*)_{\mathrm{reg}}^B$ is not simplicial.

For the regular fan $\Sigma(\Delta^*)_{\mathrm{reg}}^A$:

$$\eta_A^1 = (\ 1,0,0,1,0,0,-2,-1,1,\ 0,\ 0,\ 0) ,$$
$$\eta_A^2 = (-2,0,\ 0,0,\ 1,\ 1,\ 1,\ 1,-2,0,0,0) ,$$
$$\eta_A^3 = (-2,0,1,0,1,1,\ 0,\ 1,0,-2,\ 0,\ 0) ,$$
$$\eta_A^4 = (\ 2,0,\ 0,0,-1,-1,\ 0,-2,\ 1,1,0,0) ,$$
$$\eta_A^5 = (-2,1,0,0,1,1,\ 1,\ 0,0,\ 0,-2,\ 0) ,$$
$$\eta_A^6 = (\ 1,0,\ 0,0,-1,-1,-1,\ 0,\ 1,0,1,0) ,$$
$$\eta_A^7 = (-2,1,1,0,1,1,\ 0,\ 0,0,\ 0,\ 0,-2) ,$$
$$\eta_A^8 = (\ 1,0,-1,0,-1,-1,\ 0,\ 0,\ 0,1,0,1) . \tag{B.1}$$

For the regular fan $\Sigma(\Delta^*)_{\mathrm{reg}}^B$:
$$\eta_B^1 = (\ 0,0,\ 0,1,\ 0,\ 0,-2,0,\ 0,\ 0,\ 1,0,0) ,$$
$$\eta_B^2 = (-1,0,0,\ 1,\ 0,\ 0,0,\ 1,1,0,0,\ 0,-2) ,$$

$$\eta_B^3 = (-1, 0,\ 0, 0,\ 1,\ 1,\ 0, 0, -2,\ 0,\ 0, 0, 1),$$
$$\eta_B^4 = (\ 1, 0, 0, -1, -1, -1, 1,\ 0, 1, 0, 0,\ 0,\ 0),$$
$$\eta_B^5 = (-2, 0,\ 1, 0,\ 1,\ 1,\ 0, 1,\ 0, -2,\ 0, 0, 0),$$
$$\eta_B^6 = (\ 2, 0, 0,\ 0, -1, -1, 0, -2, 1, 1, 0,\ 0,\ 0),$$
$$\eta_B^7 = (-2, 1,\ 0, 0,\ 1,\ 1,\ 1, 0,\ 0,\ 0, -2, 0, 0),$$
$$\eta_B^8 = (-2, 1, 1,\ 0,\ 1,\ 1, 0,\ 0, 0, 0, 0, -2,\ 0),$$
$$\eta_B^9 = (\ 1, 0, -1, 0, -1, -1,\ 0, 0,\ 0,\ 1,\ 0, 1, 0). \tag{B.2}$$

Appendix C. Topological Data for Models with $h^{1,1} \leq 3$

We list the topological couplings for the Calabi–Yau models with $h^{1,1} \leq 3$. We follow the conventions in [8, 10], i.e., $8J_1^3 + 4J_1^2 J_2$ for the coupling means $K_{x_1 x_1 x_1}^{cl} = 8$, $K_{x_1 x_1 x_2}^{cl} = 4$ and others are zero. The superscript in each model shows the type of the model defined in (2.16). The divisors J_k and the variables $x^{(k)} = (-1)^{l_0^{(k)}} a^{l^{(k)}}$ are connected by the identification $J_k = m(\theta_{x^{(k)}})$ made in (3.27) and (3.37). According to Wall's theorem cited in Sect. 4, the topological type of the Calabi–Yau manifolds are classified by the classical Yukawa couplings (cubic form) and the invariant $c_2 \cdot J_k$ (linear form) on $H^{1,1}(X, \mathbf{Z})$.

For the interested reader we list the concrete basis $\{l^{(k)}\}$ for the Mori cone in the file appended to [24]. The basis for the Mori cone and the topological couplings in this list determine the prepotential $F(t)$ in (4.6).

Fermat type Calabi–Yau hypersurfaces

model	topological couplings	$c_2 \cdot \vec{J}$
$X_8^{\mathrm{I}}(2,2,2,1,1)_{-168}^2$	$8J_1^3 + 4J_1^2 J_2$	$(56, 24)$
$X_{12}^{\mathrm{I}}(6,2,2,1,1)_{-252}^2$	$4J_1^3 + 2J_1^2 J_2$	$(52, 24)$
$X_{12}^{\mathrm{II}}(4,3,2,2,1)_{-144}^2$	$2J_1^3 + 3J_1^2 J_2 + 3J_1 J_2^2 + 3J_2^3$	$(32, 42)$
$X_{14}^{\mathrm{I}}(7,2,2,2,1)_{-240}^2$	$2J_1^3 + 7J_1^2 J_2 + 21J_1 J_2^2 + 63J_2^3$	$(44, 126)$
$X_{18}^{\mathrm{I}}(9,6,1,1,1)_{-540}^2$	$9J_1^3 + 3J_1^2 J_2 + J_1 J_2^2$	$(102, 36)$
$X_{12}^{\mathrm{I}}(6,3,1,1,1)_{-324}^3$	$18J_1^3 + 6J_1^2 J_2 + 2J_1 J_2^2$ $+18J_1^2 J_3 + 6J_1 J_2 J_3 + J_2^2 J_3$ $+18J_1 J_3^2 + 3J_2 J_3^2 + 9J_3^3$	$(96, 36, 102)$
$X_{12}^{\mathrm{II}}(3,3,3,2,1)_{-132}^3$	$6J_1^3 + 4J_1^2 J_2 + 8J_1^2 J_3 + 4J_1 J_2 J_3$ $+8J_1 J_3^2 + 4J_2 J_3^2 + 8J_3^3$	$(48, 24, 56)$
$X_{15}^{\mathrm{II}}(5,3,3,3,1)_{-144}^3$	$3J_1^3 + 5J_1^2 J_2 + 5J_1 J_2^2 + 5J_2^3$ $+10J_1^2 J_3 + 15J_1 J_2 J_3 + 15J_2^2 J_3$ $+30J_1 J_3^2 + 45J_2 J_3^2 + 90J_3^3$	$(42, 50, 120)$
$X_{18}^{\mathrm{II}}(9,3,3,2,1)_{-192}^3$	$3J_1^3 + 2J_1^2 J_2 + 4J_1^2 J_3 + 2J_1 J_2 J_3$ $+4J_1 J_3^2 + 2J_2 J_3^2 + 4J_3^3$	$(42, 24, 52)$
$X_{24}^{\mathrm{I}}(12,8,2,1,1)_{-480}^3$	$8J_1^3 + 2J_1^2 J_2 + 4J_1^2 J_3$ $+J_1 J_2 J_3 + 2J_1 J_3^2$	$(92, 24, 48)$

Non-Fermat type Calabi–Yau hypersurfaces

model	topological couplings	$c_2 \cdot \vec{J}$
$X_9^{II}(3,2,2,1,1)^2_{-168}$	$6J_1^3 + 9J_1^2 J_2 + 13 J_1 J_2^2 + 17 J_2^3$	$(48,74)$
$X_7^{I}(2,2,1,1,1)^2_{-186}$	$14J_1^3 + 7J_1^2 J_2 + 3 J_1 J_2^2$	$(68,36)$
$X_8^{II}(3,2,1,1,1)^2_{-208}$	$36J_1^3 + 12J_1^2 J_2 + 4 J_1 J_2^2 + J_2^3$	$(96,34)$
$X_8^{I}(3,1,1,1,1)^2_{-240}$	$63J_1^3 + 21J_1^2 J_2 + 7 J_1 J_2^2 + 2 J_2^3$	$(126,44)$
$X_{14}^{II}(7,3,2,1,1)^2_{-260}$	$9J_1^3 + 3J_1^2 J_2 + J_1 J_2^2$	$(66,24)$
$X_{15}^{II}(5,4,3,2,1)^3_{-126}$	$8J_1^3 + 14J_1^2 J_2 + 24 J_1 J_2^2 + 37 J_2^3$ $+4J_1^2 J_3 + 7 J_1 J_2 J_3 + 10 J_2^2 J_3$ $+2J_1 J_3^2 + 2 J_2 J_3^2$	$(44,82,24)$
$X_{10}^{II}(3,2,2,2,1)^3_{-144}$	$90J_1^3 + 30J_1^2 J_2 + 10 J_1 J_2^2 + 3 J_2^3$ $+45J_1^2 J_3 + 15 J_1 J_2 J_3 + 5 J_2^2 J_3$ $+15J_1 J_3^2 + 5 J_2 J_3^2 + 5 J_3^3$	$(120,42,50)$
$X_{10}^{II}(3,3,2,1,1)^3_{-168}$	$15J_1^3 + 20J_1^2 J_2 + 26 J_1 J_2^2$ $+32J_2^3 + 10 J_1^2 J_3 + 13 J_1 J_2 J_3$ $+16J_2^2 J_3 + 6 J_1 J_3^2 + 6 J_2 J_3^2$	$(66,92,48)$
$X_{20}^{II}(10,4,3,2,1)^3_{-192}$	$18J_1^3 + 12J_1^2 J_2 + 8 J_1 J_2^2 + 5 J_2^3$ $+9J_1^2 J_3 + 6 J_1 J_2 J_3 + 4 J_2^2 J_3$ $+3J_1 J_3^2 + 2 J_2 J_3^2 + J_3^3$	$(72,50,34)$
$X_{10}^{I}(4,2,2,1,1)^3_{-192}$	$40J_1^3 + 20J_1^2 J_2 + 10 J_1 J_2^2 + 4 J_2^3$ $+10J_1^2 J_3 + 5 J_1 J_2 J_3 + 2 J_2^2 J_3$	$(100,52,24)$
$X_{16}^{II}(8,3,2,2,1)^3_{-200}$	$36J_1^3 + 12J_1^2 J_2 + 4 J_1 J_2^2 + J_2^3$ $+18J_1^2 J_3 + 6 J_1 J_2 J_3 + 2 J_2^2 J_3$ $+6J_1 J_3^2 + 2 J_2 J_3^2 + 2 J_3^3$	$(96,34,44)$
$X_{12}^{II}(5,3,2,1,1)^3_{-204}$	$50J_1^3 + 30J_1^2 J_2 + 18 J_1 J_2^2 + 9 J_2^3$ $+60J_1^2 J_3 + 36 J_1 J_2 J_3 + 21 J_2^2 J_3$ $+72J_1 J_3^2 + 43 J_2 J_3^2 + 86 J_3^3$	$(104,66,128)$
$X_{18}^{I}(9,4,2,2,1)^3_{-240}$	$8J_1^3 + 18J_1^2 J_2 + 36 J_1 J_2^2 + 72 J_2^3$ $+4J_1^2 J_3 + 9 J_1 J_2 J_3 + 18 J_2^2 J_3$ $+2J_1 J_3^2 + 4 J_2 J_3^2$	$(68,132,36)$
$X_9^{I}(4,2,1,1,1)^3_{-240}$	$72J_1^3 + 18J_1^2 J_2 + 4 J_1 J_2^2$ $+36J_1^2 J_3 + 9 J_1 J_2 J_3 + 2 J_2^2 J_3$ $+18J_1 J_3^2 + 4 J_2 J_3^2 + 8 J_3^3$	$(132,36,68)$

Non-Fermat type Calabi–Yau hypersurfaces

model	topological couplings	$c_2 \cdot \vec{J}$
$X_{16}^{\mathbf{I}}(8,3,3,1,1)^3_{-256}$	$6J_1^3 + 16J_1^2 J_2 + 42 J_1 J_2^2 + 104 J_2^3$ $+2 J_1^2 J_3 + 5 J_1 J_2 J_3 + 10 J_2^2 J_3$	$(60, 164, 24)$
$X_{16}^{\mathbf{I}}(8,5,1,1,1)^3_{-456}$	$50 J_1^3 + 10 J_1^2 J_2 + 2 J_1 J_2^2$ $+80 J_1^2 J_3 + 16 J_1 J_2 J_3 + 3 J_2^2 J_3$ $+128 J_1 J_3^2 + 25 J_2 J_3^2 + 203 J_3^3$	$(164, 36, 266)$

References

1. Candelas, P., de la Ossa, X., Green, P., Parks, L.: Nucl. Phys. **B359**, 21 (1991)
2. Morrison, D.: Picard–Fuchs Equations and Mirror Maps For Hypersurfaces. In: Essays on Mirror Manifolds. ed. S.-T.Yau, Hong Kong International Press, 1992
3. Klemm, A., Theisen, S.: Nucl. Phys. **B389**, 153 (1993)
4. Libgober, A., Teitelboim, J.: Duke Math. J., Int. Res. Notices **1**, 29 (1993)
5. Font, A.: Nucl. Phys. **B391**, 358 (1993)
6. Klemm, A., Theisen, S.: Theor. Math. Phys. **95**, 583 (1993)
7. Candelas, P., de la Ossa, X., Font, A., Katz, S., Morrison, D.: Nucl. Phys. **B416**, 481 (1994)
8. Hosono, S., Klemm, A., Theisen, S., Yau, S.-T.: Commun. Math. Phys. **167**, 301 (1995)
9. Candelas, P., de la Ossa, X., Font, A., Katz, S., Morrison, D.: Nucl. Phys. **B429**, 629 (1994)
10. Hosono, S., Klemm, A., Theisen, S., Yau, S.-T.: Nucl. Phys. **B433**, 501 (1995)
11. Berglund, P., Katz, S., Klemm, A.: Mirror Symmetry and the Moduli Space for Generic Hypersurfaces in Toric Varieties. hep-th/9506091
12. Kontsevich, M., Manin, Yu.: Commun. Math. Phys. **164**, 525 (1994)
13. Lian, B.H., Yau, S.-T.: Arithmetic properties of the mirror map and quantum coupling. hep-th/9411234
14. Klemm, A., Lerche, W., Myer, P.: K3-Fibrations and Heterotic-Type II String Duality. hep-th/9506112
15. Lian, B.H., Yau, S.-T.: Mirror Maps, Modular Relations and Hypergeometric Series I,II. hep-th/9507151, 9507153
16. Kachru, S., Vafa, C.: Exact Results for $N = 2$ Compactifications of Heterotic Strings. hep-th/9505105
17. Kachru, S., Klemm, A., Mayr, P., Lerche, W., Vafa, C.: Nonperturbative Results on the Point Particle Limit of $N = 2$ Heterotic String Compactification. hep-th/9508155
18. Lerche, W., Vafa, C., Warner, N.: Nucl. Phys. **B329**, 163 (1990)
19. Batyrev, V.: J., Algebraic Geometry **3**, 493 (1994)
20. Roan, S.-S.: Int. J. Math. **2**, 439 (1991)
21. Distler, J., Greene, B.: Nucl. Phys. **B309**, 295 (1988)
22. Batyrev, V.: Duke Math. J. **69**, 349 (1993)
23. Gel'fand, I.M., Zelevinski, A.V., Kapranov, M.M.: Funct. Anal. Appl. **28**, 94 (1989)
24. Hosono, S., Lian, B.H., Yau, S.-T.: Optional appendix to alg-geom/9511001
25. Klemm, A., Schimmrigk, R.: Nucl. Phys. **B411**, 559 (1994)
26. Kreuzer, M., Skarke, H.: Nucl. Phys. **B388**, 113 (1993)
27. Candelas, P., Dale, A.M., Lütken, C.A., Schimmrigk, R.: Nucl. Phys. **B298**, 493 (1988)
    Candelas, P., Lütken, C.A., Schimmrigk, R.: Nucl. Phys. **B306**, 113 (1988)
28. Batyrev, V., van Straten, D.: Generalized Hypergeometric Functions and Rational Curves on Calabi–Yau Complete Intersections in Toric Varieties. Alg-geom/9307010

29. Batyrev, V., Borisov, L.: On Calabi–Yau Complete Intersections in Toric Varieties. alg-geom/9412017
30. Dais, D.I.: Enumerative Combinatorics of Invariants of Certain Complex Threefolds with Trivial Canonical Bundle. MPI-preprint (1994)
31. Dolgachev, I.: In: Group actions and vector fields. Lecture Notes in Math. **Vol.956**, Berlin-Heidelberg-NewYork: Springer-Verlag (1991) P. 34
32. Greene, B., Plesser, M.: Nucl. Phys. **B338**, 15 (1990)
33. Candelas, P., de la Ossa, X., Katz, S.: Nucl. Phys. **B450**, 267 (1995)
34. Berglund, P., Hübsch, T.: Nucl. Phys. **B393**, 377 (1993)
35. Oda, T.: Convex bodies and Algebraic Geometry, An Introduction to the Theory of Toric Varieties. A Series of Modern Surveys in Mathematics, Berlin-Heidelberg-New York: Springer-Verlag, 1985
36. Billera, L., Filliman, P., Strumfels, B.: Adv. in Math. **83**, 155–179 (1990)
37. Oda, T., Park, H.S.: Tôhoku Math. J. **43**, 375–399 (1991)
38. Fulton, W.: Introduction to Toric Varieties. Ann. of Math. Studies **131**, Princeton, NJ: Princeton University Press, 1993
39. See for example, Cox, D., Little, J., O'shea, D.: Ideals, Varieties, and Algorithms. UTM Berlin-Heidelberg-New York: Springer-Verlag, 1991
40. Strumfels, B.: Tôhoku Math. J. **43**, 249 (1991)
41. Griffiths, P.: Ann. of Math. (2) **80**, 227 (1964)
42. Batyrev, V.: Quantum Cohomology Ring of Toric Manifolds. alg-geom/9310004
43. Aspinwall, P., Greene, B., Morrison, D.: Phys. Lett. **B303**, 249 (1993)
44. Griffiths, P., Harris, J.: Principles of Algebraic Geometry. New York: Wiley-Interscience, 1978
45. Witten, E.: Nucl. Phys. **B403**, 159 (1993)
46. Morrison, D., Plesser. : Nucl. Phys. **B404**, 279 (1995)
47. Aspinwall, P., Morrison, D.: Commun. Math. Phys. **151**, 245 (1993)
48. Witten, E.: Mirror Manifolds and Topological Field Theory. In: "Essays on Mirror Symmetry", ed. S.-T. Yau, Hong Kong: International Press, 1992
49. Strominger, A.: Commun. Math. Phys. **133**, 163 (1990)
50. Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C.: (with an appendix by S.Katz), Nucl. Phys. **B405**, 279 (1993)
51. Strominger, A.: Massless Black Holes and Conifolds in String Theory. hep-th/9504090
52. Greene, B., Morrison, D., Strominger, A.: Black Hole Condensation and the Unification of String Vacua. hep-th/9504145
53. Wall, C.T.: Invent. Math. **1**, 355 (1966)
54. Lynker, M., Schimmrigk, R.: Phys. Lett. **B249**, 237 (1990)
55. Yomemura, T.: Tôhoku Math. J. **42**, 351 (1990)