THE EXACT DISTRIBUTION OF VOTAW'S CRITERIA*

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1. Introduction

Let X_j, $j=1, 2, \cdots, (p+q)$ be $(p+q)$ stochastic variables having a multivariate Normal density $N(\mu, \Sigma)$. Let L denote the likelihood ratio criterion for testing the hypothesis H that the covariance matrix Σ is bipolar. That is, Σ is of the form

$$\Sigma=\begin{bmatrix}\Sigma_1 & \Sigma_2\\ \Sigma_2' & \Sigma_3\end{bmatrix}$$

where Σ_1 is a $p\times p$ matrix with diagonal elements equal to σ_{aa} and other elements to $\sigma_{aa'}$, Σ_2 is a $p\times q$ matrix with all elements equal to σ_{ab} and Σ_3 is a $q\times q$ matrix with diagonal elements σ_{bb} and other elements $\sigma_{bb'}$. Votaw [8] used Wilks' [10] method and derived the t-th moment of L, that is $E(L^t)$, under the hypothesis H. Roy [6] has shown that, under H and when a simple random sample of size n is taken from $N(\mu, \Sigma)$,

$$(1.1)\qquad E(L^t)=\{(p-1)^{p-1}(q-1)^{q-1}\}^t \times\frac{\Gamma\{(q-1)(n-1)/2\}\Gamma\{(p-1)(n-1)/2\}}{\Gamma\{(p-1)[t+(n-1)/2]\}\Gamma\{(q-1)[t+(n-1)/2]\}} \times\prod_{j=0}^{p+q-3}\left[\frac{\Gamma\{t+(n-3)/2-j/2\}}{\Gamma\{(n-3)/2-j/2\}}\right].$$

and has also given a good approximation to the distribution of L. Consul [3] has used some reduction formulae for hypergeometric series and obtained the exact distribution of L for the cases $p=q=2$, $p=q=3$ and $p=3, q=2$ in terms of hypergeometric series. For the cases $p=5$, $q=2, 3$ he obtained the distribution of L in terms of an infinite series of hypergeometric series. Here we will obtain the exact distribution of L, in the most general case, in G and H-functions and also in a simple computable series. In order to obtain the distribution in G and H-functions we will use the t-th moment of L in (1.1) and use Mellin inversion

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formula and for getting the distribution in simple computable series we will use the residue theorem and some properties of Riemann Zeta function.

2. The density of L in G and H-functions

Since $0<L<1$ the density of L is uniquely determined by the moment sequence $E(L^t)$ and is obtained as,

$$f(x)=\frac{1}{2\pi i}\int_{L'}E(L^t)x^{-t-1}dt$$
$$=x^{-1}\left\{\frac{1}{2\pi i}\int_{L'}E(L^t)x^{-t}dt\right\}, \qquad 0<x<1, \tag{2.1}$$

where L' is a suitably selected contour and $i=(-1)^{1/2}$. Throughout this article the density of L will be denoted by $f(x)$. Now,

$$E(L^t)=C_1\frac{\prod_{j=0}^{p+q-3}\Gamma\{t+(n-3)/2-j/2\}[(p-1)^{p-1}(q-1)^{q-1}]^t}{\Gamma\{(p-1)[t+(n-1)/2]\}\Gamma\{(q-1)[t+(n-1)/2]\}} \tag{2.2}$$

where

$$C_1=\frac{\Gamma\{(q-1)(n-1)/2\}\Gamma\{(p-1)(n-1)/2\}}{\prod_{j=0}^{p+q-3}\Gamma\{(n-3)/2-j/2\}}. \tag{2.3}$$

Hence the density is

$$f(x)=C_1x^{-1}H^{p+q-2,0}_{2,p+q-2}\left[\frac{x}{(p-1)^{p-1}(q-1)^{q-1}}\left|\begin{matrix}[(p-1)(n-1)/2,\,p-1],\;[(q-1)(n-1)/2,\,q-1]\\ [(n-3)/2,\,1],\;[(n-3)/2-1/2,\,1],\cdots,[(n-3)/2-(p+q-3)/2,\,1]\end{matrix}\right.\right], \quad 0<x<1 \tag{2.4}$$

in the well-known notations for the H-functions [2].

The H-function was first introduced and studied by Fox [5]. Braaksma [2] studied in detail the asymptotic expansions and analytic continuations for the H-function. A series expansion of an H-function is available in [2] but obviously the conditions for the expansion are not satisfied by the parameters in (2.4). A general expansion of an H-function, under all the conditions of its existence is not available in the literature so far and hence the density obtained in Section 3 is a new result on H-functions as well. We will also obtain the density in terms of Meijer's G-function. To this end, we will simplify the tth moment in (1.1) by using the Gauss-Legendre multiplication formula for the Gamma functions, namely,

$$\Gamma(mz)=(2\pi)^{(1-m)/2}m^{mz-1/2}\prod_{j=0}^{m-1}\Gamma(z+j/m)\,. \tag{2.5}$$

Now (1.1) becomes

$$E(L^t)=K_{p,q}\frac{\prod_{j=0}^{p+q-3}\Gamma[t+(n-1)/2-1-j/2]}{\prod_{j=0}^{p-2}\Gamma[t+(n-1)/2+j/(p-1)]\prod_{j=0}^{q-2}\Gamma[t+(n-1)/2+j/(q-1)]} \tag{2.6}$$

where

$$\begin{aligned}K_{p,q}=\Gamma[(q-1)(n-1)/2]\Gamma[(p-1)(n-1)/2]\Big/&\Big\{(2\pi)^{2-(p+q)/2}(p-1)^{(p-1)(n-1)/2-1/2}(q-1)^{(q-1)(n-1)/2-1/2}\\&\times\prod_{j=0}^{p+q-3}\Gamma[(n-3)/2-j/2]\Big\}\,.\end{aligned} \tag{2.7}$$

Now the density function is obtained by inverting (2.6) and is given as

$$\begin{aligned}f(x)=K_{p,q}x^{-1}G^{p+q-2,0}_{p+q-2,p+q-2}\Big[x\,\Big|\,&{(n-1)/2+1/(p-1),\cdots,\atop (n-1)/2-1,\,(n-1)/2-1-1/2,\cdots,}\\&{(n-1)/2+(p-2)/(p-1),\,(n-1)/2+1/(q-1),\cdots,\atop (n-1)/2-1-(p+q-3)/2}\\&(n-1)/2+(q-2)/(q-1)\Big]\,,\end{aligned} \tag{2.8}$$

$0<x<1$, where G is Meijer's G-function ([4], p. 207). Since $|x|<1$ and the lower parameters, namely, $(n-1)/2-1,\cdots,(n-1)/2-(p+q-3)/2-1$ in this case, differ by integers, a general series expansion of the G-function is not available in the literature. Hence the results obtained in Section 3 will also be new results on G-functions as well.

3. The density in simple functions

Here we will obtain the density for the general case $p\geqq 2$, $q\geqq 2$ except for the case $p=2$, $q=2$ which is simple and is given in Consul [3]. For convenience and simplicity we consider the cases p-odd, q-odd; p-odd, q-even and p-even, q-even separately. The case p-even, q-odd is available from the case p-odd, q-even. According to the residue theorem the density $f(x)$ is the sum of the residues of $E(L^t)x^{-t-1}$ at the poles of $\prod_{j=1}^{p+q-3}\Gamma[t+(n-1)/2-1-j/2]$ where the contour is selected as in the H and G-functions of (2.4) and (2.8). In order to evaluate the residues we will use the following procedure.

3A. *Case* I: *p-odd* $\geqq 3$, *q-odd* $\geqq 3$.

Consider the Gammas in (2.6) excluding the constant $K_{p,q}$. They can be simplified to the form

$$E'(L^t)=\frac{\prod_{j=1}^{(p+q-6)/2}\Gamma\{h-(p+q)/2+j\}\prod_{j=1}^{(p+q-6)/2}\Gamma\{h-(p+q+1)/2+j\}}{(h-1)^2(h-2)(h-1/2)^2(h-3/2)^2(h-5/2)} \tag{3.1}$$
$$\times\frac{1}{\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{h+j/(p-1)\}\prod_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\Gamma\{h+j/(q-1)}$$

on cancelling the common Gammas and taking $h=t+(n-1)/2$.

The poles coming from (3.1) can be obtained by equating to zero the various factors of

$$\begin{aligned}&\{h-(p+q)/2+1\}\{h-(p+q)/2+2\}^2\\ &\quad\cdots\{h-(p+q)/2+(p+q-2)/2\}^{(p+q-2)/2}\\ &\times\{h-(p+q)/2+(p+q)/2\}^{(p+q-6)/2}\\ &\times\{h-(p+q)/2+(p+q+2)/2\}^{(p+q-6)/2}\cdots\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\{h-(p+q+1)/2+1\}\{h-(p+q+1)/2+2\}^2\\ &\quad\cdots\{h-(p+q+1)/2+(p+q-2)/2\}^{(p+q-2)/2}\\ &\times\{h-(p+q+1)/2+(p+q)/2\}^{(p+q-2)/2}\\ &\times\{h-(p+q+1)/2+(p+q+2)/2\}^{(p+q-6)/2}\\ &\times\{h-(p+q+1)/2+(p+q+4)/2\}^{(p+q-6)/2}\cdots.\end{aligned} \tag{3.3}$$

The indices in (3.2) and (3.3) represent the orders of the poles. The residue corresponding to a pole belonging to (3.2) and of jth order is given by

$$a_{ij}(x)=\frac{1}{(j-1)!}\frac{\partial^{j-1}}{\partial t^{j-1}}[\{h-(p+q)/2+i\}^j E'(L^t)x^{-t}]\,, \tag{3.4}$$

at $h=(p+q)/2-i$,

$$=\frac{1}{(j-1)!}x^{-t}\left\{\frac{\partial}{\partial t}+(-\log x)\right\}^{j-1}[\{h-(p+q)/2+i\}^j E'(L^t)]$$

at $h=(p+q)/2-i$, where the operator is defined as follows:

$$\begin{aligned}\left\{\frac{\partial}{\partial t}+a\right\}^r&\equiv\left[\frac{\partial^r}{\partial t^r}+\binom{r}{1}a\frac{\partial^{r-1}}{\partial t^{r-1}}+\cdots+\binom{r}{r}a^r\right]\\ &\equiv\sum_{k=0}^{r}\binom{r}{k}a^k\frac{\partial^{r-k}}{\partial t^{r-k}}\,.\end{aligned} \tag{3.5}$$

Similarly we will define

$$b_{ij}(x)=\frac{1}{(j-1)!}x^{-t}\left\{\frac{\partial}{\partial t}+(-\log x)\right\}^{j-1} \times[\{h-(p+q+1)/2+i\}^{j}E'(L^{t})] \quad \text{at } h=(p+q+1)/2-i\,. \tag{3.6}$$

Consider the sets

$$\begin{cases} a=\{(i,j)\mid j=i,\ i=1,2,\cdots,(p+q-6)/2\} \\ a'=\{(i,j)\mid j=i,\ i=(p+q-4)/2,(p+q-2)/2\} \\ a''=\{(i,j)\mid j=(p+q-6)/2,\ i=(p+q)/2,(p+q+2)/2,\cdots\} \end{cases} \tag{3.7}$$

$$\begin{cases} b=\{(i,j)\mid j=i,\ i=1,2,\cdots,(p+q-6)/2\} \\ b'=\{(i,j)\mid j=i,\ i=(p+q-4)/2\} \\ b''=\{(i,j)\mid j=(p+q-2)/2,\ i=(p+q-2)/2,(p+q)/2\} \\ b'''=\{(i,j)\mid j=(p+q-6)/2,\ i=(p+q+2)/2,(p+q+4)/2,\cdots\}\,. \end{cases} \tag{3.8}$$

Then the density is given by

$$f(x)=K_{p,q}x^{-1}\left\{\sum_{a\cup a'\cup a''}a_{ij}(x)+\sum_{b\cup b'\cup b''\cup b'''}b_{ij}(x)\right\},\qquad 0<x<1\,. \tag{3.9}$$

$$=K_{p,q}x^{(n-3)/2}\left\{\sum_{a\cup a'\cup a''}\sum_{k=0}^{j-1}\left[\binom{j-1}{k}(-\log x)^{k}A_{0i}^{(j-1-k)}\right]\frac{x^{i-(p+q)/2}}{(j-1)!}\right.$$
$$+\sum_{b\cup b'\cup b''\cup b'''}\sum_{k=0}^{j-1}\left[\binom{j-1}{k}(-\log x)^{k}C_{0i}^{(j-1-k)}\right]$$
$$\left.\times\frac{x^{i-(p+q+1)/2}}{(j-1)!}\right\}\qquad 0<x<1\,,$$

where

$$A_{i}=\{h-(p+q)/2+i\}^{j}E'(L^{t})\,, \tag{3.10}$$

$$C_{i}=\{h-(p+q+1)/2+i\}^{j}E'(L^{t})\,, \tag{3.11}$$

$$A_{i}^{(r)}=\frac{\partial^{r}}{\partial t^{r}}A_{i}\,,\qquad C_{i}^{(r)}=\frac{\partial^{r}}{\partial t^{r}}C_{i}\,, \tag{3.12}$$

and A_{0i}, $A_{0i}^{(r)}$ denote respectively the values of A_i, $A_i^{(r)}$ at $h=(p+q)/2-i$ while C_{0i}, $C_{0i}^{(r)}$ denote respectively the values of C_i, $C_i^{(r)}$ at $h=(p+q+1)/2-i$.

In order to evaluate $A_{0i}^{(r)}$ and $C_{0i}^{(r)}$ we define

(3.13) $$B_i = \frac{\partial}{\partial t}\log A_i \quad \text{and} \quad D_i = \frac{\partial}{\partial t}\log C_i\,,$$

so that

(3.14) $$\frac{\partial}{\partial t}A_i = A_iB_i = A_i^{(1)}\,,$$

(3.15) $$A_i^{(m)} = A_i^{(m-1)}B_i + \binom{m-1}{1}A_i^{(m-2)}B_i^{(1)} + \cdots + \binom{m-1}{m-1}A_iB_i^{(m-1)}$$

where $B_i^{(r)}$ denotes the rth derivative of B_i. We can write similar expressions for $C_i^{(1)}$ and $C_i^{(m)}$. (3.15) gives a recurrence relation from which $A_{0i}^{(m)}$ is obtained on taking $h=(p+q)/2-i$. Similarly $C_{0i}^{(m)}$ can be obtained from $C_i^{(m)}$ on taking $h=(p+q+1)/2-i$. Hence the density in (3.9) is completely determined. A_{0i}, B_{0i}, $B_{0i}^{(r)}$ for the set $a\cup a'\cup a''$ and C_{0i}, D_{0i}, $D_{0i}^{(r)}$ for the set $b\cup b'\cup b''\cup b'''$ are derived below.

(i) $(i, j)\in a$: We have

(3.16) $$A_i = [\Gamma\{h-(p+q)/2+i+1\}]^{i+1}\prod_{j=i+2}^{(p+q-6)/2}\Gamma\{h-(p+q)/2+j\} \times \prod_{j=1}^{(p+q-6)/2}\Gamma\{h-(p+q+1)/2+j\}\Big/ \Big\{(h-2)(h-1)^2(h-5/2)(h-3/2)^2(h-1/2)^2 \times \prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma[h+j/(p-1)]\prod_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\Gamma[h+j/(q-1)] \times \prod_{j=1}^{i-1}[h-(p+q)/2+j]^j\Big\}.$$

Hence,

(3.17) $$A_{0i} = \prod_{j=i+2}^{(p+q-6)/2}\Gamma(-i+j)\prod_{j=1}^{(p+q-6)/2}\Gamma(-i-1/2+j)\Big/ \Big[\{(p+q)/2-i-2\}\{(p+q)/2-i-1\}^2\{(p+q)/2-i-5/2\} \times\{(p+q)/2-i-3/2\}^2\{(p+q)/2-i-1/2\}^2 \times\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q)/2-i+j/(p-1)\} \times\prod_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\Gamma\{(p+q)/2-i+j/(q-1)\}\prod_{j=1}^{i-1}(-i+j)^j\Big].$$

Now,

(3.18) $$B_i = \frac{\partial}{\partial t}\log A_i = (i+1)\phi\{h-(p+q)/2+i+1\}$$

$$+\sum_{j=i+2}^{(p+q-6)/2}\phi\{h-(p+q)/2+j\}+\sum_{j=1}^{(p+q-6)/2}\phi\{h-(p+q+1)/2+j\}$$
$$-1/(h-2)-2/(h-1)-1/(h-5/2)-2/(h-3/2)-2/(h-1/2)$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{h+j/(p-1)\}-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\phi\{h+j/(q-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{i-1}\{j/[h-(p+q)/2+j]\},$$

where

$$\phi(z)=(z-1)\sum_{m=0}^{\infty}1/[(m+1)(m+z)], \qquad z\neq 0, -1, \cdots, \tag{3.19}$$

which is evidently the well-known ϕ-function ([4], p. 15 (3)) excluding the Euler's constant γ. In our problem, due to the special nature of z, $\phi(z)$ can be put into a finite sum by using the results of ([4], p. 16). Hence,

$$\begin{aligned}(3.20)\quad B_{0i}=&\sum_{j=i+2}^{(p+q-6)/2}\phi(-i+j)+\sum_{j=1}^{(p+q-6)/2}\phi(-i-1/2+j)\\ &-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q)/2-i+j/(p-1)\}\\ &-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\phi\{(p+q)/2-i+j/(q-1)\}-1/\{(p+q)/2-i-2\}\\ &-2/\{(p+q)/2-i-1\}-1/\{(p+q)/2-i-5/2\}\\ &-2/\{(p+q)/2-i-3/2\}-2/\{(p+q)/2-i-1/2\}\\ &-\sum_{j=1}^{i-1}j/(-i+j).\end{aligned}$$

Now differentiating B_i in (3.18) r times, we have

$$\begin{aligned}(3.21)\quad B_i^{(r)}=(-1)^{r-1}r!\Big[&(i+1)g\{r+1, h-(p+q)/2+i+1\}\\ &+\sum_{j=i+2}^{(p+q-6)/2}g\{r+1, h-(p+q)/2+j\}\\ &+\sum_{j=1}^{(p+q-6)/2}g\{r+1, h-(p+q+1)/2+j\}\\ &-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}g\{r+1, h+j/(p-1)\}-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}g\{r+1, h+j/(q-1)\}\\ &+1/(h-2)^{r+1}+2/(h-1)^{r+1}+1/(h-5/2)^{r+1}+2/(h-3/2)^{r+1}\\ &+2/(h-1/2)^{r+1}+\sum_{j=1}^{i-1}j/\{h-(p+q)/2+j\}^{r+1}\Big]\end{aligned}$$

where

$$(3.22)\qquad g(s, c)=\sum_{m=0}^{\infty} 1/(m+c)^{s}, \qquad R(s)>0,\ c\neq 0, -1, -2, \cdots,$$

which is the well-known generalized Zeta function ([7], p. 36). Hence,

$$(3.23)\qquad B_{0i}^{(r)}=(-1)^{r-1}r!\Big[(i+1)g(r+1, 1)+\sum_{j=i+2}^{(p+q-6)/2} g(r+1, -i+j)$$
$$+\sum_{j=1}^{(p+q-6)/2} g(r+1, -i-1/2+j)$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2} g\{r+1, (p+q)/2-i+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2} g(r+1, (p+q)/2-i+j/(q-1)\}$$
$$+1/\{(p+q)/2-i-2\}^{r+1}+2/\{(p+q)/2-i-1\}^{r+1}$$
$$+1/\{(p+q)/2-i-5/2\}^{r+1}+2/\{(p+q)/2-i-3/2\}^{r+1}$$
$$+2/\{(p+q)/2-i-1/2\}^{r+1}+\sum_{j=1}^{i-1} j/(-i+j)^{r+1}\Big].$$

Without going into the details of calculations we write down A_{0i}, B_{0i}, $B_{0i}^{(r)}$ for $(i, j)\in a', a''$ and C_{0i}, D_{0i}, $D_{0i}^{(r)}$ for $(i, j)\in b, b', b'', b'''$ respectively.

(ii) $(i, j)\in a'$:

$$(3.24)\qquad A_{0i}=(-1)^{(p+q)/2-i}\prod_{j=1}^{(p+q-6)/2}\Gamma(-i-1/2+j)\Big/$$
$$\Big[\prod_{j=1}^{(p+q-6)/2}(-i+j)^{j}\prod_{j=(p+q-4)/2}^{i-1}(-i+j)^{(p+q-6)/2}$$
$$\times\{(p+q)/2-i-1/2\}^{2}\{(p+q)/2-i-3/2\}^{2}$$
$$\times\{(p+q)/2-i-5/2\}\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q)/2-i+j/(p-1)\}$$
$$\times\prod_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\Gamma\{(p+q)/2-i+j/(q-1)\}\Big].$$

$$(3.25)\qquad B_{0i}=(-2)^{(p+q-2)/2-i}+\sum_{j=1}^{(p+q-6)/2}\phi(-i-1/2+j)-\sum_{j=1}^{(p+q-6)/2} j/(-i+j)$$
$$-\{(p+q-6)/2\}\sum_{j=(p+q-4)/2}^{i-1} 1/(-i+j)-2/\{(p+q)/2-i-1/2\}$$
$$-2/\{(p+q)/2-i-3/2\}-1/\{(p+q)/2-i-5/2\}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q)/2-i+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\phi\{(p+q)/2-i+j/(q-1)\}.$$

$$(3.26)\quad B_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{2 \text{ or } (-1)^{r+1} \text{ according as } i=(p+q-4)/2 \text{ or } (p+q-2)/2\}+\{(p+q-6)/2\}g(r+1,1)$$

$$+\sum_{j=1}^{(p+q-6)/2} g(r+1,-i-1/2+j)+\sum_{j=1}^{(p+q-6)/2} j/(-i+j)^{r+1}$$

$$+\{(p+q-6)/2\}\sum_{j=(p+q-4)/2}^{i-1}\{1/(-i+j)^{r+1}\}$$

$$+2/\{(p+q)/2-i-1/2\}^{r+1}+2/\{(p+q)/2-i-3/2\}^{r+1}$$

$$\times 1/\{(p+q)/2-i-5/2\}^{r+1}$$

$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2} g\{r+1,(p+q)/2-i+j/(p-1)\}$$

$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2} g\{r+1,(p+q)/2-i+j/(q-1)\}\Big].$$

(iii) $(i, j)\in a''$:

$$(3.27)\quad A_{0i}=\prod_{j=1}^{(p+q-6)/2}\Gamma(-i-1/2+j)\Big/$$

$$\Big[\prod_{j=1}^{(p+q-6)/2}(-i+j)^{j}\prod_{j=(p+q-4)/2}^{i-1}(-i+j)^{(p+q-6)/2}\{(p+q)/2-i-1\}^{2}$$

$$\times\{(p+q)/2-i-2\}\{(p+q)/2-i-1/2\}^{2}$$

$$\times\{(p+q)/2-i-3/2\}^{2}\{(p+q)/2-i-5/2\}$$

$$\times\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q)/2-i+j/(p-1)\}$$

$$\prod_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\Gamma\{(p+q)/2-i+j/(q-1)\}\Big].$$

$$(3.28)\quad B_{0i}=\sum_{j=1}^{(p+q-6)/2}\phi(-i-1/2+j)-\sum_{j=1}^{(p+q-6)/2} j/(-i+j)-\{(p+q-6)/2\}$$

$$\times\sum_{j=(p+q-4)/2}^{i-1} 1/(-i+j)-2/\{(p+q)/2-i-1\}$$

$$-1/\{(p+q)/2-i-2\}-2/\{(p+q)/2-i-1/2\}$$

$$-2/\{(p+q)/2-i-3/2\}-1/\{(p+q)/2-i-5/2\}$$

$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q)/2-i+j/(p-1)\}$$

$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\phi\{(p+q)/2-i+j/(q-1)\}.$$

$$(3.29)\quad B_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{(p+q-6)/2\}g(r+1,1)$$

$$+\sum_{j=1}^{(p+q-6)/2} g(r+1,-i-1/2+j)+\sum_{j=1}^{(p+q-6)/2} j/(-i+j)^{r+1}$$

$$+\{(p+q-6)/2\}\sum_{j=(p+q-4)/2}^{i-1}\{1/(-i+j)^{r+1}\}$$
$$+2/\{(p+q)/2-i-1\}^{r+1}+1/\{(p+q)/2-i-2\}^{r+1}$$
$$+2/\{(p+q)/2-i-1/2\}^{r+1}+2/\{(p+q)/2-i-3/2\}^{r+1}$$
$$+1/\{(p+q)/2-i-5/2\}^{r+1}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2} g\{r+1,(p+q)/2-i+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2} g\{r+1,(p+q)/2-i+j/(q-1)\}\Big].$$

(iv) $(i, j)\in b$:

$$(3.30)\quad C_{0i}=\prod_{j=i+2}^{(p+q-6)/2}\Gamma(-i+j)\prod_{j=1}^{(p+q-6)/2}\Gamma(1/2-i+j)\Big/$$
$$\Big[\prod_{j=1}^{i-1}(-i+j)^{j}\{(p+q+1)/2-i-1\}^{2}\{(p+q+1)/2-i-2\}$$
$$\times\{(p+q+1)/2-i-1/2\}^{2}\{(p+q+1)/2-i-3/2\}^{2}$$
$$\times\{(p+q+1)/2-i-5/2\}$$
$$\times\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q+1)/2-i+j/(p+1)\}$$
$$\times\prod_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\Gamma\{(p+q+1)/2-i+j/(q-1)\}\Big].$$

$$(3.31)\quad D_{0i}=\sum_{j=i+2}^{(p+q-6)/2}\phi(-i+j)+\sum_{j=1}^{(p+q-6)/2}\phi(-i+1/2+j)-\sum_{j=1}^{i-1}j/(-i+j)$$
$$-2/\{(p+q+1)/2-i-1\}-1/\{(p+q+1)/2-i-2\}$$
$$-2/\{(p+q+1)/2-i-1/2\}-2/\{(p+q+1)/2-i-3/2\}$$
$$-1/\{(p+q+1)/2-i-5/2\}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\phi\{(p+q+1)/2-i+j/(q-1)\}.$$

$$(3.32)\quad D_{0i}^{(r)}=(-1)^{r-1}r!\Big[(i+1)g(r+1,1)+\sum_{j=i+2}^{(p+q-6)/2}g(r+1,-i+j)$$
$$+\sum_{j=1}^{(p+q-6)/2}g(r+1,1/2-i+j)+\sum_{j=1}^{i-1}j/(-i+j)^{r+1}$$
$$+2/\{(p+q+1)/2-i-1\}^{r+1}+1/\{(p+q+1)/2-i-2\}^{r+1}$$
$$+2/\{(p+q+1)2-i-1/2\}^{r+1}+2/\{(p+q+1)/2-i-3/2\}^{r}$$
$$+1/\{(p+q+1)/2-i-5/2\}^{r+1}$$

$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2} g\{r+1,\ (p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2} g\{r+1,\ (p+q+1)/2-i+j/(q-1)\}\Big].$$

(v) $(i, j)\in b'$:

$$(3.33)\quad C_{0i}=\prod_{j=1}^{(p+q-6)/2}\Gamma\{5/2-(p+q)/2+j\}\Big/\Big[(3/2)^2(1/2)2^2 1^2\prod_{j=1}^{(p+q-6)/2}\{5/2-(p+q+1)/2+j\}^j \times\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{5/2+j/(p-1)\}\prod_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\Gamma\{5/2+j/(q-1)\}\Big].$$

$$(3.34)\quad D_{0i}=\sum_{j=1}^{(p+q-6)/2}\phi\{5/2-(p+q)/2+j\}-\sum_{j=1}^{(p+q-6)/2}j/\{5/2-(p+q+1)/2+j\}$$
$$-2/(3/2)-1/(1/2)-2/2-2/1-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{5/2+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\phi\{5/2+j/(q-1)\}.$$

$$(3.35)\quad D_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{(p+q-6)/2\}g(r+1,1)$$
$$+\sum_{j=1}^{(p+q-6)/2}g\{r+1,\ 5/2-(p+q)/2+j\}$$
$$+\sum_{j=1}^{(p+q-6)/2}j/\{5/2-(p+q+1)/2+j\}+2/(3/2)^{r+1}+1/(1/2)^{r+1}$$
$$+2/(2)^{r+1}+2/(1)^{r+1}-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}g\{r+1,\ 5/2+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}g\{r+1,\ 5/2+j/(q-1)\}\Big].$$

(vi) $(i, j)\in b''$:

$$(3.36)\quad C_{0i}=\prod_{j=1}^{(p+q-6)/2}\Gamma(1/2-i+j)\Big/\Big[\{(p+q+1)/2-i-1\}^2\{(p+q+1)/2-i-2\}$$
$$\times\{(p+q+1)/2-i-5/2\}$$
$$\times\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q+1)/2-i+j/(p-1)\}$$
$$\times\prod_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\Gamma\{(p+q+1)/2-i+j/(q-1)\}\Big].$$

$$(3.37)\quad D_{0i}=2(-1)^{(p+q)/2-i}+\sum_{j=1}^{(p+q-6)/2}\phi(1/2-i+j)-2/\{(p+q+1)/2-i-1\}$$
$$-1/\{(p+q+1)/2-i-2\}-1/\{(p+q+1)/2-i-5/2\}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\phi\{(p+q+1)/2-i+j/(q-1)\}\,.$$

$$(3.38)\quad D_{0i}^{(r)}=(-1)^r r!\Big[\{2 \text{ or } 2(-1)^{r+1} \text{ according as } i=(p+q-2)/2$$
$$\text{or } (p+q)/2\}+\{(p+q-6)/2\}g(r+1,1)$$
$$+\sum_{j=1}^{(p+q-6)/2}g(r+1,1/2-i+j)+2/\{(p+q+1)/2-i-1\}^{r+1}$$
$$+1/\{(p+q+1)/2-i-2\}^{r+1}+1/\{(p+q+1)/2-i-5/2\}^{r+1}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}g\{r+1,(p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}g\{r+1,(p+q+1)/2-i+j/(q-1)\}\Big]\,.$$

(vii) $(i, j)\in b'''$:

$$(3.39)\quad C_{0i}=\prod_{j=1}^{(p+q-6)/2}\Gamma(1/2-i+j)\Big/$$
$$\Big[\prod_{j=1}^{(p+q-6)/2}(-i+j)^j\prod_{j=(p+q-4)/2}^{i-1}(-i+j)^{(p+q-6)/2}\{(p+q+1)/2-i-1\}^2$$
$$\times\{(p+q+1)/2-i-2\}\{(p+q+1)/2-i-1/2\}^2$$
$$\times\{(p+q+1)/2-i-3/2\}^2\{(p+q+1)/2-i-5/2\}$$
$$\times\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q+1)/2-i+j/(p-1)\}$$
$$\times\prod_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2}\Gamma\{(p+q+1)/2-i+j/(q-1)\}\Big]\,.$$

$$(3.40)\quad D_{0i}=\sum_{j=1}^{(p+q-6)/2}\phi(1/2-i+j)-\sum_{j=1}^{(p+q-6)/2}j/(-i+j)-\{(p+q-6)/2\}$$
$$\times\sum_{j=(p+q-4)/2}^{i-1}\{1/(-i+j)\}-2/\{(p+q+1)/2-i-1\}$$
$$-1/\{(p+q+1)/2-i-2\}-2/\{(p+q+1)/2-i-1/2\}$$
$$-2/\{(p+q+1)/2-i-3/2\}-1/\{(p+q+1)/2-i-5/2\}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q+1)/2-i+j/(p-1)\}$$

$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2} \phi\{(p+q+1)/2-i+j/(q-1)\}\,.$$

(3.41) $$D_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{(p+q-6)/2\}g(r+1,1)$$
$$+\sum_{j=1}^{(p+q-6)/2} g(r+1,1/2-i+j)+\sum_{j=1}^{(p+q-6)/2}\{j/(-i+j)^{r+1}\}$$
$$+\{(p+q-6)/2\}\sum_{j=(p+q-4)/2}^{i-1}\{1/(-i+j)^{r+1}\}$$
$$+2/\{(p+q+1)/2-i-1\}^{r+1}+1/\{(p+q+1)/2-i-2\}^{r+1}$$
$$+2/\{(p+q+1)/2-i-1/2\}^{r+1}+2/\{(p+q+1)/2-i-3/2\}^{r+1}$$
$$+1/\{(p+q+1)/2-i-5/2\}^{r+1}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2} g\{r+1,(p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{\substack{j=1\\ j\neq(q-1)/2}}^{q-2} g\{r+1,(p+q+1)/2-i+j/(q-1)\}\Big]\,.$$

Special cases

(a) For $p=3$, $q=3$ the density function (3.9) reduces to,

(3.42) $$f(x)=K_{3,3}x^{(n-3)/2}\Big[\Sigma_{a'}\sum_{k=0}^{j-1}\left\{\binom{j-1}{k}(-\log x)^k A_{0i}^{(j-1-k)}\right\}$$
$$\times\frac{x^{i-3}}{(j-1)!}+\Sigma_{b'\cup b''}\sum_{k=0}^{j-1}\left\{\binom{j-1}{k}(-\log x)^k C_{0i}^{(j-1-k)}\right\}\frac{x^{i-7/2}}{(j-1)!}\Big]\,,$$

$0<x<1$, where, $a'=\{(i,j)|j=i,\ i=1,2\}$, $b'=\{(i,j)|j=i=1\}$, $b''=\{(i,j)|j=2,\ i=2,3\}$ and $K_{3,3}$ is given in (2.7); that is

$$f(x)=K_{3,3}x^{(n-3)/2}[A_{01}x^{-2}+A_{02}(B_{02}-\log x)x^{-1}+C_{01}x^{-5/2}$$
$$+C_{02}(D_{02}-\log x)x^{-3/2}+C_{03}(D_{03}-\log x)x^{-1/2}]\,,$$

where,

$$A_{01}=-2^5/9,\quad A_{02}=2^5/3,\quad B_{02}=5/3,\quad C_{01}=2/9,\quad C_{02}=8,$$
$$C_{03}=4/3,\quad D_{02}=-3\quad\text{and}\quad D_{03}=43/6\,.$$

Hence,

$$f(x)=\frac{\{\Gamma(n-1)\}^2x^{n/2-4}}{\Gamma(n-4)\Gamma(n-6)2(4!)^2}$$
$$\times[-16x^{1/2}+(80-48\log x)x^{3/2}+1-(108+36\log x)x$$
$$+(43-6\log x)x^2]\,,$$

which is a result recently obtained by Consul [3].

(b) For $p=5$, $q=3$, the density function (3.9) gives,

$$(3.43)\quad f(x)=K_{5,3}\,x^{(n-3)/2}\Big[A_{01}x^{-3}+(A_{02}^{(1)}-A_{02}\log x)x^{-2}$$
$$+\left\{\binom{2}{0}A_{03}^{(2)}+\binom{2}{1}(-\log x)A_{03}^{(1)}+\binom{2}{2}(-\log x)^2A_{03}\right\}\frac{x^{-1}}{2!}$$
$$+\sum_{i=4}^{\infty}A_{0i}x^{i-4}+C_{01}x^{-7/2}+\{C_{02}^{(1)}-C_{02}\log x\}x^{-5/2}$$
$$+\left\{\binom{2}{0}C_{03}^{(2)}+\binom{2}{1}(-\log x)C_{03}^{(1)}+\binom{2}{2}(-\log x)^2C_{03}\right\}\frac{x^{-3/2}}{2!}$$
$$+\left\{\binom{3}{0}C_{04}^{(3)}+\binom{3}{1}(-\log x)C_{04}^{(2)}+\binom{3}{2}(-\log x)^2C_{04}^{(1)}\right.$$
$$\left.+\binom{3}{3}(-\log x)^3C_{04}\right\}\frac{x^{-1/2}}{3!}+\sum_{i=5}^{\infty}C_{0i}x^{i-9/2}\Big],$$

where

$$a=\{(i,j)\mid j=i=1\},\quad a'=\{(i,j)\mid j=i,\ i=2,3\},$$
$$a''=\{(i,j)\mid j=1,\ i=4,5,\cdots\},\quad b=\{(i,j)\mid j=i=1\},$$
$$b'=\{(i,j)\mid j=i=2\},\quad b''=\{(i,j)\mid j=3,\ i=3,4\}$$

and

$$b'''=\{(i,j)\mid j=1,\ i=5,6,\cdots\}.$$

Consul [3] has obtained a result for $p=5$, $q=3$ in terms of an infinite series involving Gauss's hypergeometric function ${}_2F_1(\cdot)$. Due to the difference in representations it does not seem to be easy to verify his result. Since our result is in simple computable form we will not try to put it into an infinite series involving hypergeometric series.

Cumulative distribution function

It is easy to show that for $\alpha>0$, $k=0,1,\cdots$ $0<x<1$,

$$(3.44)\quad \int_0^u x^{\alpha}(-\log x)^k dx=\frac{u^{\alpha+1}}{k+1}\sum_{r=0}^{k}[(k+1)k\cdots(k-r+1)$$
$$\times(-\log u)^{k-r}/(\alpha+1)^{r+1}].$$

Hence the cumulative distribution function is obtained on integrating (3.9) from 0 to u. That is,

$$(3.45)\quad F(u)=K_{p,q}\Big[\sum_{a\cup a'\cup a''}\frac{1}{(j-1)!}\sum_{k=0}^{j-1}\left\{\binom{j-1}{k}A_{0i}^{(j-1-k)}\frac{u^{i+(n-3)/2-(p+q)/2+1}}{k+1}\right.$$
$$\left.\times\sum_{r=0}^{k}\frac{(k+1)k\cdots(k-r+1)(-\log u)^{k-r}}{\{i+(n-3)/2-(p+q)/2+1\}^{r+1}}\right\}$$

$$+\sum_{b\cup b'\cup b''\cup b'''}\frac{1}{(j-1)!}\sum_{k=0}^{j-1}\left\{\binom{j-1}{k}C_{0i}^{(j-1-k)}\frac{u^{i+(n-3)/2-(p+q+1)/2+1}}{k+1}\right.$$
$$\left.\left.\times\sum_{r=0}^{k}\frac{(k+1)k\cdots(k-r+1)(-\log u)^{k-r}}{\{i+(n-3)/2-(p+q+1)/2+1\}^{r+1}}\right\}\right],$$

for $0<u<1$, $i+(n-3)/2-(p+q+1)/2>$ for all i.

3B. *Case* II: p-odd$\geqq 3$, q-even$\geqq 2$

The Gammas in (2.6) excluding the constant $K_{p,q}$ can be simplified to the form:

$$(3.46)\quad E'(L^t)=\prod_{j=0}^{(p+q-5)/2}\Gamma\{h-(p+q)/2+j\}\prod_{j=1}^{(p+q-5)/2}\Gamma\{h-(p+q+1)/2+j\}\Big/$$
$$\left[(h-3/2)(h-1/2)(h-2)(h-1)^2\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{h+j/(p-1)\}\right.$$
$$\left.\times\prod_{j=1}^{q-2}\Gamma\{h+j/(q-1)\}\right],$$

on cancelling the common Gammas and taking $h=(n-1)/2+t$. The poles coming from (3.46) are given by equating to zero the various factors of

$$(3.47)\quad \{h-(p+q)/2+1\}\{h-(p+q)/2+2\}^2\cdots\{h-(p+q)/2$$
$$+(p+q-3)/2\}^{(p+q-3)/2}\{h-(p+q)/2+(p+q-1)/2\}^{(p+q-3)/2}$$
$$\times\{h-(p+q)/2+(p+q+1)/2\}^{(p+q-5)/2}\{h-(p+q)/2$$
$$+(p+q+3)/2\}^{(p+q-5)/2}\cdots$$

and

$$(3.48)\quad \{h-(p+q+1)/2+1\}\{h-(p+q+1)/2+2\}^2\cdots\{h-(p+q+1)/2$$
$$+(p+q-1)/2\}^{(p+q-1)/2}\{h-(p+q+1)/2+(p+q+1)/2\}^{(p+q-5)/2}$$
$$\times\{h-(p+q+1)/2+(p+q+3)/2\}^{(p+q-5)/2}\cdots$$

where the indices in (3.47) and (3.48) represent the orders of the poles.

Proceeding as in the previous section we find that the density is given by

$$(3.49)\quad f(x)=K_{p,q}x^{(n-3)/2}\left[\sum_{a\cup a'\cup a''}\sum_{k=0}^{j-1}\left\{\binom{j-1}{k}(-\log x)^kA_{0i}^{(j-1-k)}\right\}\frac{x^{i-(p+q)/2}}{(j-1)!}\right.$$
$$\left.+\sum_{b\cup b'\cup b''}\sum_{k=0}^{j-1}\left\{\binom{j-1}{k}(-\log x)^kC_{0i}^{(j-1-k)}\right\}\frac{x^{i-(p+q+1)/2}}{(j-1)!}\right],$$
$$0<x<1,$$

where

$$(3.50)\quad \begin{cases} a=\{(i,j)\mid j=i,\ i=1,2,\cdots,(p+q-5)/2\}\,, \\ a'=\{(i,j)\mid j=(p+q-3)/2,\ i=(p+q-3)/2,(p+q-1)/2\}\,, \\ a''=\{(i,j)\mid j=(p+q-5)/2,\ i=(p+q+1)/2,(p+q+3)/2,\cdots\}\,, \end{cases}$$

$$(3.51)\quad \begin{cases} b=\{(i,j)\mid j=i,\ i=1,2,\cdots,(p+q-5)/2\}\,, \\ b'=\{(i,j)\mid j=i,\ i=(p+q-3)/2,(p+q-1)/2\}\,, \\ b''=\{(i,j)\mid j=(p+q-5)/2,\ i=(p+q+1)/2,(p+q+3)/2,\cdots\}\,, \end{cases}$$

and A_{0i}, B_{0i}, $B_{0i}^{(r)}$, C_{0i}, D_{0i}, $D_{0i}^{(r)}$ corresponding to the above sets are given below.

(i) $(i,j)\in a$:

$$(3.52)\quad A_{0i}=\prod_{j=i+2}^{(p+q-5)/2}\Gamma(-i+j)\prod_{j=1}^{(p+q-5)/2}\Gamma(-i-1/2+j)\Big/ \Big[\{(p+q)/2-i-2\}\{(p+q)/2-i-1\}^2\{(p+q)/2-i-3/2\} \times\{(p+q)/2-i-1/2\}\prod_{j=1}^{i-1}(-i+j)^j \times\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q)/2-i+j/(p-1)\} \times\prod_{j=1}^{q-2}\Gamma\{(p+q)/2-i+j/(q-1)\}\Big].$$

$$(3.53)\quad B_{0i}=\sum_{j=i+2}^{(p+q-5)/2}\phi(-i+j)+\sum_{j=1}^{(p+q-5)/2}\phi(-i-1/2+j)-1/\{(p+q)/2-i-2\} -2/\{(p+q)/2-i-1\}-1/\{(p+q)/2-i-3/2\} -1/\{(p+q)/2-i-1/2\}-\sum_{j=1}^{i-1}j/(-i+j) -\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q)/2-i+j/(p-1)\} -\sum_{j=1}^{q-2}\phi\{(p+q)/2-i+j/(q-1)\}.$$

$$(3.54)\quad B_{0i}^{(r)}=(-1)^{r-1}r!\Big[(i+1)g(r+1,1)+\sum_{j=i+2}^{(p+q-5)/2}g(r+1,-i+j) +\sum_{j=1}^{(p+q-5)/2}g(r+1,-i-1/2+j)+1/\{(p+q)/2-i-2\}^{r+1} +2/\{(p+q)/2-i-1\}^{r+1}+1/\{(p+q)/2-i-3/2\}^{r+1} +1/\{(p+q)/2-i-1/2\}^{r+1} -\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}g\{r+1,(p+q)/2-i+j/(p-1)\}$$

$$-\sum_{j=1}^{q-2} g\{r+1, (p+q)/2-i+j/(q-1)\}$$
$$\left.+\sum_{j=1}^{i-1} j/(-i+j)^{r+1}\right].$$

(ii) $(i, j) \in a'$:

$$(3.55)\quad A_{0i}=(-1)^{(p+q+1)/2-i} \prod_{j=1}^{(p+q-5)/2} \Gamma(-i-1/2+j) \Big/$$
$$\left[\prod_{j=1}^{(p+q-5)/2} (-i+j)^j \prod_{j=(p+q-3)/2}^{i-1} (-i+j)^{(p+q-5)/2}\right.$$
$$\times \{(p+q)/2-i-2\} \{(p+q)/2-i-1\}^2$$
$$\times \prod_{\substack{j=1 \\ j\neq(p-1)/2}}^{p-2} \Gamma\{(p+q)/2-i+j/(p-1)\}$$
$$\left.\times \prod_{j=1}^{q-2} \Gamma\{(p+q)/2-i+j/(q-1)\}\right].$$

$$(3.56)\quad B_{0i}=(-1)^{(p+q-1)/2-i}+\sum_{j=1}^{(p+q-5)/2} \phi(-i-1/2+j)-\sum_{j=1}^{(p+q-5)/2} j/(-i+j)$$
$$-\{(p+q-5)/2\} \sum_{j=(p+q-3)/2}^{i-1} 1/(-i+j)-1/\{(p+q)/2-i-2\}$$
$$-2/\{(p+q)/2-i-1\}-\sum_{\substack{j=1 \\ j\neq(p-1)/2}}^{p-2} \phi\{(p+q)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2} \phi\{(p+q)/2-i+j/(q-1)\}.$$

$$(3.57)\quad B_{0i}^{(r)}=(-1)^{r-1} r! \Big[\{1 \text{ or } (-1)^{r+1} \text{ according as } i=(p+q-3)/2$$
$$\text{or } (p+q-1)/2\}+\{(p+q-5)/2\} g(r+1, 1)$$
$$+\sum_{j=1}^{(p+q-5)/2} g(r+1, -i-1/2+j)+\sum_{j=1}^{(p+q-5)/2} j/(-i+j)^{r+1}$$
$$+\{(p+q-5)/2\} \sum_{j=(p+q-3)/2}^{i-1} \{1/(-i+j)^{r+1}\}$$
$$+1/\{(p+q)/2-i-2\}^{r+1}+2/\{(p+q)/2-i-1\}^{r+1}$$
$$-\sum_{\substack{j=1 \\ j\neq(p-1)/2}}^{p-2} g\{r+1, (p+q)/2-i+j/(p-1)\}$$
$$\left.-\sum_{j=1}^{q-2} g\{r+1, (p+q)/2-i+j/(q-1)\}\right].$$

(iii) $(i, j) \in a''$:

$$(3.58)\quad A_{0i}=\prod_{j=1}^{(p+q-5)/2} \Gamma(-i-1/2+j) \Big/$$
$$\left[\prod_{j=1}^{(p+q-5)/2} (-i+j)^j \prod_{j=(p+q-3)/2}^{i-1} (-i+j)^{(p+q-5)/2} \{(p+q)/2-i-2\}\right.$$

$$\times\{(p+q)/2-i-1\}^2\{(p+q)/2-i-3/2\}$$
$$\times\{(p+q)/2-i-1/2\}\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q)/2-i+j/(p-1)\}$$
$$\times\prod_{j=1}^{q-2}\Gamma\{(p+q)/2-i+j/(q-1)\}\Big].$$

(3.59) $$B_{0i}=\sum_{j=1}^{(p+q-5)/2}\phi(-i-1/2+j)-\sum_{j=1}^{(p+q-5)/2}j/(-i+j)-\{(p+q-5)/2\}$$
$$\times\sum_{j=(p+q-3)/2}^{i-1}1/(-i+j)-1/\{(p+q)/2-i-2\}$$
$$-2/\{(p+q)/2-i-1\}-1/\{(p+q)/2-i-3/2\}$$
$$-1/\{(p+q)/2-i-1/2\}-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2}\phi\{(p+q)/2-i+j)/(q-1)\}.$$

(3.60) $$B_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{(p+q-5)/2\}g(r+1,1)$$
$$+\sum_{j=1}^{(p+q-5)/2}g(r+1,-i-1/2+j)+\sum_{j=1}^{(p+q-5)/2}j/(-i+j)^{r+1}$$
$$+\{(p+q-5)/2\}\sum_{j=(p+q-3)/2}^{i-1}\{1/(-i+j)^{r+1}\}$$
$$+1/\{(p+q)/2-i-2\}^{r+1}+2/\{(p+q)/2-i-1\}^{r+1}$$
$$+1/\{(p+q)/2-i-3/2\}^{r+1}+1/\{(p+q)/2-i-1/2\}^{r+1}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}g\{r+1,(p+q)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2}g\{r+1,(p+q)/2-i+j/(q-1)\}\Big].$$

(iv) $(i,j)\in b$:

(3.61) $$C_{0i}=\prod_{j=1}^{(p+q-5)/2}\Gamma(-i+1/2+j)\prod_{j=i+2}^{(p+q-5)/2}\Gamma(-i+j)\Big/$$
$$\Big[\prod_{j=1}^{i-1}(-i+j)^j\{(p+q+1)/2-i-2\}\{(p+q+1)/2-i-1\}^2$$
$$\times\{(p+q+1)/2-i-3/2\}\{(p+q+1)/2-i-1/2\}$$
$$\times\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q+1)/2-i+j/(p-1)\}$$
$$\times\prod_{j=1}^{q-2}\Gamma\{(p+q+1)/2-i+j/(q-1)\}\Big].$$

(3.62) $$D_{0i}=\sum_{j=i+2}^{(p+q-5)/2}\phi(-i+j)+\sum_{j=1}^{(p+q-5)/2}\phi(-i+1)/2+j)$$

$$-\sum_{j=1}^{i-1} j/(-i+j)-1/\{(p+q+1)/2-i-2\}$$
$$-2/\{(p+q+1)/2-i-1\}-1/\{(p+q+1)/2-i-3/2\}$$
$$-1/\{(p+q+1)/2-i-1/2\}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2} \phi\{(p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2} \phi\{(p+q+1)/2-i+j/(q-1)\}\,.$$

(3.63) $$D_{0i}^{(r)}=(-1)^{r-1}r!\Big[(i+1)g(r+1,1)+\sum_{j=i+2}^{(p+q-5)/2} g(r+1,-i+j)$$
$$+\sum_{j=1}^{(p+q-5)/2} g(r+1,-i+1/2+j)+\sum_{j=1}^{i-1} j/(-i+j)^{r+1}$$
$$+1/\{(p+q+1)/2-i-2\}^{r+1}+2/\{(p+q+1)/2-i-1\}^{r+1}$$
$$+1/\{(p+q+1)/2-i-3/2\}^{r+1}+1/\{(p+q+1)/2-i-1/2\}^{r+1}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2} g\{r+1,(p+q+1)2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2} g\{r+1,(p+q+1)/2-i+j/(q-1)\}\Big]\,.$$

(v) $(i, j)\in b'$:

(3.64) $$C_{0i}=(-1)^{(p+q+1)/2-i}\prod_{j=1}^{(p+q-5)/2}\Gamma(1/2-i+j)\Big/$$
$$\Big[\{(p+q+1)/2-i-3/2\}\{(p+q+1)/2-i-1/2\}$$
$$\times\prod_{j=1}^{(p+q-5)/2}(-i+j)^{j}\prod_{j=(p+q-3)/2}^{i-1}(-i+j)^{(p+q-5)/2}$$
$$\times\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q+1)/2-i+j/(p-1)\}$$
$$\times\prod_{j=1}^{q-2}\Gamma\{(p+q+1)/2-i+j/(q-1)\}\Big]\,.$$

(3.65) $$D_{0i}=\{-2 \text{ or } 1 \text{ according as } i=(p+q-3)/2 \text{ or } (p+q-1)/2\}$$
$$+\sum_{j=1}^{(p+q-5)/2}\phi(1/2-i+j)-1/\{(p+q+1)/2-i-3/2\}$$
$$-1/\{(p+q+1)/2-i-1/2\}-\sum_{j=1}^{(p+q-5)/2} j/(-i+j)$$
$$-\{(p+q-5)/2\}\sum_{j=(p+q-3)/2}^{i-1}\{1/(-i+j)\}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q+1)/2-i+j/(p-1)\}$$

$$-\sum_{j=1}^{q-2}\phi\{(p+q+1)/2-i+j/(q-1)\}\,.$$

(3.66) $$D_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{2 \text{ or } (-1)^{r+1} \text{ according as } i=(p+q-3)/2 \text{ or } (p+q-1)/2\}+\{(p+q-5)/2\}g(r+1,1)$$
$$+\sum_{j=1}^{(p+q-5)/2} g(r+1,1/2-i+j)+1/\{(p+q+1)/2-i-3/2\}^{r+1}$$
$$+1/\{(p+q+1)/2-i-1/2\}^{r+1}+\sum_{j=1}^{(p+q-5)/2} j/(-i+j)^{r+1}$$
$$+\{(p+q-5)/2\}\sum_{j=(p+q-3)/2}^{i-1}\{1/(-i+j)^{r+1}\}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2} g\{r+1,(p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2} g\{r+1,(p+q+1)/2-i+j/(q-1)\}\Big]\,.$$

(vi) $(i,j)\in b''$:

(3.67) $$C_{0i}=\prod_{j=1}^{(p+q-5)/2}\Gamma(1/2-i+j)\Big/\Big[\prod_{j=1}^{(p+q-5)/2}(-i+j)^{j}\prod_{j=(p+q-3)/2}^{i-1}(-i+j)^{(p+q-5)/2}\{(p+q+1)/2-i-2\}$$
$$\times\{(p+q+1)/2-i-1\}^{2}\{(p+q+1)/2-i-3/2\}$$
$$\times\{(p+q+1)/2-i-1/2\}$$
$$\times\prod_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\Gamma\{(p+q+1)/2-i+j/(p-1)\}$$
$$\times\prod_{j=1}^{q-2}\Gamma\{(p+q+1)/2-i+j/(q-1)\}\Big]\,.$$

(3.68) $$D_{0i}=\sum_{j=1}^{(p+q-5)/2}\phi(1/2-i+j)-\sum_{j=1}^{(p+q-5)/2} j/(-i+j)-\{(p+q-5)/2\}$$
$$\times\sum_{j=(p+q-3)/2}^{i-1}\{1/(-i+j)\}-1/\{(p+q+1)/2-i-2\}$$
$$-2/\{(p+q+1)/2-i-1\}-1/\{(p+q+1)/2-i-3/2\}$$
$$-1/\{(p+q+1)/2-i-1/2\}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2}\phi\{(p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2}\phi\{(p+q+1)/2-i+j/(q-1)\}\,.$$

(3.69) $$D_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{(p+q-5)/2\}g(r+1,1)$$

$$+\sum_{j=1}^{(p+q-5)/2} g(r+1, 1/2-i+j)+\sum_{j=1}^{(p+q-5)/2} j/(-i+j)^{r+1}$$
$$+\{(p+q-5)/2\}\sum_{j=(p+q-3)/2}^{i-1}\{1/(-i+j)^{r+1}\}$$
$$+1/\{(p+q+1)/2-i-2\}^{r+1}+2/\{(p+q+1)/2-i-1\}^{r+1}$$
$$+1/\{(p+q+1)/2-i-3/2\}^{r+1}+1/\{(p+q+1)/2-i-1/2\}^{r+1}$$
$$-\sum_{\substack{j=1\\ j\neq(p-1)/2}}^{p-2} g\{r+1, (p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2} g\{r+1, (p+q+1)/2-i+j/(q-1)\}\Big].$$

Special cases

(a) For $p=3$, $q=2$ the density function (3.49) yields

(3.70) $$f(x)=K_{3,2}x^{(n-3)/2}[A_{01}x^{-3/2}+A_{02}x^{-1/2}+C_{01}x^{-2}+C_{02}(D_{02}-\log x)x^{-1}], \quad 0<x<1,$$

where $A_{01}=-8$, $A_{02}=8/3$, $C_{01}=4/3$, $C_{02}=4$, $D_{02}=1$ and $K_{3,2}$ is given in (2.7). That is,

$$f(x)=\frac{\Gamma(n-1)\Gamma\{(n-1)/2\}}{2\Gamma(n-4)\Gamma\{(n-5)/2\}}x^{(n-7)/2}$$
$$\times[-2x^{1/2}+(2/3)x^{3/2}+(1/3)+(1-\log x)x],$$

which is the result recently given by Consul [3].

(b) When $p=3$, $q=4$ the density function (3.49) gives the following result.

(3.71) $$f(x)=K_{3,4}x^{(n-3)/2}\Big[A_{01}x^{-5/2}+(A_{02}^{(1)}-A_{02}\log x)x^{-3/2}$$
$$+(A_{03}^{(1)}-A_{03}\log x)x^{-1/2}+\sum_{i=4}^{\infty}A_{0i}x^{i-7/2}+C_{01}x^{-3}$$
$$+(C_{02}^{(1)}-C_{02}\log x)x^{-2}+\left\{\binom{2}{0}C_{03}^{(2)}+\binom{2}{1}(-\log x)C_{03}^{(1)}\right.$$
$$\left.+\binom{2}{2}(-\log x)^2C_{03}\right\}x^{-1}\Big/2!+\sum_{i=4}^{\infty}C_{0i}x^{i-4}\Big], \qquad 0<x<1,$$

and the sets are given as follows:

$$a=\{(i, j)\mid j=i=1\}, \qquad a'=\{(i, j)\mid j=2, \ i=2, 3\},$$
$$a''=\{(i, j)\mid j=1, \ i=4, 5, \cdots\}, \qquad b=\{(i, j)\mid j=i=1\},$$
$$b'=\{(i, j)\mid j=i, \ i=2, 3\}, \qquad b''=\{(i, j)\mid j=1, \ i=4, 5, \cdots\}.$$

The cumulative distribution function

The cumulative distribution function in this case is given as

$$(3.72)\quad F(u)=K_{p,q}\Big\{\sum_{a\cup a'\cup a''}\frac{1}{(j-1)!}\sum_{k=0}^{j-1}\Big[\binom{j-1}{k}A_{0i}^{(j-1-k)}\frac{u^{i+(n-3)/2-(p+q)/2+1}}{k+1}\times\sum_{r=0}^{k}\frac{(k+1)k\cdots(k-r+1)(-\log u)^{k-r}}{\{i+(n-3)/2-(p+q)/2+1\}^{r+1}}\Big]$$
$$+\sum_{b\cup b'\cup b''}\frac{1}{(j-1)!}\sum_{k=0}^{j-1}\Big[\binom{j-1}{k}C_{0i}^{(j-1-k)}\frac{u^{i+(n-3)/2-(p+q+1)/2+1}}{k+1}\times\sum_{r=0}^{k}\frac{(k+1)k\cdots(k-r+1)(-\log u)^{k-r}}{\{i+(n-3)/2-(p+q+1)/2+1\}^{r+1}}\Big]\Big\},$$

valid for $0<u<1$, $i+(n-3)/2-(p+q+1)/2>0$ for all i.

3C. *Case* III: p-even, q-even, $p\geqq 2$, $q\geqq 2$ $(p+q>4)$

In this case the Gammas in (2.6), excluding the constant $K_{p,q}$ and after concelling the common Gammas, take the form

$$(3.73)\quad E'(L^t)=\prod_{j=1}^{(p+q-6)/2}\Gamma\{h-(p+q)/2+j\}\prod_{j=1}^{(p+q-2)/2}\Gamma\{h-(p+q+1)/2+j\}\Big/$$
$$(h-1)^2(h-2)\prod_{j=1}^{p-2}\Gamma\{h+j/(p-1)\}\prod_{j=1}^{q-2}\Gamma\{h+j/(q-1)\}.$$

The poles of (3.73) are obtained by equating to zero the various factors of

$$(3.74)\quad \{h-(p+q)/2+1\}\{h-(p+q)/2+2\}^2\cdots\{h-(p+q)/2+(p+q-2)/2\}^{(p+q-2)/2}\{h-(p+q)/2+(p+q)/2\}^{(p+q-6)/2}\times\{h-(p+q)/2+(p+q+2)/2\}^{(p+q-6)/2}\cdots$$

and

$$(3.75)\quad \{h-(p+q+1)/2+1\}\{h-(p+q+1)/2+2\}^2\cdots\{h-(p+q+1)/2+(p+q-2)/2\}^{(p+q-2)/2}\{h-(p+q+1)/2+(p+q)/2\}^{(p+q-2)/2}\cdots$$

where the indices give the orders of the poles. As before the density is given by

$$(3.76)\quad f(x)=K_{p,q}x^{(n-3)/2}\Big\{\sum_{a\cup a'\cup a''}\sum_{k=0}^{j-1}\Big[\binom{j-1}{k}(-\log x)^k A_{0i}^{(j-1-k)}\Big]\frac{x^{i-(p+q)/2}}{(j-1)!}$$
$$+\sum_{b\cup b'}\sum_{k=0}^{j-1}\Big[\binom{j-1}{k}(-\log x)^k C_{0i}^{(j-1-k)}\Big]\frac{x^{i-(p+q+1)/2}}{(j-1)!}\Big\},$$
$$0<x<1,$$

where

$$(3.77)\quad \begin{cases} a=\{(i,j)\mid j=i,\ i=1,2,\cdots,(p+q-6)/2\}\,,\\ a'=\{(i,j)\mid j=i,\ i=(p+q-4)/2,(p+q-2)/2\}\,,\\ a''=\{(i,j)\mid j=(p+q-6)/2,\ i=(p+q)/2,(p+q+2)/2,\cdots\}\,,\end{cases}$$

$$(3.78)\quad \begin{cases} b=\{(i,j)\mid j=i,\ i=1,\cdots,(p+q-2)/2\}\,,\\ b'=\{(i,j)\mid j=(p+q-2)/2,\ i=(p+q)/2,(p+q+2)/2,\cdots\}\,,\end{cases}$$

and A_{0i}, B_{0i}, $B_{0i}^{(r)}$, C_{0i}, D_{0i} and $D_{0i}^{(r)}$ corresponding to the above sets are given below.

(i) $(i,j)\in a$:

$$(3.79)\quad A_{0i}=\prod_{j=i+2}^{(p+q-6)/2}\Gamma(-i+j)\prod_{j=1}^{(p+q-2)/2}\Gamma(-i-1/2+j)\Big/ \Big[\{(p+q)/2-i-1\}^2\{(p+q)/2-i-2\}\prod_{j=1}^{i-1}(-i+j)^j \times\prod_{j=1}^{p-2}\Gamma\{(p+q)/2-i+j/(p-1)\} \times\prod_{j=1}^{q-2}\Gamma\{(p+q)/2-i+j/(q-1)\}\Big]\,.$$

$$(3.80)\quad B_{0i}=\sum_{j=i+2}^{(p+q-6)/2}\phi(-i+j)+\sum_{j=1}^{(p+q-2)/2}\phi(-i-1/2+j)-\sum_{j=1}^{i-1}j/(-i+j) -2/\{(p+q)/2-i-1\}-1/\{(p+q)/2-i-2\} -\sum_{j=1}^{p-2}\phi\{(p+q)/2-i+j/(p-1)\} -\sum_{j=1}^{q-2}\phi\{(p+q)/2-i+j/(q-1)\}\,.$$

$$(3.81)\quad B_{0i}^{(r)}=(-1)^{r-1}r!\Big[(i+1)g(r+1,1)+\sum_{j=i+2}^{(p+q-6)/2}g(r+1,-i+j) +\sum_{j=1}^{(p+q-2)/2}g(r+1,-i-1/2+j)+\sum_{j=1}^{i-1}j/(-i+j)^{r+1} +2/\{(p+q)/2-i-1\}^{r+1}+1/\{(p+q)/2-i-2\}^{r+1} -\sum_{j=1}^{p-2}g\{r+1,(p+q)/2-i+j/(p-1)\} -\sum_{j=1}^{q-2}g\{r+1,(p+q)/2-i+j/(q-1)\}\Big]\,.$$

(ii) $(i,j)\in a'$:

$$(3.82)\quad A_{0i}=(-1)^{(p+q)/2-i}\prod_{j=1}^{(p+q-2)/2}\Gamma(-i-1/2+j)\Big/$$

$$\left[\prod_{j=1}^{(p+q-6)/2}(-i+j)^j \prod_{j=(p+q-4)/2}^{i-1}(-i+j)^{(p+q-6)/2} \times\prod_{j=1}^{p-2}\Gamma\{(p+q)/2-i+j/(p-1)\} \times\prod_{j=1}^{q-2}\Gamma\{(p+q)/2-i+j/(q-1)\}\right].$$

(3.83) $$B_{0i}=\{-2 \text{ or } 1 \text{ according as } i=(p+q-4)/2 \text{ or } (p+q-2)/2\} + \sum_{j=1}^{(p+q-2)/2}\phi(-i-1/2+j)-\sum_{j=1}^{(p+q-6)/2} j/(-i+j)-\{(p+q-6)/2\} \times \sum_{j=(p+q-4)/2}^{i-1}\{1/(-i+j)\}-\sum_{j=1}^{p-2}\phi\{(p+q)/2-i+j/(p-1)\} -\sum_{j=1}^{q-2}\phi\{(p+q)/2-i+j/(q-1)\}.$$

(3.84) $$B_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{2 \text{ or } (-1)^{r+1} \text{ according as } i=(p+q-4)/2 \text{ or } (p+q-2)/2\}+\sum_{j=1}^{(p+q-2)/2} g(r+1, -i-1/2+j) +\{(p+q-6)/2\}g(r+1,1)+\sum_{j=1}^{(p+q-6)/2} j/(-i+j)^{r+1} +\{(p+q-6)/2\}\sum_{j=(p+q-4)/2}^{i-1}\{1/(-i+j)^{r+1}\} -\sum_{j=1}^{p-2} g\{r+1, (p+q)/2-i+j/(p-1)\} -\sum_{j=1}^{q-2} g\{r+1, (p+q)/2-i+j/(q-1)\}\Big].$$

(iii) $(i, j) \in a''$:

(3.85) $$A_{0i}=\prod_{j=1}^{(p+q-2)/2}\Gamma(-1/2-i+j)\Big/ \left[\prod_{j=1}^{(p+q-6)/2}(-i+j)^j \prod_{j=(p+q-4)/2}^{i-1}(-i+j)^{(p+q-6)/2} \times\{(p+q)/2-i-1\}^2\{(p+q)/2-i-2\} \times\prod_{j=1}^{p-2}\Gamma\{(p+q)/2-i+j/(p-1)\} \times\prod_{j=1}^{q-2}\Gamma\{(p+q)/2-i+j/(q-1)\}\right].$$

(3.86) $$B_{0i}=\sum_{j=1}^{(p+q-2)/2}\phi(-1/2-i+j)-\sum_{j=1}^{(p+q-6)/2} j/(-i+j)-\{(p+q-6)/2\} \times \sum_{j=(p+q-4)/2}^{i-1}\{1/(-i+j)\}-2/\{(p+q)/2-i-1\}$$

$$-1/\{(p+q)/2-i-2\}-\sum_{j=1}^{p-2}\phi\{(p+q)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2}\phi\{(p+q)/2-i+j/(q-1)\}\,.$$

$$(3.87)\quad B_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{(p+q-6)/2\}g(r+1,1)$$
$$+\sum_{j=1}^{(p+q-2)/2}g(r+1,-i-1/2+j)+\sum_{j=1}^{(p+q-6)/2}j/(-i+j)^{r+1}$$
$$+\{(p+q-6)/2\}\sum_{j=(p+q-4)/2}^{i-1}\{1/(-i+j)^{r+1}\}$$
$$+2/\{(p+q)/2-i-1\}^{r+1}+1/\{(p+q)/2-i-2\}^{r+1}$$
$$-\sum_{j=1}^{p-2}g\{r+1,(p+q)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2}g\{r+1,(p+q)/2-i+j/(q-1)\}\Big]\,.$$

(iv) $(i,j)\in b$:

$$(3.88)\quad C_{0i}=\prod_{j=i+2}^{(p+q-2)/2}\Gamma(-i+j)\prod_{j=1}^{(p+q-6)/2}\Gamma(1/2-i+j)\Big/$$
$$\Big[\{(p+q+1)/2-i-1\}^2\{(p+q+1)/2-i-2\}$$
$$\times\prod_{j=1}^{i-1}(-i+j)^j\prod_{j=1}^{p-2}\Gamma\{(p+q+1)/2-i+j/(p-1)\}$$
$$\times\prod_{j=1}^{q-2}\Gamma(p+q+1)/2-i+j/(q-1)\}\Big]\,.$$

$$(3.89)\quad D_{0i}=\sum_{j=i+2}^{(p+q-2)/2}\phi(-i+j)+\sum_{j=1}^{(p+q-6)/2}\phi(1/2-i+j)$$
$$-2/\{(p+q+1)/2-i-1\}-1/\{(p+q+1)/2-i-2\}$$
$$-\sum_{j=1}^{i-1}j/(-i+j)-\sum_{j=1}^{p-2}\phi\{(p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2}\phi\{(p+q+1)/2-i+j/(q-1)\}\,.$$

$$(3.90)\quad D_{0i}^{(r)}=(-1)^{r-1}r!\Big[(i+1)g(r+1,1)+\sum_{j=i+2}^{(p+q-2)/2}g(r+1,-i+j)$$
$$+\sum_{j=1}^{(p+q-6)/2}g(r+1,1/2-i+j)+\sum_{j=1}^{i-1}j/(-i+j)^{r+1}$$
$$+2/\{(p+q+1)/2-i-1\}^{r+1}+1/\{(p+q+1)/2-i-2\}$$
$$-\sum_{j=1}^{p-2}g\{r+1,(p+q+1)/2-i+j/(p-1)\}$$
$$-\sum_{j=1}^{q-2}g\{r+1,(p+q+1)/2-i+j/(q-1)\}\Big]\,.$$

(v) $(i, j) \in b'$:

$$(3.91)\quad C_{0i}=\prod_{j=1}^{(p+q-6)/2} \Gamma(-i+1/2+j)\Big/ \Big[\{(p+q+1)/2-i-1\}^2\{(p+q+1)/2-i-2\} \times \prod_{j=1}^{(p+q-2)/2} (-i+j)^j \prod_{j=(p+q)/2}^{i-1} (-i+j)^{(p+q-2)/2} \times \prod_{j=1}^{p-2} \Gamma\{(p+q+1)/2-i+j/(p-1)\} \times \prod_{j=1}^{q-2} \Gamma\{(p+q+1)/2-i+j/(q-1)\}\Big].$$

$$(3.92)\quad D_{0i}=\sum_{j=1}^{(p+q-6)/2} \phi(1/2-i+j)-2/\{(p+q+1)/2-i-1\} -1/\{(p+q+1)/2-i-2\}-\sum_{j=1}^{(p+q-2)/2} j/(-i+j)-\{(p+q-2)/2\} \times \sum_{j=(p+q)/2}^{i-1} \{1/(-i+j)\}-\sum_{j=1}^{p-2} \phi\{(p+q+1)/2-i+j/(p-1)\} -\sum_{j=1}^{q-2} \phi\{(p+q+1)/2-i+j/(q-1)\}.$$

$$(3.93)\quad D_{0i}^{(r)}=(-1)^{r-1}r!\Big[\{(p+q-2)/2\}g(r+1, 1) +\sum_{j=1}^{(p+q-6)/2} g(r+1, 1/2-i+j)+2/\{(p+q+1)/2-i-1\}^{r+1} +1/\{(p+q+1)/2-i-2\}^{r+1}+\sum_{j=1}^{(p+q-2)/2} j/(-i+j)^{r+1} +\{(p+q-2)/2\}\sum_{j=(p+q)/2}^{i-1} \{1/(-i+j)^{r+1}\} -\sum_{j=1}^{p-2} g\{r+1, (p+q+1)/2-i+j/(p-1)\} -\sum_{j=1}^{q-2} g\{r+1, (p+q+1)/2-i+j/(q-1)\}\Big].$$

Special case

We will be giving the density in the case where $p=4$, $q=2$ explicitly, for the purpose of illustration.

$$(3.94)\quad f(x)=K_{4,2}x^{(n-3)/2}\Big\{A_{01}x^{-2}+A_{02}(B_{02}-\log x)x^{-1}+C_{01}x^{-5/2} +C_{02}(D_{02}-\log x)x^{-3/2}+\sum_{i=3}^{\infty} C_{0i}(D_{0i}-\log x)x^{i-7/2}\Big\},$$

$$0<x<1,$$

where

$$A_{01}=\prod_{j=1}^{2}\Gamma(-3/2+j)\Big/\prod_{j=1}^{2}\Gamma(2+j/3)\,,$$

$$A_{02}=-\prod_{j=1}^{2}\Gamma(-5/2+j)\Big/\prod_{j=1}^{2}\Gamma(1+j/3)\,,$$

$$B_{02}=5/6-4\log 2+3\log 3\,,$$

$$C_{01}=8\Big/\left\{9\prod_{j=1}^{2}\Gamma(5/2+j/3)\right\},$$

$$C_{02}=8\Big/\prod_{j=1}^{2}\Gamma(3/2+j/3)\,,$$

$$D_{02}=-7/2-2\log 2+3\log 3\,,$$

$$C_{0i}=1\Big/\left[(5/2-i)^2(3/2-i)\prod_{j=1}^{2}\Gamma(7/2-i+j/3)\times\prod_{j=1}^{2}(-i+j)^j\prod_{j=3}^{i-1}(-i+j)^2\right],$$

$$D_{0i}=-2/(5/2-i)-1/(3/2-i)-\sum_{j=1}^{2}j/(-i+j)-2\sum_{j=3}^{i-1}1/(-i+j)-\sum_{j=1}^{2}\phi(7/2-i+j/3)\,,\qquad i=3,4,\cdots.$$

The cumulative distribution function

In this case the cumulative distribution function is

$$F(u)=K_{p,q}\left\{\sum_{a\cup a'\cup a''}\frac{1}{(j-1)!}\sum_{k=0}^{j-1}\left[\binom{j-1}{k}A_{0i}^{(j-1-k)}\frac{u^{i+(n-3)/2-(p+q)/2+1}}{k+1}\times\sum_{r=0}^{k}\frac{(k+1)k\cdots(k-r+1)(-\log u)^{k-r}}{\{i+(n-3)/2-(p+q)/2+1\}^{r+1}}\right]\right.$$
$$\left.+\sum_{b\cup b'}\frac{1}{(j-1)!}\sum_{k=0}^{j-1}\left[\binom{j-1}{k}C_{0i}^{(j-1-k)}\frac{u^{i+(n-3)/2-(p+q+1)/2+1}}{k+1}\times\sum_{r=0}^{k}\frac{(k+1)k\cdots(k-r+1)(-\log u)^{k-r}}{\{i+(n-3)/2-(p+q+1)/2+1\}^{r+1}}\right]\right\},\tag{3.95}$$

valid for $0<u<1$, $i+(n-3)/2-(p+q+1)/2>0$, for all i.

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