ON THE LIMITING DISTRIBUTION OF A SEQUENCE OF ESTIMATORS WITH UNIFORMITY PROPERTY*

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1. Introduction

Let us define some notations in the following way:

$(\mathcal{X}, \mathcal{A}, \mu)$	;	a σ-finite measure space
Θ	;	a parameter space which is a closed subset of k-dimensional Euclidean space $\mathcal{R}^k$
$\{P_\theta; \theta \in \Theta\}$	;	a family of probability measures on $(\mathcal{X}, \mathcal{A})$, dominated by μ $$p(\cdot, \theta) = dP_\theta/d\mu$$
$(\mathcal{X}_n, \mathcal{A}_n, \mu_n)$	;	the usual n-fold measure space of $(\mathcal{X}, \mathcal{A}, \mu)$
$P_{n\theta}$	;	the n-fold probability measure of P with the density $p(\cdot, \theta)$, i.e. $$p_n(x, \theta) = \prod_{i=1}^{n} p(x_i, \theta), \qquad x = (x_1, \cdots, x_n)$$
$t_n(x)$	;	a k-dimensional measurable function on $\mathcal{R}^k$, $x = (x_1, \cdots, x_n)$
$T_n = t_n(X)$	;	an estimator of θ where $X = (X_1, \cdots, X_n)$ is a random vector
$\{X_n\}$	;	a sequence of random vectors which are independent and identically distributed with P_θ
$\mathcal{L}[T \mid P]$	;	the distribution function of the random variable T under a true probability measure P

Wolfowitiz [6] and Kaufman [3] argued about the sequence of estimators with uniformity property:

$$\mathcal{L}[n^{1/2}(T_n - \theta) \mid P_\theta](y) \to L_\theta(y) \tag{1.1}$$

uniformly for $(y, \theta) \in \mathcal{R}^k \times C$ where C is any compact subset of Θ^0, the interior of Θ. Such a sequence of estimators, $\{T_n\}$, is said to have uniformity property and the family of such sequences is denoted by $\mathcal{E}$.

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* This concept is introduced by Wolfowitz [6] and is called "Property U" in [7].

We can see in Theorem 2.2 below that the sequence of maximum likelihood estimators (m.l.e.), say, $\{\hat{\theta}_n\}$ has the uniformity property under suitable conditions. Kaufman obtained an optimality of the sequence of m.l.e.'s $\{\hat{\theta}_n\}$ among the family $\mathcal{E}$; for any $\{T_n\} \in \mathcal{E}$

$$(1.2) \qquad \lim P_\theta\{n^{1/2}(T_n-\theta) \in S\} \leqq \lim P_\theta\{n^{1/2}(\hat{\theta}_n-\theta) \in S\}\,,$$

where S is any symmetric (about the origin) convex subset in $\mathcal{R}^k$. This implies that the sequence of m.l.e.'s is more concentrated about the true parameter θ than others in $\mathcal{E}$.

The purpose of this paper is to improve the proof of (1.2) given by Kaufman [3] to be the more elegant and prospective one in the following sense: under the same conditions as in Kaufman [5] the limiting distribution function L_θ of the sequence of estimators with uniformity property can be exactly represented as a convolution

$$(1.3) \qquad L_\theta = G_\theta * \Phi_\theta\,,$$

where G_θ is some distribution function and Φ_θ is the normal distribution function with mean 0 and covariance matrix $I(\theta)^{-1}$, $I(\theta)$ being the Fisher's information matrix. Then the optimality of m.l.e. (1.2) is easily verified to be the immediate result of (1.3) and Anderson's Theorem[1]. Although T_{nk} in Kaufman is the smoothing of $\bar{T}_n$ by the uniform distribution function on the k-square with the size K, we shall however use, in place of it, T_n^* which is smoothed by the normal distribution depending on the sample size n, in order to simplify the proof[2].

In Section 2, we shall state some known results without proofs. In the later half of this section we shall prove two preparatory lemmas (Lemma 2.2 and 2.3 below) which are the corresponding lemmas when T_{nk} is replaced with T_n^*. In Section 3 we shall verify the main theorem (1.3).

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2. Preliminary lemmas

Throughout this paper, we assume the same regularity conditions on $\{p(\cdot,\theta)\}$ as (4.1)-(4.11) in Kaufman ([3], p. 167).

THEOREM 2.1 (Wolfowitz [6] and Kaufman [3]). *For* $\{T_n\} \in \mathcal{E}$ *with*

[1] Let G and Φ be probability distribution functions. If Φ has a unimodal probability density function, then

$$\int_S dG * \Phi \leqq \int_S d\Phi\,, \qquad \text{for } S \text{, the same as in (1.2).}$$

[2] For T_{nk}, see Kaufman ([3], p. 163) and for $\bar{T}_n$ and T_n^*, see Lemmas 2.1 and 2.2 below.

the limiting distribution L_θ, *it holds that*

(i) $L_\theta[(\cdots, -\infty, \cdots)]=0$ *and* $L_\theta[(+\infty, \cdots, +\infty)]=1$ *for* $\theta \in \Theta^0$ *and each convergence is uniform in* C,

(ii) *for each fixed* $\theta \in \Theta^0$, $L_\theta(\cdot)$ *is continuous in* $\mathcal{R}^k$, *and for each fixed* $y \in R^k$, $L_\cdot(y)$ *is continuous in* Θ^0,

(iii) L_θ *is absolutely continuous with respect to Lebesgue measure,*

(iv) $L_\theta(y)$ *is strictly increasing in the sense that if* $y<y'$ (*i.e.* $y_j \leqq y'_j$, $j=1, \cdots, k$ *and* $y_j<y'_j$ *for some* j), *then*

$$L_\theta(y)<L_\theta(y'),$$

(v) $L_\theta(S)$ *is continuous in* Θ^0 *for any simple set*[1] S *where*

$$L_\theta(S)=\int_S dL_\theta,$$

and

(vi) *for any simple set* S,

$$P_\theta\{n^{1/2}(T_n-\theta) \in S\} \to L_\theta(S) \tag{2.1}$$

uniformly in C.

THEOREM 2.2. *There exists a sequence of m.l.e.'s* $\{\hat{\theta}_n\}$ *such that for* $n=1, 2, \cdots$, $\hat{\theta}_n(x)$ *is a root of the likelihood equation with probability* $p_n(\theta)$ *where* $p_\theta(\theta) \to 1$ *uniformly in* C, *and that*

$$\mathcal{L}[n^{1/2}(\hat{\theta}_n-\theta) \mid P_\theta](y) \to \Phi_\theta(y)^{2)} \tag{2.2}$$

uniformly for $(y, \theta) \in R^k \times C$.

THEOREM 2.3. *The sequence of m.l.e.'s* $\{\hat{\theta}_n\}$ *is asymptotically sufficient for* $\{P_\theta; \theta \in \Theta^0\}$ *in the sense that there exists a family* $\{Q_{n\theta}; \theta \in \Theta^0\}$ *of probability measures on* $(\mathcal{X}_n, \mathcal{A}_n)$ *with the densities* $\{q_n(\cdot, \theta)\}$ *with respect to* μ_n, *such that* $\hat{\theta}_n$ *is sufficient for* $\{Q_{n\theta}; \theta \in \Theta^0\}$ *i.e.*

$$q_n(x, \theta)=g_n(\hat{\theta}_n(x), \theta)h_n(x) \tag{2.3}$$

and such that

$$\| P_{n\theta}-Q_{n\theta} \|=\int | p_n(x, \theta)-q_n(x, \theta) | d\mu_n \to 0 \tag{2.4}$$

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1) A set S is called "simple" in this paper if for any finite interval $I \subset R^k$ and $\delta>0$, there exist D and D', finite unions of disjoint intervals such that $D \subset S \cap I \subset D'$ and $\lambda(D'-D)<\delta$, where λ is the Lebesgue measure. For example, convex set is simple.

2) We can see that $I(\theta)^{-1}$ (which is the covariance matrix of the distribution Φ_θ) is continuous in Θ^0. See [5] for the proof of this theorem, for example.

uniformly in C. Furthermore $g_n(y, \cdot)$ *is Borel measurable in* Θ^0.

Remark. Let

$$\tau_n(B)=\int_{\hat{\theta}_n^{-1}(B)} h_n(x)d\mu_n ,$$

and

$$D_n=\left\{y;\int_{I_n+y} d\tau_n(Z)>0\right\}$$

where $I_n=\{y; \|y\|<\delta_n/2\}$, $\delta_n=o(n^{-1/2})$, and $\|y\|=\max\{|y_j|; j=1,\cdots,k\}$. Then we can choose q_n such that, for bounded B, Borel set, $\tau_n(B)<\infty$, and for any $M>0$ and compact $C\subset\Theta^0$, there exists a positive integer $N=N(M, C)$ such that for $n\geqq N$,

$$\{y; n^{1/2}\|y-\theta\|<M\}\subset D_n, \qquad \text{for any } \theta\in C, \tag{2.5}$$

$$\text{i.e.}\quad \{y; d(y, C)<n^{-1/2}M\}\subset D_n .$$

Since the relation (2.4) implies that any asymptotic property holding for $\{P_{n\theta}\}$ holds also for $\{Q_{n\theta}\}$, we can utilize $\{Q_{n\theta}; \theta\in\Theta^0\}$ obtained in Theorem 2.3 in order to investigate properties of the limiting distribution L_θ.

For $\{T_n\}\in\mathcal{E}$ and $B\in\mathcal{B}^k$, Borel σ-field of R^k, let

$$\nu_n(B, y)=Q_{n\theta}\{T_n\in B\mid\hat{\theta}_n=y\}$$

which is independent of θ because of sufficiency of $\hat{\theta}_n$ for $\{Q_{n\theta}; \theta\in\Theta^0\}$.

LEMMA 2.1 (Kaufman [3], p. 160). *Given* $\{T_n\}\in\mathcal{E}$ *and associated* $\{\nu_n\}$, *let* $\{\delta_n\}$ *be a sequence of positive numbers tending to zero with the order of* $o(n^{-1/2})$, *and let* $\{W_n\}$ *be a sequence of random variables in* R^k *with uniform distribution over the square* $I_n=\{y; \|y\|<\delta_n/2\}$.

Define, for each n,

$$\bar{\theta}_n=\hat{\theta}_n+W ,$$

$$w_n(y, \theta)=\delta_n^{-k}\int_{I_n+y} d\mathcal{L}[\hat{\theta}_n\mid Q_{n\theta}](z)=\delta_n^{-k}Q_{n\theta}\{\theta_n\in I_n+y\}$$

for $y\in R^k$ *and* $\theta\in\Theta^0$, *and*

$$\bar{\nu}_n(B, y)=\frac{\int_{I_n+y}\nu_n(B, z)d\tau_n(z)}{\int_{I_n+y}d\tau_n(z)}$$

for $B\in\mathcal{B}^k$ *and* $y\in D_n=\{y;\text{denominator}\neq 0\}$, *and* $\bar{\nu}_n(B, y)=0$ *if* $y\notin D_n$.

Then,

(i) *The distribution function of* $\bar{\theta}_n$ *is absolutely continuous with the density* $w_n(\cdot, \theta)$ *with respect to Lebesgue measure.*
(ii) *There exists a* $\{\delta_n\}$, *mentioned above, such that*

$$n^{-k/2}W_n(n^{-1/2}u+\theta.\ \theta)\rightarrow\phi(u, \theta)$$

uniformly for $(u, \theta) \in R^k \times C$ *where* $\phi(u, \theta)$ *is the* $\mathfrak{N}_k(0, I(\theta)^{-1})$-*density.*
(iii) *For any convex symmetric set* S *and* $\theta \in \Theta^0$

$$\lim_{n\to\infty} Q_{n\theta}\{n^{1/2}(\bar{\theta}_n-\theta) \in S\} = \lim_{n\to\infty} Q_{n\theta}\{n^{1/2}(\hat{\theta}_n-\theta) \in S\} .$$

(iv) *For each* n *and* $y \in D_n$, $\bar{\nu}_n(\cdot, y)$ *is a probability measure on* $(R^k\cdot\mathcal{B}^k)$ *and defines an estimator* $\bar{T}_n$ *which has* $\bar{\nu}_n(\cdot, y)$ *as the conditional distribution for* $\bar{\theta}_n=y$, *and therefore*

$$Q_{n\theta}\{\bar{T}_n \in B\} = \int \bar{\nu}_n(B, y)w_n(y, \theta)dy .$$

(v) *For any simple set* S *and* $\theta \in \Theta^0$,

$$\lim_{n\to\infty} Q_{n\theta}\{n^{1/2}(\bar{T}_n-\theta) \in S\} = \lim_{n\to\infty} Q_{n\theta}\{n^{1/2}(T_n-\theta) \in S\}$$

the convergence being uniform for $\theta \in C$.

The following two lemmas correspond to the lemmas due to Kaufman although we use T_n^* in place of T_{nk}.

LEMMA 2.2. *Using* $\{\bar{\nu}_n\}$ *in* Lemma 2.1, *define*

$$\nu_n^*(B, y) = \{1+\gamma_n(y)\} \int \bar{\nu}_n(B+n^{-\beta/2}t, y+n^{-\beta/2}t)\phi(t)dt \tag{2.6}$$

where

$$1+\gamma_n(y)=1\Big/\int_{n^{1/2}(D_n-y)} \phi(t)dt , \qquad 0<\beta<1 , \tag{2.7}$$

and ϕ *is the density of* $\mathfrak{N}_k(0, E)$, E *being the identity matrix of order* k, *then*
(i) *for each* n *and* $y \in R^k$, $\nu_n^*(\cdot, y)$ *is a probability measure on* $(R^k, \mathcal{B}^k)$ *and defines an estimator* T_n^*, *which has* $\nu_n^*(\cdot, y)$ *as the conditional distribution for given* $\bar{\theta}_n=y$, *and therefore*

$$Q_{n\theta}(T_n^* \in B)=\int \nu_n^*(B, y)w_n(y, \theta)dy ,$$

and
(ii) *for any simple set* S *and* $\theta \in \Theta^0$,

$$\lim Q_{n\theta}\{n^{1/2}(T_n^*-\theta) \in S\} = \lim Q_{n\theta}\{n^{1/2}(T_n-\theta) \in S\} ,$$

the convergence being uniform for $\theta \in C$.

PROOF.

(i) It is obvious that for fixed $y \in R^k$, $\nu_n^*(\cdot, y)$ is a probability measure and for fixed B, $\nu_n^*(B, \cdot)$ is measurable.

$$Q_{n\theta}(T_n^* \in B) = \int \nu_n^*(B, y) w_n(y, \theta) dy .$$

(ii) Let ε be any small positive number, and fix it. Let M_1 be a positive number such that

$$\int_{\|t\|>M_1} \phi(t)dt < \varepsilon/8 . \tag{2.8}$$

It follows from Lemma 2.1 (v) that there exists a positive number N_1 such that for $n \geqq N_1$ and $\|t\| \leqq M$

$$|Q_{n,\theta+n^{-\beta/2}t}\{n^{1/2}[\bar{T}_n-(\theta+n^{-\beta/2}t)] \in S\} - Q_{n\theta}\{n^{1/2}(T_n-\theta) \in S\}| < \varepsilon/8 ,$$

and hence that

$$\left| Q_{n\theta}\{n^{1/2}(T_n-\theta) \in S\} - \int_{\|t\| \leqq M_1} \phi(t)dt \cdot Q_{n,\theta+n^{-\beta/2}t}\{n^{1/2}[\bar{T}_n-(\theta+n^{-\beta/2}t)] \in S\} \right| \leqq \varepsilon/4 . \tag{2.9}$$

Since $\phi(u, \theta)$ is continuous in θ (see the footnote 2, p. 3), it holds that there exists $M_2 > 0$ such that for any $\theta \in C$

$$\int_{\|u\|>M_2} \phi(u, \theta)du < \varepsilon/8 . \tag{2.10}$$

Since C is a compact subset of $\Theta^0(\subset R^k)$, we can see easily that

$$\delta = d(C, (\Theta^0)^c) > 0$$

and so, that let

$$C' = \{\theta ; d(\theta, C) \leqq \delta/2\} ,$$

then C' is compact and $C \subset C' \subset \Theta^0$. If in (2.4) we take compact set C' and $M_2 > 0$, it follows that there exists a positive integer $N = N(M_1, M_2, C')$ such that for $n \geqq N$, $\|u\| \leqq M_2$, $\|t\| \leqq M_1$, and $\theta \in C$,

$$\theta + n^{-\beta/2}t \in C' \qquad \text{and} \qquad y = n^{-1/2}u + \theta + n^{-\beta/2}t \in D_n ,$$

and hence that

$$1 \geqq 1/\{1+\gamma_n(n^{-1/2}u+\theta)\} = \int_{n^{\beta/2}(D_n - n^{-1/2}u - \theta)} \phi(t)dt \geqq \int_{\|t\| \leqq M_1} \phi(t)dt .$$

Hence we have that $\gamma_n(n^{-1/2}u+\theta)\to 0$, uniformly in $\|u\|\leqq M_2$ and $\theta\in C$, i.e. there exists $N_2=N_2(\varepsilon)$ such that for $n\geqq N_2$, $\|u\|\leqq M_2$ and $\theta\in C$,

$$0\leqq\gamma_n(n^{-1/2}u+\theta)<\varepsilon/8\,. \tag{2.11}$$

By Scheffe's theorem[1], it follows from Lemma 2.1 (ii) that for measurable functions $\{f_n\}$, bounded by 1 and $\|t\|\leqq M_1$, there exists a positive integer N_3 such that for $n\geqq N_3$ and $\theta\in C$,

$$\left|\int f_n(u)n^{-k/2}w_n(n^{-1/2}u+\theta+n^{-\beta/2}t,\ \theta+n^{-\beta/2}t)du - \int f_n(u)\phi(u,\theta)du\right|<\varepsilon/8\,. \tag{2.12}$$

Therefore we have that for $n\geqq\max(N_1, N_2, N_3)$ and $\theta\in C$

$$\begin{aligned}
&\left|\int_{\|t\|\leqq M_1}\phi(t)dtQ_{n,\theta+n^{-\beta/2}t}\{n^{1/2}[\bar{T}_n-(\theta+n^{-\beta/2}t)]\in S\} - Q_{n\theta}\{n^{1/2}(T_n^*-\theta)\in S\}\right|\\
&=\left|\int_{\|t\|\leqq M_1}\phi(t)dt\int\bar{\nu}_n(n^{-1/2}S+\theta+n^{-\beta/2}t,\ y)w_n(y,\ \theta+n^{-\beta/2}t)dy - \int w_n(y',\theta)dy'\nu_n^*(n^{-1/2}S+\theta,\ y')\right|\\
&=\left|\int_{\|t\|\leqq M_1}\phi(t)dt\int\bar{\nu}_n(n^{-1/2}S+\theta+n^{-\beta/2}t,\ n^{-1/2}u+\theta+n^{-\beta/2}t)\right.\\
&\qquad\cdot n^{-k/2}w_n(n^{-1/2}u+\theta+n^{-\beta/2}t,\ \theta+n^{-\beta/2}t)du\\
&\qquad\left.-\int n^{-k/2}w_n(n^{-1/2}u+\theta,\ \theta)du\,\nu_n^*(n^{-1/2}S+\theta,\ n^{-1/2}u+\theta)\right|\\
&\qquad(\text{by letting } y=n^{-1/2}u+\theta+n^{-\beta/2}t \text{ and } y'=n^{-1/2}u+\theta)\\
&\leqq\left|\int_{\|t\|\leqq M_1}\phi(t)dt\int\bar{\nu}_n(n^{-1/2}S+\theta+n^{-\beta/2}t,\ n^{-1/2}u+\theta+n^{-\beta/2}t)\phi(u,\theta)du\right.\\
&\qquad\left.-\int\phi(u,\theta)du\,\nu_n^*(n^{-1/2}S+\theta,\ n^{-1/2}u+\theta)\right|+2\cdot\varepsilon/8\\
&\qquad(\text{from (2.12) and } 0\leqq\bar{\nu}_n\leqq 1)\\
&\leqq\varepsilon/4+\left|\int_{\|t\|\leqq M_1}\phi(t)dt\int_{\|u\|\leqq M_2}\bar{\nu}_n(n^{-1/2}S+\theta+n^{-\beta/2}t,\ n^{-1/2}u+\theta+n^{-\beta/2}t)\phi(u,\theta)du\right.\\
&\qquad\left.-\int_{\|u\|\leqq M_2}\phi(u,\theta)du\,\nu_n^*(n^{-1/2}S+\theta,\ n^{-1/2}u+\theta)\right|+2\cdot\varepsilon/8\\
&\qquad(\text{from (2.10) and } 0\leqq\bar{\nu}_n,\ \nu_n^*\leqq 1)\\
&=\varepsilon/2+\left|\int_{\|t\|\leqq M_1}\phi(t)dt\int_{\|u\|\leqq M_2}\bar{\nu}_n(n^{-1/2}S+\theta+n^{-\beta/2}t,\ n^{-1/2}u+\theta+n^{-\beta/2}t)\phi(u,\theta)du\right.
\end{aligned}$$

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1) For example, see Rao ([4], p. 104).

$$-\int_{\|u\|\leqq M_2} \phi(u,\theta)du\{1+\gamma_n(n^{-1/2}u+\theta)\}$$
$$\cdot\int \phi(t)dt\,\bar{\nu}_n(n^{-1/2}S+\theta+n^{-\beta/2}t,\ n^{-1/2}u+\theta+n^{-\beta/2}t)\Big|$$

(from (2.6))

$$\leqq \varepsilon/2+\Big|\int_{\|t\|>M_1}\phi(t)dt\int_{\|u\|\leqq M_2}\bar{\nu}_n(n^{-1/2}S+\theta+n^{-\beta/2}t,$$
$$n^{-1/2}u+\theta+n^{-\beta/2}t)\phi(u,\theta)du\Big|+\varepsilon/8$$

(from (2.11))

$$\leqq \varepsilon/2+\varepsilon/8+\varepsilon/8=3\varepsilon/4 \qquad \text{(from (2.8))}.$$

That is,

$$\Big|\int_{\|t\|\leqq M_1}\phi(t)dt\,Q_{n,\theta+n^{-\beta/2}t}\{n^{-1/2}[\bar{T}_n-(\theta+n^{-\beta/2}t)]\in S\}$$
$$-Q_{n\theta}\{n^{1/2}(T_n^*-\theta)\in S\}\Big|. \tag{2.13}$$

Consequently, from (2.9) and (2.13) it holds that for $n\geqq\max(N_1, N_2, N_3)$ and $\theta\in C$

$$|Q_{n\theta}\{n^{1/2}(T_n-\theta)\in S\}-Q_{n\theta}\{n^{1/2}(T_n^*-\theta)\in S\}|<\varepsilon.$$

The proof is complete.

LEMMA 2.3. *The random variables $U_n=n^{1/2}(\bar{\theta}_n-\theta)$ and $V_n=n^{1/2}(T_n^*-\bar{\theta}_n)$ are asymptotically independent in the following sense:*

$$|Q_{n\theta}\{U_n\in B_1, V_n\in B_2\}-Q_{n\theta}\{U_n\in B_1\}Q_{n\theta}\{V_n\in B_2\}|\to 0$$

uniformly for $B_1, B_2\in\mathscr{B}^k$.

PROOF. Fix $\theta\in\Theta^0$. It follows from Lemma 2.1 (ii) that there exist a positive integer N_1 and $M_1>0$ such that for $n\geqq N_1$

$$Q_{n\theta}\{\|U_n\|>M_1\}<\varepsilon/4. \tag{2.14}$$

From the definitions of T_n^* and $\bar{\theta}_n$, we have

$$Q_{n\theta}\{V_n\in B_2\,|\,U_n=u\}$$
$$=\nu_n^*(n^{-1/2}B_2+n^{-1/2}u+\theta,\ n^{-1/2}u+\theta)$$
$$=\{1+\gamma_n(n^{-1/2}u+\theta)\}\int\bar{\nu}_n(n^{-1/2}B_2+\theta+n^{-1/2}u+n^{-\beta/2}t,$$
$$\theta+n^{-1/2}u+n^{-\beta/2}t)\phi(t)dt$$
$$=\{1+\gamma_n(n^{-1/2}u+\theta)\}\int\bar{\nu}_n(n^{-1/2}B_2+\theta+n^{-\beta/2}t',\ \theta+n^{-\beta/2}t')\phi(t'-n^{-(1-\beta)/2}u)dt'$$

(letting $t+n^{-(1-\beta)/2}u=t'$).

Since ϕ is uniformly continuous, $0<\beta<1$, and $\gamma_n(n^{-1/2}u+\theta)\to 0$ uniformly in $\|u\|\leqq M_1$ (and $\theta\in C$), we can show that there exist $M_2>0$ and a positive integer N_2 such that for $n\geqq N_2$ and $\|u_i\|\leqq M_1$ $i=1, 2$,

$$\int_{\|t\|>M_2}\phi(t-n^{-\beta/2}u)dt<\varepsilon/8\,, \tag{2.15}$$

$$|\phi(t-n^{-\beta/2}u_1)-\phi(t-n^{-\beta/2}u_2)|<(2M_2)^{-k}\varepsilon/8\,, \tag{2.16}$$

and

$$0\leqq\gamma_n(n^{-1/2}u+\theta)<\varepsilon/8\,. \tag{2.17}$$

Hence, it follows that for $n\geqq N_2$ and $\|u_i\|\leqq M_1$, $i=1, 2$,

$$\begin{aligned}
&|Q_{n\theta}\{V_n\in B_2\,|\,U_n=u_1\}-Q_{n\theta}\{V_n\in B_2\,|\,U_n=u_2\}|\\
&=\Big|\{1+\gamma_n(n^{-1/2}u_1+\theta)\}\int\bar{\nu}_n(n^{-1/2}B_2+\theta+n^{-\beta/2}t,\ \theta+n^{-\beta/2}t)\phi(t-n^{-(1-\beta)/2}u_1)dt\\
&\quad-\{1+\gamma_n(n^{-1/2}u_2+\theta)\}\int\bar{\nu}_n(n^{-1/2}B_2+\theta+n^{-\beta/2}t,\ \theta+n^{-\beta/2}t)\\
&\qquad\cdot\phi(t-n^{-(1-\beta)/2}u_2)dt\Big|\\
&\leqq\Big|\int\bar{\nu}_n(n^{-1/2}B_2+\theta+n^{-\beta/2}t,\ \theta+n^{-\beta/2}t)\{\phi(t-n^{-(1-\beta)/2}u_1)\\
&\qquad-\phi(t-n^{-(1-\beta)/2}u_2)\}dt\Big|+\varepsilon/4\\
&\qquad\text{(from (2.17))}\\
&<\varepsilon/4+\int_{\|t\|\leqq M_2}\Big|\phi(t-n^{-\beta/2}u_1)-(t-n^{-\beta/2}u_2)\Big|dt+\varepsilon/8\\
&<3\varepsilon/8+(2M_2)^k\cdot(2M_2)^{-k}\varepsilon/8\qquad\text{(from (2.16))}\\
&=\varepsilon/2\,,\qquad\text{i.e.}
\end{aligned}$$

$$|Q_{n\theta}\{V_n\in B_2\,|\,U_n=u_1\}-Q_{n\theta}\{V_n\in B_2\,|\,U_n=u_2\}|<\varepsilon/2\,. \tag{2.18}$$

Therefore it holds that for $n\geqq N=\max(N_1, N_2)$ and $\|u_1\|\leqq M_1$,

$$\begin{aligned}
&|Q_{n\theta}\{V_n\in B_2\,|\,U_n=u_1\}-Q_{n\theta}\{V_n\in B_2\}| \qquad (2.19)\\
&=\Big|\int[Q_{n\theta}\{V_n\in B_2\,|\,U_n=u_1\}-Q_{n\theta}\{V_n\in B_2\,|\,U_n=u\}]\,d\mathcal{L}[U_n\,|\,Q_{n\theta}](u)\Big|\\
&\leqq\varepsilon/4+\int_{\|u\|\leqq M_1}|Q_{n\theta}\{V_n\in B_2\,|\,U_n=u_1\}\\
&\qquad-Q_{n\theta}\{V_n\in B_2\,|\,U_n=u\}|\,d\mathcal{L}[U_n\,|\,Q_{n\theta}](u)\\
&\qquad\text{(from (2.14))}\\
&\leqq\varepsilon/4+\varepsilon/2=3\varepsilon/4\qquad\text{(from (2.18))}.
\end{aligned}$$

Finally we have for $n\geqq N$ and $B_1, B_2\in\mathcal{B}^k$,

$$
\begin{aligned}
&|Q_{n\theta}\{U_n \in B_1, V_n \in B_2\} - Q_{n\theta}\{U_n \in B_1\}Q_{n\theta}\{V_n \in B_2\}| \\
&\leqq |Q_{n\theta}\{U_n \in B_1, \|U_n\| \leqq M_1, V_n \in B_2\} \\
&\qquad\qquad -Q_{n\theta}\{U_n \in B_1, \|U_n\| \leqq M_1\}Q_{n\theta}\{V_n \in B_2\}| + \varepsilon/4 \\
&\leqq \varepsilon/4 + |Q_{n\theta}\{V_n \in B_2 \mid U_n \in B_1, \|U_n\| \leqq M_1\} - Q_{n\theta}\{V_n \in B_2\}| \\
&\leqq \varepsilon/4 + E_{Q_{n\theta}}[|Q_{n\theta}\{V_n \in B_2 \mid U_n\} - Q_{n\theta}\{V_n \in B_2\}| \mid U_n \in B_1, \|U_n\| \leqq M_1] \\
&\leqq \varepsilon/4 + 3\varepsilon/4 = \varepsilon \qquad \text{(from (2.19))}.
\end{aligned}
$$

This proves the lemma.

3. Main theorem

We assume the same conditions as in Section 2.

THEOREM 3.1. *The limiting distribution function L_θ of the sequence of estimators with uniformity property is represented as a convolution*

$$L_\theta = G_\theta * \Phi_\theta ,$$

where G_θ is the limiting distribution function of $V_n = n^{1/2}(T_n^* - \bar{\theta}_n)$ and Φ_θ is that of $U_n = n^{1/2}(\bar{\theta}_n - \theta)$, i.e. $\mathcal{N}_k(0, I(\theta)^{-1})$.

PROOF. Fix θ in Θ^0 and we shall omit the index θ for brevity. Let

$$L_n(y) = Q_{n\theta}\{n^{1/2}(T_n^* - \theta) < y\} ,$$

$$F_n(u) = Q_{n\theta}\{U_n < u\} ,$$

$$G_n(v) = Q_{n\theta}\{V_n < v\} ,$$

$$H_n(u, v) = Q_{n\theta}\{U_n < u, V_n < v\}$$

and

$$J_n(u, v) = Q_{n\theta}\{U_n < u\}Q_{n\theta}\{V_n < v\} .$$

For simplicity we shall use the same symbol for example,

$$L_n(S) = \int_S dL_n .$$

From the definitions and arguments above, we have

$$F_n(y) \to \Phi(y) = \Phi_\theta(y) ,$$

$$L_n(y) = \int_{u+v<y} dH_n(u, v) ,$$

$$G_n * F_n(y) = \int_{u+v<y} dJ_n(u, v) ,$$

$$L_n(y) \to L(y) = L_\theta(y) \quad \text{for } y \in R^k , \tag{3.1}$$

and

$$\lim \{H_n(B_1 \times B_2) - J_n(B_1 \times B_2)\} = 0 , \tag{3.2}$$

the convergence being uniform for $B_1, B_2 \in \mathscr{B}^k$.

By Helly's theorem[1], it follows that for any subsequence $\{n'\}$ of $\{n\}$ there exists some subsequence $\{m\}$ of $\{n'\}$ and G (not necessarily a probability measure) such that

$$G_m(u) \to G(u) \tag{3.3}$$

at continuity point u of G. Then we have[2] that

$$G_m * F_m(y) \to G * \Phi(y) \tag{3.4}$$

for all $y \in R^k$ because $G * \Phi$ is continuous since Φ is continuous. Fix $y \in R^k$.

From (3.1), (3.4) and the continuity of the limiting distributions L and $G * \Phi$, it follows that there exist $d>0$ and a positive integer N_1 such that for $m>N_1$

$$| L(y) - L_m(y) | < \varepsilon/8 , \tag{3.5}$$

$$| G * \Phi(y) - G_m * F_m(y) | < \varepsilon/8 , \tag{3.6}$$

$$0 \leqq L_m(y + d \cdot e) - L_m(y) < \varepsilon/8 , \tag{3.7}$$

and

$$0 \leqq G_m * F_m(y + d \cdot e) - G_m * F_m(y) < \varepsilon/8 , \tag{3.8}$$

where $e = (1 \cdots 1)'$, the $k \times 1$ vector.

Let

$$A(i_1, \cdots, i_k) = \{u = (u_1, \cdots, u_k)' ; i_j d \leqq u_j < (i_j + 1)d, j = 1, \cdots, k\}$$
$$\text{for } i_j = 0, \pm 1, \pm 2, \cdots, (j = 1, \cdots, k),$$

and let

$$\bar{B}(i_1, \cdots, i_k) = \{(u, v) ; u + v < y + d \cdot e, u \in A(i_1, \cdots, i_k)\} ,$$

$$\underline{B}(i_1, \cdots, i_k) = \{(u, v) ; u + v < y, u \in A(i_1, \cdots, i_k)\}$$

and

$$B(i_1, \cdots, i_k) = A(i_1, \cdots, i_k) \times \{v = (v_1, \cdots, v_k)' ;$$
$$v_j < y_j - i_j d, j = 1, \cdots, k\} .$$

1) See Feller ([2], p. 261).

2) See Theorem 2 of Feller ([2], p. 251).

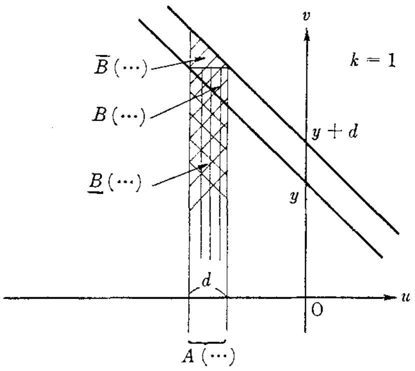

Fig. 1

Since $\underline{B}(i_1, \cdots, i_k) \subset B(i_1, \cdots, i_k) \subset \bar{B}(i_1, \cdots, i_k)$, we have

(3.9)
$$\begin{aligned} L_m(y) &= \sum_{(i_1, \cdots, i_k)} \int_{\underline{B}(i_1, \cdots, i_k)} dH_m(u, v) \\ &\leqq \sum_{(i_1, \cdots, i_k)} \int_{B(i_1, \cdots, i_k)} dH_m(u, v) \\ &\leqq \sum_{(i_1, \cdots, i_k)} \int_{\bar{B}(i_1, \cdots, i_k)} dH_m(u, v) \\ &= L_m(y + d \cdot e) , \end{aligned}$$

and, similarly,

(3.10) $$G_m * F_m(y) \leqq \sum_{(i_1, \cdots, i_k)} \int_{B(i_1, \cdots, i_k)} dJ_m(u, v) \leqq G_m * F_m(y + d \cdot e) .$$

Similarly as in (2.14), we have that there exist $M > 0$ and a positive integer N_2 such that for $m \geqq N_2$

(3.11) $$F_m\{\| u \| > M\} < \varepsilon/8 .$$

Hence we have that for $m \geqq N_2$,

(3.12) $$0 \leqq \Sigma \int_{B(i_1, \cdots, i_k) \cap \{\| u \| > M\} \times R^k} dH_m(u, v) \leqq F_m\{\| u \| > M\} < \varepsilon/8$$

and

(3.13) $$0 \leqq \Sigma \int_{B(i_1, \cdots, i_k) \cap \{\| u \| > M\} \times R^k} dJ_m(u, v) \leqq F_m\{\| u \| > M\} < \varepsilon/8 .$$

Thus from (3.5)-(3.13) we have

(3.14) $$\left| L(y) - \Sigma \int_{B(i_1, \cdots, i_k) \cap \{\| u \| \leqq M\} \times R^k} dH_m(u, v) \right| < 3 \cdot \varepsilon/8$$

and

(3.15) $$\left| G * \Phi(y) - \Sigma \int_{B(i_1, \cdots, i_k) \cap \{\| u \| \leqq M\} \times R^k} dJ_m(u, v) \right| < 3 \cdot \varepsilon/8 .$$

Furthermore it follows from Lemma 2.3 that there exists a positive integer N_3 such that for $m \geqq N_3$ and $B_1, B_2 \in \mathscr{B}^k$,

$$| H_m(B_1 \times B_2) - J_m(B_1 \times B_2) | < \varepsilon/4 \times (2l)^{-k}$$

where $l = [M/d] + 1$. Then we have for $m \geqq N_3$,

(3.16) $$\left| \Sigma \int_{B(i_1, \cdots, i_k) \cap \{\| u \| \leqq M\} \times R^k} dH_m(u, v) \right.$$
$$\left. - \Sigma \int_{B(i_1, \cdots, i_k) \cap \{\| u \| \leqq M\} \times R^k} dJ_m(u, v) \right|$$

$$\leqq \Sigma \mid H_m[B(i_1,\cdots,i_k)\cap\{\|u\|\leqq M\}\times R^k] - J_m[B(i_1,\cdots,i_k)\cap\{\|u\|\leqq M\}\times R^k]\mid$$
$$\leqq (2l)^k\cdot(2l)^{-k}\varepsilon/4=\varepsilon/4$$

(counting the number of set $B(i_1,\cdots,i_k)\cap\{\|u\|\leqq M\}\times R^k$).

Hence it follows from (3.14), (3.15) and (3.16) that for $m\geqq\max(N_1, N_2, N_3)$

$$|L(y)-G*\Phi(y)|<2\cdot 3\varepsilon/8+\varepsilon/4=\varepsilon .$$

Since ε is arbitrary, we have

$$L(y)=G*\Phi(y) . \tag{3.17}$$

The relation (3.17) implies that since G chosen before does not depend on the choice of the subsequence $\{m\}$,

$$G_n(u)\rightarrow G(u)$$

at continuity point u of G, and, therefore, that G is a probability distribution function.

The theorem is completely proved.

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