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# Nonlinear Hodge Theory on Manifolds with Boundary (*).

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**Summary.** – *The intent of this paper is first to provide a comprehensive and unifying development of Sobolev spaces of differential forms on Riemannian manifolds with boundary. Second, is the study of a particular class of nonlinear, first order, elliptic* PDEs *called Hodge systems. The Hodge systems are far reaching extensions of the Cauchy-Riemann system and solutions are referred to as Hodge conjugate fields. We formulate and solve the Dirichlet and Neumann boundary value problems for the Hodge systems and establish the* $\mathcal{L}^p$*-theory for such solutions. Among the many desirable properties of Hodge conjugate fields, we prove, in analogy with the case of holomorphic functions on the plane, the compactness principle and a strong theorem on the removability of singularities. Finally, some relevant examples and applications are indicated.*

## 1. – Introduction.

The first six sections are written to serve as a solid introduction to the $\mathcal{L}^p$-theory of differential forms, although we have tried to keep it brief. The forms are defined on a regular open region $M$ of closed $C^\infty$-smooth oriented Riemannian manifold $\mathcal{R}$ of dimension $n$, called the reference manifold. The boundary $\partial M$ of $M$ is itself a closed $(n-1)$-manifold which is empty when $M = \mathcal{R}$. We denote by $\bigwedge^l M$, $l = 0, 1, \ldots, n$, the $l$th-exterior power of the cotangent bundle. Thus, the sections of $\bigwedge^l M$, denoted by $\Gamma(\bigwedge^l M)$, are the $l$-forms on $M$. To denote a particular subspace of $\Gamma(\bigwedge^l M)$, we replace $\Gamma$ by an appropriate symbol:

$C^\infty(\bigwedge^l M)$: smooth $l$-forms,

$C_T^\infty(\bigwedge^l M)$: smooth $l$-forms with vanishing tangential component on $\partial M$,

$C_T^\infty(\bigwedge^l M)$: smooth $l$-forms with vanishing normal component on $\partial M$.

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The closures of these spaces in the usual Sobolev norm are respectively denoted

$$\mathcal{W}^{1,p}(\wedge^l M), \quad \mathcal{W}_T^{1,p}(\wedge^l M) \text{ and } \mathcal{W}_N^{1,p}(\wedge^l M)$$

Further, by $\mathcal{W}^{d,p}(\wedge^l M)$, $\mathcal{W}_T^{d,p}(\wedge^l M)$ and $\mathcal{W}^{d^*,p}(\wedge^l M)$, $\mathcal{W}_N^{d^*,p}(\wedge^l M)$, we denote the natural domains of the exterior derivative $d$ and its formal adjoint, the coexterior derivative $d^*$. Of course, there are more Sobolev spaces of differential forms of interest which are relevant to boundary value problems. We have found relations between these spaces, identified the duals and established $\mathcal{L}^p$-estimates. For $p=2$, the $\mathcal{L}^p$-theory is well understood and can be found in [Con56], [Duff52], [DS52], [Gaf54], [Hod33] and [Kod49]. In the case $p \neq 2$, there are some results in Morrey's book [Mor66]. However, the theory has yet to be fully developed. Indeed, establishing $\mathcal{L}^p$-estimates for the Hodge Decomposition and Poincaré type inequalities (see Theorems 6.3 and 6.4) demand greater effort than in the Euclidean case and are handled rather comprehensively here for the first time in many instances. Thus, our first six sections expose familiar results while filling considerable gaps between textbooks and research papers on nonlinear potential theory. Perhaps our extension of Gaffney's inequality

$$\|\omega\|_{1,p} \leq C_p(M)(\|\omega\|_p + \|d\omega\|_p + \|d^*\omega\|_p) \tag{1.1}$$

for $\omega \in \mathcal{W}_T^{1,p}(\wedge^l M) \cup \mathcal{W}_N^{1,p}(\wedge^l M)$, is the most fundamental of these estimates since it proves critical in establishing the $\mathcal{L}^p$- Hodge decompositions for manifolds with boundary

$$\begin{aligned}(1.2) \qquad \mathcal{L}^p(\wedge^l M) &= d\,\mathcal{W}_T^{1,p}(\wedge^{l-1} M) \oplus d^*\,\mathcal{W}_N^{1,p}(\wedge^{l+1} M) \oplus \mathcal{H}^p(\wedge^l M) = \\ &= d\,\mathcal{W}_T^{1,p}(\wedge^{l-1} M) \oplus d^*\,\mathcal{W}^{1,p}(\wedge^{l+1} M) \oplus \mathcal{H}_T(\wedge^l M) = \\ &= d\,\mathcal{W}^{1,p}(\wedge^{l-1} M) \oplus d^*\,\mathcal{W}_N^{1,p}(\wedge^{l+1} M) \oplus \mathcal{H}_N(\wedge^l M)\end{aligned}$$

where $\mathcal{H}^p(\wedge^l M)$, $\mathcal{H}_T(\wedge^l M)$ and $\mathcal{H}_N(\wedge^l M)$ denote the spaces of $\mathcal{L}^p$-harmonic fields, harmonic fields with vanishing tangential part and harmonic fields with vanishing normal part respectively. These decompositions serve as a guide for the proper formulation of boundary conditions for the nonlinear PDEs in the sequel. With the thorough machinery of Sect. 1 and 6 in place, we make a transition to the study of nonlinear PDEs.

Perhaps the most natural of these PDEs arises in the following classical variational problem. Given $\omega \in \mathcal{L}^p(\wedge^l M)$, find the nearest, in $\mathcal{L}^p$-norm, exact form $\phi$. That is, find $\phi$ satisfying

$$\int_M |\omega - \phi|^p = \min\left\{\int_M |\omega - \xi|^p : \xi \in d\,\mathcal{W}^{1,p}(\wedge^{l-1} M)\right\} \tag{1.3}$$

Here, we exploit the $\mathcal{L}^p$-Hodge theory of Sect. 5 to verify that such a $\phi$ exists and is unique in $d\,\mathcal{W}^{1,p}(\wedge^{l-1} M)$. The Lagrange-Euler equation for (1.3) takes the form

$$d^* |\omega - \phi|^{p-2}(\omega - \phi) = 0$$

It can be reformulated as a relation

$$|\omega - \phi|^{p-2}(\omega - \phi) = \psi$$

between the exact form $\phi$ and a coclosed form $\psi$. Of course, in the above minimization problem, $d\mathcal{W}^{1,p}(\wedge^{l-1}M)$ may be replaced by other subspaces of interest. Indeed, we examine eight natural such subspaces, each of which leads to the study of a pair $(\phi, \psi)$ coupled by

$$\omega = \mathfrak{B}(\phi, \psi), \qquad d\phi = d^*\psi = 0 \tag{1.4}$$

where $\mathfrak{B} : \wedge^l M \times \wedge^l M \to \wedge^l M$ is a given bundle map. Apparently, this provides for a nonlinear decomposition of $\omega$. The linear case $\mathfrak{B}(\phi, \psi) = \phi + \psi$ gives the familiar Hodge decompositions discussed in Sect. 5.

More surprising is the fact that these PDEs also arise in quasiconformal analysis. Indeed, given a quasiregular map $f\colon M \to N$ between two oriented Riemannian manifolds of the same dimension $n$, we fix a harmonic field $\xi$ on $N$. In this setting, the unknowns in (1.4) are $\phi = f^{\#}(\xi)$ and $\psi = f_{\#}(\xi)$, the pullbacks of $\xi$ via $f$; see Sect. 7. The case of $n = 2l$ (i.e. even dimensions) is particularly interesting since for $\xi \in \mathcal{H}(\wedge^l N)$, the system (1.4) is linear; see [DS89], [IM93] and [Man95].

Also, in nonlinear elasticity, the method of differential forms and equations of type (1.4) are becoming ever more indispensible (e.g. in the study of null Lagrangians [IL93], [RRT88] and [Iwa95]).

To effectively handle all of these applications as well as any others of this type which will likely arise, we introduce the so-called *Hodge Systems* (Sect. 8). Given a bundle map $\mathfrak{H}_p\colon \wedge^l M \to \wedge^l M$ satisfying conditions (8.15-8.17), a pair $(\phi, \psi) \in \mathcal{L}^p_{\mathrm{loc}}(\wedge^l M) \times \mathcal{L}^q_{\mathrm{loc}}(\wedge^l M)$, $p + q = pq$ is called an $\mathfrak{H}_p$-*(conjugate) couple* or *Hodge conjugate fields* in case

$$\psi = \mathfrak{H}_p(\phi), \qquad d\phi = 0 \text{ and } d^*\psi = 0 \tag{1.5}$$

In the main body of the text, we also treat the nonhomogeneous case of this equation but for purposes of this introduction (1.5) will suffice.

A central theme of our work are results obtained when the natural exponents $p$ and $q$ are replaced by $\lambda p$ and $\lambda q$ with $\lambda \geqslant \max\{1/p, 1/q\}$. Representative of our results in this direction is

THEOREM 1.1 (Regularity Theorem). – *There exist numbers $a < 1 < b$ so that each $\mathfrak{H}_p$-couple $(\phi, \psi) \in \mathcal{L}^{ap}_{\mathrm{loc}}(\wedge^l M) \times \mathcal{L}^{aq}_{\mathrm{loc}}(\wedge^l M)$ actually belongs to $\mathcal{L}^{bp}_{\mathrm{loc}}(\wedge^l M) \times \mathcal{L}^{bq}_{\mathrm{loc}}(\wedge^l M)$.*

We emphasize here that $a$ and $b$ depend only on the structural constants defining the bundle map $\mathfrak{H}_p$. To fix these ideas, notice that the simplest Hodge system is

$$\psi = \phi\,, \qquad d\phi = d^*\psi = 0 \tag{1.6}$$

which means that $\phi$ is a harmonic field. In fact, when considering (1.6) for 1-forms on the complex plane, it is simply the Cauchy- Riemann system. Continuing this analogy more deeply, we give

THEOREM 1.2 (Compactness Principle). – *Let $\Omega \subset \mathcal{R}$ be open. Each sequence of $\mathfrak{H}_p$-couples bounded in $\mathcal{L}^{ap}(\wedge^l \Omega) \times \mathcal{L}^{aq}(\wedge^l \Omega)$ contains a subsequence converging in $\mathcal{L}^{bp}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{bq}_{\mathrm{loc}}(\wedge^l \Omega)$.*

Of course, the value of this principle goes beyond this mere analogy with the normal family theorem for holomorphic functions. Indeed, it plays an essential role in verifying existence for the Dirichlet and Neumann problems.

Guided by the Hodge decompositions, we found proper boundary conditions for Hodge systems. These conditions are formulated in terms of the potential of $\phi$ or the potential of $\psi$. Thus, we should assume that either $\phi$ is exact or that $\psi$ is coexact. Because of a duality principle, we need only consider the case when $\phi$ is exact, say $\phi = d\alpha$. The Hodge system can then be written

$$\psi = \mathfrak{H}_p(d\alpha), \qquad d^* \psi = 0 \tag{1.7}$$

for $\alpha \in \mathcal{W}^{d,\lambda p}(\wedge^{l-1} M)$ and $\psi \in \mathcal{L}^{\lambda q}(\wedge^l M)$, with some $\lambda \in [a, b]$.

In the Dirichlet Problem, we prescribe the tangential component of $\alpha$ on $\partial M$ while in the Neumann problem, we prescribe the normal component of $\psi$ on $\partial M$.

THEOREM 1.3 (Existence Theorem). – *Given $\alpha_0 \in \mathcal{W}^{d,\lambda p}(\wedge^{l-1} M)$, $a \leq \lambda \leq b$, the Dirichlet Problem for* (1.7) *has a solution satisfying: $\alpha - \alpha_0 \in \mathcal{W}^{d,\lambda p}_T(\wedge^{l-1} M)$ and*

$$\|\alpha\|_{1,\lambda p} \leq C_p \|d\alpha_0\|_{\lambda p}$$

*Given $\psi_0 \in \mathcal{W}^{d^*,\lambda q}(\wedge^l M)$, $a \leq \lambda \leq b$, the Neumann Problem for* (1.7) *has a solution satisfying: $\psi - \psi_0 \in \mathcal{W}^{d^*,\lambda q}_N(\wedge^l M)$ and*

$$\|\alpha\|_{1,\lambda p} \leq C_p \|\psi_0\|^{q-1}_{\lambda q}$$

We have reserved for our final result, a dramatic extension of the Painlevé removability theorem and the results of [IM93], [Iwa92]. For the sake of simplicity, we state this result for the introduction only in the linear case which is already both new and nontrivial. Thus, we assume that $\mathfrak{H}\colon \wedge^l \mathcal{R} \to \wedge^l \mathcal{R}$ is a linear bundle automorphism.

THEOREM 1.4 (Removability Theorem). – *Let $\Omega \subset \mathcal{R}$ be open and $E \subset \mathcal{R}$ be closed. Consider the Hodge system*

$$d^* \zeta = \mathfrak{H}(d\xi) \quad \text{in } U = \Omega - E \tag{1.8}$$

*for $\xi \in \mathcal{W}^{d,2}_{\mathrm{loc}}(\wedge^{l-1} U)$ and $\zeta \in \mathcal{W}^{d^*,2}_{\mathrm{loc}}(\wedge^{l+1} U)$. If, moreover, $\xi$ and $\zeta$ are bounded then they extend to $\Omega$ as solutions to* (1.8), *provided* $\dim E \leq s$, *where $s > n - 2$ is a number dependent only on the structural constants for $\mathfrak{H}$.*

Although we do not pusue the matter here, a few words about connections with the second order PDEs are in order. Applying $d^*$ to (1.7), one can eliminate $\psi$ from the Hodge system to arrive at the equation

$$d^* \mathfrak{H}_p(d\alpha) = 0$$

for one unknown $\alpha \in \mathcal{W}^{d,\lambda p}(\wedge^{l-1} M)$. For $l = 1$ and $M$ an open region of $\mathbb{R}^n$, this comprises the familiar $A$-harmonic equation

$$\mathrm{div} A(x, \nabla u) = 0$$

where $A\colon M \times \mathbb{R}^n \to \mathbb{R}^n$; a mapping of the trivial bundle $\wedge^1 M$. We refer the reader to [BI83] and [Iwa83] for estimates of $A$-harmonic functions with applications to quasiregular mappings and to [HKM93] for a fuller treatment. The advantage in studying the first order Hodge systems is particularly evident when one wants to pass to the dual equation. For this, it is necessary to write the $A$-harmonic equation as $\mathfrak{H}(du) = d^* v$, with a 2-form $v$ as an additional unknown. One can now eliminate $u$ to obtain the dual equation $d\mathfrak{H}^{-1}(d^* v) = 0$ which inevitably involves differential forms. These duality arguments were successfuly exploited in [Iwa92].

## 2. - Preliminaries.

2.1. *Some exterior algebra.* - The current section is dedicated not only to an exposition of those aspects of exterior algebra essential to our development but more importantly to establishing notation for the sometimes cumbersome technical details associated with differential forms. The best general reference here is [Car70].

We let $E$ denote a real vector space of $n$-dimensions. An $l$-linear, alternating function $\xi\colon E \times \ldots \times E \to \mathbb{R}$ will be called an $l$-form and the space of all such forms will be indicated by $\wedge^l E$. In particular, $\wedge^1 E = E'$, the dual to $E$. For technical reasons, we set $\wedge^0 E = \mathbb{R}$ and recall the exterior algebra of forms $\wedge E \equiv \oplus_{l=0}^n \wedge^l E$. The familiar wedge product of $\xi \in \wedge^l E$ and $\zeta \in \wedge^k E$ is given by

$$(\xi \wedge \zeta)(X_1, \ldots, X_{k+l}) \equiv \sum \varepsilon \xi(X_{i_1}, \ldots, X_{i_l})\, \zeta(X_{j_1}, \ldots, X_{j_k}), \tag{2.1}$$

where the sum is taken over all permutations $\{i_1, \ldots, i_l, j_1, \ldots, j_k\}$ of $\{1, \ldots, k+l\}$ satisfying $i_1 < \ldots < i_l$ and $j_1 < \ldots < j_k$, and $\varepsilon$ is the sign of $\{i_1, \ldots, i_l, j_1, \ldots, j_k\}$. Note that $\xi \wedge \zeta = (-1)^{kl} \zeta \wedge \xi$.

When $E$ is endowed with an inner product $\langle , \rangle$ and an orthonormal basis $\mathcal{B} = \{e_1, \ldots, e_n\}$, an inner product is naturally induced for $E'$ by

$$\langle \xi, \zeta \rangle \equiv \sum_{i=1}^{n} \xi(e_i)\, \zeta(e_i) \tag{2.2}$$

and for $\wedge^l E$ by

$$\langle \xi, \zeta \rangle \equiv \det \langle \xi_i, \zeta_j \rangle \tag{2.3}$$

where $\xi = \xi_1 \wedge \ldots \wedge \xi_l$ and $\zeta = \zeta_1 \wedge \ldots \wedge \zeta_l$ for $\xi_i, \zeta_j \in E'$. Let's denote the basis dual to $\mathcal{B}$ by $\mathcal{B}' = \{e^1, \ldots, e^n\}$ (i.e. $e^i(e_j) = \delta_{ij}$). We recall that when $\mathcal{B}$ is orthonormal, (2.10) and (2.11) guarantee that the system

$$\{e^{i_1} \wedge \ldots \wedge e^{i_l}\colon 1 \leq i_1 < \ldots < i_l \leq n\} \tag{2.4}$$

forms an orthonormal basis for $\wedge^l E$ with dimension $\binom{n}{l}$. We refer to $\boldsymbol{e} = e^1 \wedge \ldots \wedge e^n$ as an orientation for $E$. Associated with e is the so-called Hodge star operator.

$$* : \wedge^l E \to \wedge^{n-l} E, \qquad l = 0, 1, \ldots, n \tag{2.5}$$

uniquely determined by

$$* 1 = \boldsymbol{e} \quad \text{and} \quad \xi \wedge * \zeta = \langle \xi, \zeta \rangle \boldsymbol{e} \tag{2.6}$$

for all $\xi, \zeta \in \wedge^l E$ and $l = 0, 1, \ldots, n$. Here, we observe that $*$ is an isometry and that $**$ is simply multiplication by $(-1)^{l(n-l)}$ on $\wedge^l E$. We take a moment now to exploit the structure elucidated above to discuss some intrinsic geometric properties of forms associated with subspaces of $E$. For this, let $V$ be a subspace of $E$ and let $\pi : E \to V$ denote orthogonal projection. Given an arbitrary $\omega \in \wedge^l E$, we define the tangential part of $\omega$ (w.r.t. $V$) $\omega_T \in \wedge^l E$ by

$$\omega_T(X_1, \ldots, X_l) \equiv \omega(\pi X_1, \ldots, \pi X_l) \tag{2.7}$$

for $X_1, \ldots, X_l \in E$. We observe here that $\omega_T$ is still an element of $\wedge^l E$ but with the property that for $X_1, \ldots, X_l \in V$, we have

$$\omega_T(X_1, \ldots, X_l) = \omega(X_1, \ldots, X_l) \tag{2.8}$$

This induces $\omega$'s normal part $\omega_N = \omega - \omega_T$. By $\wedge_T^l E$ ($\wedge_N^l E$), we indicate the subspace of $\wedge^l E$ with $\omega_T = 0$ ($\omega_N = 0$). Thus, the $l$-forms orthogonally decompose according to

$$\wedge^l E = (\wedge_N^l E) \oplus (\wedge_T^l E) \tag{2.9}$$

Often useful is the following

LEMMA 2.1. – *Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $E$ with $\{e_1, \ldots, e_k\}$ $(k < n)$ a basis of the subspace $V$ and let $\omega = \sum \omega_{i_1, \ldots, i_n} e^{i_1} \wedge \ldots \wedge e^{i_n}$, where the sum is taken over all ordered $l$-tuples $1 \leq i_1 < \ldots < i_l \leq n$. Then*

$$\omega_T = \sum_{i_l \leq k} \omega_{i_1, \ldots, i_l} e^{i_1} \wedge \ldots \wedge e^{i_l} \in \wedge_N^l E \tag{2.10}$$

$$\omega_N = \sum_{i_l > k} \omega_{i_1, \ldots, i_l} e^{i_1} \wedge \ldots \wedge e^{i_l} \in \wedge_T^l E \tag{2.11}$$

For the remainder of this section, we let $\mathcal{L} : E \to F$ be a linear map between the inner product spaces $(E, \langle\,,\rangle_E)$ and $(F, \langle\,,\rangle_F)$. Denote by $\mathcal{L}^{\#} : F' \to E'$ the map dual to $\mathcal{L}$ (i.e. $(\mathcal{L}^{\#} \xi)(X) \equiv \xi(\mathcal{L} X)$ for $\xi \in F'$ and $X \in E$). The concept of the dual map extends naturally to $l$-forms. This map, still denoted by $\mathcal{L}^{\#} : \wedge^l F \to \wedge^l E$ and called the pullback of

$\mathcal{L}$, is defined according to

$$\mathcal{L}^{\#}(\xi_1 \wedge \ldots \wedge \xi_l) \equiv \mathcal{L}^{\#}\xi_1 \wedge \ldots \wedge \mathcal{L}^{\#}\xi_l \tag{2.12}$$

for $\xi_1, \ldots, \xi_l \in F'$.

Notice that "pulling back" is possessed of many nice properties including

$$\mathcal{L}^{\#}(\xi \wedge \zeta) = \mathcal{L}^{\#}\xi \wedge \mathcal{L}^{\#}\zeta \tag{2.13}$$

$$(\mathcal{L}K)^{\#} = \mathcal{K}^{\#}\mathcal{L}^{\#} \tag{2.14}$$

$$(\mathcal{L}^{\#})^{-1} = (\mathcal{L}^{-1})^{\#} \tag{2.15}$$

$$(\lambda\mathcal{L})^{\#} = \lambda^l \mathcal{L}^{\#}\colon \wedge^l F \to \wedge^l E \quad \text{for } \lambda \in \mathbb{R} \tag{2.16}$$

When $\dim E = \dim F$, this pullback provides for the general formulation of determinant of $\mathcal{L}$, uniquely given by

$$\mathcal{L}^{\#}\boldsymbol{f} = (\det \mathcal{L})\, \boldsymbol{e} \tag{2.17}$$

where $\boldsymbol{e}$ and $\boldsymbol{f}$ are the orientation forms for $E$ and $F$, respectively. Also, the transpose of $\mathcal{L}$, $\mathcal{L}^t\colon F \to E$ is uniquely defined by the rule

$$\langle \mathcal{L}^t Y, X\rangle_E = \langle Y, \mathcal{L}X\rangle_F \tag{2.18}$$

for all $X \in E$ and $Y \in F$. The interplay between the operations $*$, det, pullback, transpose, tangential and normal parts is both satisfying and important for the sequel. For later reference, we give a short list of facts representing this interplay

$$(\mathcal{L}^t)^{\#} = (\mathcal{L}^{\#})^t \tag{2.19}$$

$$\det \mathcal{L}^t = \det \mathcal{L} \tag{2.20}$$

$$(\mathcal{L}^{\#})^t * = (\det \mathcal{L}) * (\mathcal{L}^{\#})^{-1} \tag{2.21}$$

$$*\,\omega_N = (*\,\omega)_T \quad \text{and} \quad *\,\omega_T = (*\,\omega)_N \tag{2.22}$$

Further, with $i\colon V \to E$ denoting inclusion, we get

$$i^{\#}\omega = i^{\#}\omega_T$$

Another useful pullback of a linear map $\mathcal{L}\colon E \to F$ between inner product spaces is defined by the rule

$$\mathcal{L}_{\#} = (-1)^{\ln - l} * \mathcal{L}^{\#} * \colon \wedge^l F \to \wedge^l E \tag{2.23}$$

We then have

$$
(2.24)\qquad \begin{cases} (\mathcal{L}K)_\# = \mathcal{K}_\# \mathcal{L}_\# \\ (\mathcal{L}_\#)^{-1} = (\mathcal{L}^{-1})_\# \\ (\lambda \mathcal{L})_\# = \lambda^{n-l} \mathcal{L}_\# \colon \wedge^l F \to \wedge^l E \qquad \text{for all } \lambda \in \mathbb{R} \end{cases}
$$

With this notation, the well known Laplace expansion of the determinant reads as

$$
\mathcal{L}_\# (\mathcal{L}^\#)^t = (\det \mathcal{L})\ \mathrm{Id} \colon \wedge^l E \to \wedge^l E
$$

or

$$
(\mathcal{L}^\#)^t \mathcal{L}_\# = (\det \mathcal{L})\ \mathrm{Id} \colon \wedge^l F \to \wedge^l F
$$

Fixing orthogonal bases for $E$ and $F$, $\dim E = \dim F = n$, $\mathcal{L}^\#$ and $\mathcal{L}_\#$ are represented with respect to the induced orthogonal system by a matrix whose entries are the $l \times l$ and $(n-l) \times (n-l)$-minors respectively of the matrix for $\mathcal{L}$. Notice that these matrices have size $\binom{n}{l} = \binom{n}{n-l}$.

2.2. *The Riemannian manifold setting.* – For the duration of this paper, $M$ will denote a regular open region of a closed (without boundary), oriented, Riemannian manifold $(\mathcal{R}, g)$ of dimension $n \geq 2$. We mention now that $\mathcal{R}$ will play only an auxilliary role serving as a reference to various geometric structures on $M$. For this reason, we refer to $(\mathcal{R}, g)$ as a reference manifold. Also, let us recall that a regular open region $M \subset \mathcal{R}$ is one for which there exists a finite atlas $\mathcal{A}$ on the reference manifold $\mathcal{R}$ consisting entirely of coordinate charts $(U, \kappa) \in \mathcal{A}$ so that $\kappa$ is a $C^\infty$-diffeomorphism *onto* $\mathbb{R}^n$ and $\kappa(U \cap M) = \mathbb{R}^n_+$ whenever $U$ meets $\partial M$. Observe that the notation

$$
\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \colon x_n > 0\}
$$

is being used here. We refer to such an atlas $\mathcal{A}$ as a *regular atlas* and to those $(U, \kappa)$ for which $U$ meets $\partial M$ as coordinate neighborhoods at the boundary.

For any regular region $M$, there is a sufficiently small number $\varepsilon > 0$ so that there emanates, from each point $s \in \partial M$, a unique open geodesic arc $\gamma_s$ of length $2\varepsilon$ which is orthogonal to $\partial M$ having half of $\gamma_s$ in $M$ and half outside of $M$. These geodesic arcs form a 1-dimensional $C^\infty$-foliation of a region, called a collar neighborhood of $\partial M$, with size $\varepsilon$

$$
\mathcal{N}_\varepsilon = \mathcal{N}_\varepsilon(\partial M) = \bigcup_{s \in \partial M} \gamma_s
$$

To each point $a \in \mathcal{N}_\varepsilon$ we assign cylindrical coordinates $(s, t)$, where $s \in \partial M$ indicates the unique geodesic $\gamma_s$ passing through $a$ and the number $t \in (-\varepsilon, \varepsilon)$ is the geodesic distance from $a$ to $s$ which is positive if $a \in M$ and negative if $a$ is not in $M$. Of course, the coordinate function $t\colon \mathcal{N}_\varepsilon \to \mathbb{R}$ has no critical points (i.e. $dt \neq 0$). This gives rise to a $C^\infty$-diffeomorphism

$$\Phi\colon \mathcal{N}_\varepsilon \to \partial M \times (-\varepsilon, \varepsilon), \qquad \Phi(a) = (s, t) \tag{2.25}$$

Using such a collar neighborhood and associated coordinates, it is understood that there exists a $C^\infty$-perturbation $F\colon \mathcal{R} \times (-1, 1) \to \mathcal{R}$ of the identity $id_{\mathcal{R}}\colon \mathcal{R} \to \mathcal{R}$ with the properties

$$\text{Each } F_t = F(\bullet, t)\colon \mathcal{R} \to \mathcal{R} \text{ is a diffeomorphism with } F_0 = \mathrm{id}_{\mathcal{R}} \tag{2.26}$$

$$\begin{cases} \overline{M} \subset F_t(M), & t > 0 \\ F_t(\overline{M}) \subset M, & t < 0 \end{cases} \tag{2.27}$$

Further, the reflection through $\partial M$ in $\mathcal{N}_\varepsilon$

$$\mathrm{r}\colon \mathcal{N}_\varepsilon \to \mathcal{N}_\varepsilon \tag{2.28}$$

is an orientation reversing diffeomorphism of $\mathcal{N}_\varepsilon$ with itself given by $r(s, t) = (s, -t)$, in cylindrical coordinates.

We will denote by $T\mathcal{R}$ the tangent bundle over $\mathcal{R}$ and if we need to specify precisely the fibre over $a \in \mathcal{R}$, we write $T_a\mathcal{R}$. Each fibre $T_a\mathcal{R}$ is furnished with an inner product, induced by the tensor $g$, which we denote by $\langle X, Y\rangle$, for $X, Y \in T_a\mathcal{R}$. Observe that in this notation we ignore the dependence of the inner product on the point $a \in \mathcal{R}$. By $\wedge^l \mathcal{R}$, we indicate the $l$-th exterior power of the cotangent bundle $T^*\mathcal{R}$. Precisely, we mean that the fibre over $a \in \mathcal{R}$ is given by $\wedge^l_a \mathcal{R} = \wedge^l (T_a\mathcal{R})$. See Subsect. 2.1. We use the symbol $\langle \xi, \zeta\rangle$ for the inner product of the $l$-covectors $\xi, \zeta \in \wedge^l_a\mathcal{R}$. The Whitney sum $\wedge \mathcal{R} = \oplus_{l=0}^n \wedge^l \mathcal{R}$ will be called the exterior algebra bundle, whose fibre $\wedge_a \mathcal{R} = \oplus_{l=0}^n \wedge^l_a \mathcal{R}$ is endowed with the inner product defined by letting the spaces $\wedge^l_a\mathcal{R}$, $l = 0, 1, \ldots, n$ be mutually orthogonal. Since most often we will be considering the bundle $\wedge^l\mathcal{R}$, we abbreviate this notation to $\wedge^l$ when no confusion is possible. Finally, we use analogous notation for $\partial M$ when we consider it as a manifold rather than simply a subset of $\mathcal{R}$ (e.g. $\wedge^l \partial M = \wedge^l (T\partial M)$).

2.3. *Exterior forms.* – Let $E$ be a bundle over a manifold $N$ and let $\Omega$ be an arbitrary subset of $N$. By $\Gamma(\Omega, E)$ we denote the sections of $E$ defined on $\Omega$. To simplify this a bit, in those cases when $\Omega = N$, we write $\Gamma(E)$ for $\Gamma(N, E)$. For example, $\Gamma(\partial M, \wedge^l)$ denotes sections of $\wedge^l = \wedge^l \mathcal{R}$ which are defined on $\partial M$ while $\Gamma(\wedge^l \partial M) = \Gamma(\partial M, \wedge^l \partial M)$ and $\Gamma(\wedge^l) = \Gamma(\mathcal{R}, \wedge^l \mathcal{R})$ denote the $l$-forms on the manifolds $\partial M$ and $\mathcal{R}$ respectively. As one more point of emphasis for these subtle but important distictions we give

$$\Gamma(\Omega, \wedge^l) = \Gamma(\wedge^l \Omega)$$

when $\Omega$ is an open subset of $\mathcal{R}$.

Notice that the wedge product, Hodge star and inner product discussed in Subsect. 2.1 extend pointwise to $l$-forms. For example, $\langle\,,\rangle\colon \Gamma(\Omega, \wedge^l) \times \Gamma(\Omega, \wedge^l) \to$

$\to \Gamma(\wedge^0 \Omega)$ is defined by the rule

$$\langle \xi, \zeta \rangle(a) = \langle \xi(a), \zeta(a) \rangle \tag{2.29}$$

for $\xi, \zeta \in \Gamma(\Omega, \wedge^l)$ and $a \in \Omega$. Recall that the symbol $\langle , \rangle$ on the right hand side stands for the inner product on $\wedge_a^l \mathcal{R}$. Following Subsect. 2.1, we may define tangential and normal parts of a form $\xi \in \Gamma(\overline{M}, \wedge^l)$ pointwise on $\partial M$ by

$$\xi_T(b) = \xi(b)_T, \qquad \xi_N(b) = \xi(b)_N \tag{2.30}$$

for $b \in \partial M$. Notice that $\xi_T$ and $\xi_N$ are elements of $\Gamma(\partial M, \wedge^l)$, and

$$\xi|_{\partial M} = \xi_T + \xi_N \tag{2.31}$$

When we wish to denote particular subspaces of $\Gamma(\Omega, \wedge^l)$, we will use familiar notation for the space and $(\Omega, \wedge^l)$ to indicate the domain as well as the degree of the forms under consideration. For example, the arbitrarily differentiable $l$-forms on the regular region $M \subset \mathcal{R}$ are denoted by $C^\infty(M, \wedge^l) = C^\infty(\wedge^l M)$, those with compact support by $C_0^\infty(\wedge^l M)$ and those which are $C^\infty$-smooth up to the boundary by $C^\infty(\overline{M}, \wedge^l)$. Also, throughout this work, we let $C_T^\infty(\overline{M}, \wedge^l)$ and $C_N^\infty(\overline{M}, \wedge^l)$ denote the smooth $l$-forms with vanishing tangential and normal parts respectively.

For a $C^\infty$-mapping $f: X \to Y$ between manifolds, we may define the associated pullback of forms $f^\#, f_\#: \Gamma(\wedge^l Y) \to \Gamma(\wedge^l X)$ according to

$$(f^\# \omega)(a) = [Df(a)]^\# \omega(b)$$

and

$$(f_\# \omega)(a) = [Df(a)]_\# \omega(b)$$

for $\omega \in \Gamma(\wedge^l Y)$ and all $a \in X$, $b = f(a)$. Here, $Df(a)$: $T_a X \to T_b Y$ denotes the differential of $f$ and $[Df(a)]^\#, [Df(a)]_\#: \wedge_b^l Y \to \wedge_a^l X$ are the pullbacks of the linear map $Df(a)$ as given in Subsect. 2.1.

Of fundamental concern to us will be the exterior derivative

$$d: C^\infty(\wedge^l M) \to C^\infty(\wedge^{l+1} M) \tag{2.32}$$

For which, we have the formula

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^l \alpha \wedge d\beta$$

where $l$ stands for the degree of $\alpha$. The formal adjoint of $d$, also called the Hodge codifferential, is given by

$$d^* = (-1)^{nl+1} * d * : C^\infty(\wedge^{l+1} M) \to C^\infty(\wedge^l M) \tag{2.33}$$

We note the commutation rules $f^\# d = df^\#$ and $f_\# d^* = d^* f_\#$. Of course, $d$ and $d^*$ are understood for more general spaces of differential forms but we reserve such discussion until Sobolev classes are introduced. Finally, we note that the duality between these differential operators is emphasized by the integration by parts formula

$$\int_M \langle du, v\rangle - \int_M \langle u, d^* v\rangle = \int_{\partial M} u_T \wedge * v_N \tag{2.34}$$

where $u \in C^\infty(\overline{M}, \wedge^l)$ and $v \in C^\infty(\overline{M}, \wedge^{l+1})$.

## 3. - Sobolev classes of differential forms.

Here and subsequently, the measure on $\mathcal{R}$ will be the one induced by the volume form $\mu = *1$. Differential forms which are equal a.e. will be regarded as indistinguishable. It is customary to omit notation of the measure under the integral sign and we shall follow this custom when it is clear that no confusion will arise.

3.1. *The Lebesgue spaces.* - If $\Omega$ is a measurable subset of $\mathcal{R}$ and $1 \leq p \leq \infty$ is fixed, we denote by $\mathcal{L}^p(\Omega, \wedge^l)$ the space of all measurable $l$-forms $\omega \in \Gamma(\Omega, \wedge^l)$ for which

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega|^p\right)^{1/p} < \infty, \qquad \text{for } p < \infty \tag{3.1}$$

$$\|\omega\|_{\infty,\Omega} = \operatorname{esssup}_\Omega |\omega| < \infty \tag{3.2}$$

Most of the time we will be dealing with the spaces $\mathcal{L}^p(\wedge^l M)$ in which case we omit reference to the set $\Omega = M$ and simply write $\|\omega\|_p = \|\omega\|_{p,M}$. If $1 \leq p, q \leq \infty$ is a Hölder conjugate pair, then the scalar product of $\alpha \in \mathcal{L}^p(\Omega, \wedge^l)$ and $\beta \in \mathcal{L}^q(\Omega, \wedge^l)$ is defined by

$$(\alpha, \beta)_\Omega = \int_\Omega \alpha \wedge * \beta = \int_\Omega \beta \wedge * \alpha = \int_\Omega \langle \alpha, \beta\rangle \tag{3.3}$$

In the style of previous conventions, we abbreviate $(\alpha, \beta)_M$ with $(\alpha, \beta)$. Of course, the full $\mathcal{L}^p$ space is not the only one that will be of interest to us. In particular, we will require the local spaces

$$\mathcal{L}^p_{\mathrm{loc}}(\wedge^l M) = \{\omega \in \Gamma(\wedge^l M) : \|\omega\|_{p,\Omega} < \infty \text{ for each compact } \Omega \subset M\} \tag{3.4}$$

DEFINITION 3.1. - *A differential form $\omega \in \mathcal{L}^1_{\mathrm{loc}}(\wedge^l M)$ is said to have generalized exterior derivative in case there exists a locally integrable $(l+1)$-form on M, denoted by $d\omega$, such that*

$$(\omega, d^*\eta) = (d\omega, \eta) \tag{3.5}$$

*for every test form $\eta \in C_0^\infty(\wedge^{l+1} M)$. If both $\omega$ and $d\omega$ are integrable on M and* (3.5) *holds for any $\eta \in C^\infty(\wedge^{l+1}\mathcal{R})$, then we write $\omega_T = 0$ and say that $\omega$ has vanishing tangential part. The notion of generalized exterior coderivative and vanishing normal part are defined analogously. Lastly, we refer to*

$$\ker d = \{\omega \in \mathcal{L}^1_{\mathrm{loc}}(\wedge^l M) : d\omega = 0\}$$

*as the closed l-forms and to*

$$\ker d^* = \{\omega \in \mathcal{L}^1_{\mathrm{loc}}(\wedge^l M) \colon d^* \omega = 0\}$$

*as the coclosed l-forms.*

3.2. *The Sobolev spaces.* – For $k = 1, 2, \ldots$ and $1 \leqslant p \leqslant \infty$, Sobolev space $\mathcal{W}^{k,p}(\wedge^l M)$, is defined in the usual fashion by first choosing a finite atlas for the reference manifold $\mathcal{R}$, say $\mathcal{A} = \{(U_i, \kappa_i) \colon i = 1, \ldots, m\}$, as well as a $C^\infty$-partition of unity $\{\chi_i \in C_0^\infty(U_i) \colon \sum \chi_i \equiv 1\}$ subordinate to $\mathcal{A}$. This allows us to decompose $\omega \in \Gamma(\wedge^l M)$ according to $\omega = \sum \omega_i$, where $\omega_i = \chi_i \omega$ and consider the pullback of $\omega_i$ via the mapping $\kappa_i^{-1} \colon \mathbb{R}^n \to U_i$

$$\omega_i^\#(x) = \sum_I \omega_i^I(x)\, dx_I, \qquad x \in \mathcal{U}_i = \kappa_i(U_i \cap M) \subset \mathbb{R}^n \tag{3.6}$$

Here, the sum is taken over all ordered $l$-tuples $I = (i_1, \ldots, i_l)$ and $dx_I = dx_{i_1} \wedge \ldots \wedge \wedge dx_{i_l}$. If each function $\omega_i^I$ has generalized partial derivatives $D^\alpha \omega_i^I \in \mathcal{L}^p(\mathcal{U}_i)$ for all $|\alpha| \leqslant k$, then the Sobolev norm of $\omega$ is defined by

$$\|\omega\|_{k,p} = \sum_{i=1}^m \|\omega_i\|_{k,p}$$

where

$$\|\omega_i\|_{k,p} = \sum_{|\alpha| \leqslant k} \sum_I \|D^\alpha \omega_i^I\|_{p, \mathcal{U}_i}$$

It is not difficult to see that the Sobolev spaces corresponding to different atlases and partitions of unity are all equivalent.

An important feature of $\mathcal{W}^{1,p}(\wedge^l M)$ is that every such form is the restriction to $M$ of an element of $\mathcal{W}^{1,p}(\wedge^l \mathcal{R})$. Even better, there is a bounded linear operator, called an extension operator

$$\sim \colon \mathcal{W}^{1,p}(\wedge^l M) \to \mathcal{W}^{1,p}(\wedge^l \mathcal{R}) \tag{3.7}$$

which satisfies $\tilde{\omega}|_M = \omega$ for all $\omega \in \mathcal{W}^{1,p}(\wedge^l M)$. To see that this is the case, let $\mathcal{A}$ be a regular atlas. If $U_i \subset M$, then $\omega_i = \chi_i \omega$ can be regarded as a form of class $\mathcal{W}^{1,p}(\wedge^l \mathcal{R})$ equal to zero outside of $U_i$. If, however, $U_i$ meets $\partial M$, then we recall formula (3.6) and extend $\omega_i^\#$ to $\mathbb{R}^n$ by requiring that $\omega_i^I(x_1, \ldots, x_n) = \omega_i^I(x_1, \ldots, -x_n)$. We denote this extension to $\mathbb{R}^n$ by $\tilde{\omega}_i^\#$. Next, return to the manifold $\mathcal{R}$ by pulling back $\tilde{\omega}_i^\#$ via the map $\kappa_i \colon U_i \to \mathbb{R}^n$. In this way, we obtain a form $\tilde{\omega}_i \in \mathcal{W}_0^{1,p}(\wedge^l U_i)$. Finally, we put $\tilde{\omega} = \sum \tilde{\omega}_i \in \mathcal{W}^{1,p}(\wedge^l \mathcal{R})$.

Armed with the extension operator, one can prove basic approximation properties of the Sobolev Spaces. For example, using the Meyers and Serrin approximation theorem, we obtain

COROLLARY 3.2. – $C^\infty(\wedge^l \mathcal{R})$ *(restricted to* $M$*) is dense in* $W^{1,p}(\wedge^l M)$*, for all* $1 \leqslant p < \infty$.

Let us denote by $\mathcal{W}_0^{1,p}(\wedge^l M)$, $1 \leqslant p \leqslant \infty$, the space of $l$-forms on $M$ whose zero extension to the reference manifold belongs to $\mathcal{W}^{1,p}(\wedge^l \mathcal{R})$. We then have

COROLLARY 3.3. – *$C_0^\infty(\wedge^l M)$ is dense in $\mathcal{W}_0^{1,p}(\wedge^l M)$, for all $1 \leqslant p < \infty$.*

Due to Definition 3.1, we may speak of vanishing tangential and normal parts of forms in $\mathcal{W}^{1,p}(\wedge^l M)$. Accordingly, the spaces of such forms will be denoted by $\mathcal{W}_T^{1,p}(\wedge^l M)$ and $\mathcal{W}_N^{1,p}(\wedge^l M)$. It is immediate that

$$\mathcal{W}_0^{1,p}(\wedge^l M) = \mathcal{W}_T^{1,p}(\wedge^l M) \cap \mathcal{W}_N^{1,p}(\wedge^l M)$$

A very useful tool in dealing with boundary values of differential forms is furnished by special coordinate systems $\{U, \kappa = (x_1, \ldots, x_n)\}$, $\kappa(U \cap M) = \mathbb{R}_+^n$, chosen so that for each $b \in \partial M \cap U$, the tangent vector $\partial/\partial x_n$ is orthogonal to $T_b(\partial M)$. The existence of such systems is easily established by using cylindrical coordinates. In addition, one can require that the vectors $\partial/\partial x_1, \ldots, \partial/\partial x_n$ form a frame which is orthonormal for $T_b \mathcal{R}$ at a specified point $b \in \partial M$, but not necessarily at a collection of points clustered about $b$.

Suppose that $\omega \in W_T^{1,p}(\wedge^l M)$ and $\chi \in C_0^\infty(U)$ is a member of a partition of unity as above. The form $\chi\omega$ splits into tangential and normal parts

$$\chi\omega = \sum_{1 \leqslant i_1 < \ldots < i_l < n} \alpha_{i_1, \ldots, i_l} dx_{i_1} \wedge \ldots \wedge dx_{i_l} + \sum_{i \leqslant i_1 < \ldots < i_l = n} \beta_{i_1, \ldots, i_l} dx_{i_1} \wedge \ldots \wedge dx_{i_l}$$

Here, the coefficients $\alpha_{i_1, \ldots, i_l}$ belong to $W_0^{1,p}(U \cap M)$ while the coefficients $\beta_{i_1, \ldots, i_l}$ only belong to $W^{1,p}(U \cap M)$. We then see that $\alpha_{i_1, \ldots, i_l}$ and $\beta_{i_1, \ldots, i_l}$ can be approximated by functions of class $C_0^\infty(U \cap M)$ and $C_0^\infty(U)$ respectively. Thus, there exists a sequence of $l$-forms in $C_0^\infty(\wedge^l U)$ with vanishing tangential part on $\partial M$ which converges to $\chi\omega$ in the Sobolev class $W^{1,p}(U \cap M, \wedge^l)$. Using a partition of unity, we then obtain

COROLLARY 3.4. – *The space $C_T^\infty(\overline{M}, \wedge^l)$ is dense in $W_T^{1,p}(\wedge^l M)$. Also, in view of Hodge star duality, $C_N^\infty(\overline{M}, \wedge^l)$ is dense in $W_N^{1,p}(\wedge^l M)$, for all $1 \leqslant p < \infty$.*

3.3. *Partly Sobolev classes of first order.* – One special feature of the differential equations we shall discuss is that the partial differentiation occurs only via the operators $d$ and/or $d^*$. Therefore, the natural spaces of differential forms in which to look for solutions will not require that all partials exist. Such spaces, called partly Sobolev classes, have a place of central importance in this paper. In this section we define and summarize briefly the basic properties of such classes.

For the space $\mathcal{W}^{d,p}(\wedge^l M)$, we require only that both a form and its generalized exterior derivative (see Definition 3.1) are $\mathcal{L}^p$-integrable

$$\mathcal{W}^{d,p}(\wedge^l M) = \{\omega \in \mathcal{L}^p(\wedge^l M) \colon d\omega \in \mathcal{L}^p(\wedge^{l+1} M)\} \tag{3.8}$$

This space is equipped with the norm

$$\|\omega\|_{d,p} = \|\omega\|_p + \|d\omega\|_p \tag{3.9}$$

Similarly, we define

$$\mathcal{W}^{d^*,p}(\wedge^l M) = \{\omega \in \mathcal{L}^p(\wedge^l M) \colon d^*\omega \in \mathcal{L}^p(\wedge^{l-1} M)\} \tag{3.10}$$

which is provided the norm

$$\|\omega\|_{d^*,p} = \|\omega\|_p + \|d^*\omega\|_p \tag{3.11}$$

REMARK 3.5. – *It is a straightforward consequence of Definition* 3.1 *that for* $\omega \in \mathcal{W}^{d,p}(\wedge^l M)$, $1 \leq p \leq \infty$, *the form* $d\omega$ *has generalized exterior derivative equal to zero (i.e.* $dd\omega = 0$*). Moreover, if* $\omega_T = 0$, *then* $(d\omega)_T = 0$ *as well. Similar considerations apply to the exterior coderivative.*

Note that both $\mathcal{W}^{d,p}(\wedge^l M)$ and $\mathcal{W}^{d^*,p}(\wedge^l M)$ are modules over the ring $C^\infty(\overline{M})$, since for each $\chi \in C^\infty(\overline{M})$, we have

$$d(\chi\omega) = \chi\, d\omega + d\chi \wedge \omega \in \mathcal{L}^p(\wedge^{l+1} M) \quad \text{if } \omega \in \mathcal{W}^{d,p}(\wedge^l M) \tag{3.12}$$

and

$$d^*(\chi\omega) = \chi\, d^*\omega - (-1)^{nl+n} * (d\chi \wedge *\,\omega) \in \mathcal{L}^p(\wedge^{l-1} M) \quad \text{if } \omega \in \mathcal{W}^{d^*,p}(\wedge^l M) \tag{3.13}$$

For $f\colon X \to Y$ a $C^\infty$-diffeomorphism of compact Riemannian manifolds (with or without boundary), we recall the formula $d(f^\# \omega) = f^\#(d\omega)$. Because of this, the pullback operation

$$f^\# \colon \mathcal{W}^{d,p}(\wedge^l Y) \to \mathcal{W}^{d,p}(\wedge^l X) \tag{3.14}$$

is a Banach space isomorphism. This observation is the key to seeing that the extension operator $\sim$, originally defined in (3.7) for $\mathcal{W}^{1,p}(\wedge^l M)$, is actually acting on $\mathcal{W}^{d,p}(\wedge^l M)$ as a bounded linear operator with values in $\mathcal{W}^{d,p}(\wedge^l \mathcal{R})$.

A slight change in the proof of Corollary 3.2 gives

COROLLARY 3.6. – $C^\infty(\wedge^l \mathcal{R})$ *is dense in* $\mathcal{W}^{d,p}(\wedge^l M)$, *for all* $1 \leq p < \infty$.

PROOF. – As before, we choose a finite atlas for $\mathcal{R}$, say $\mathcal{A} = \{(U_i, \kappa_i) \colon \kappa_i \colon U_i \to \mathbb{R}^n\}$ and a partition of unity $\{\chi_i \in C_0^\infty(U_i) \colon \sum \chi_i = 1\}$. Fix $\omega \in \mathcal{W}^{d,p}(\wedge^l M)$ and let $\tilde{\omega}$ denote its extension to $\mathcal{R}$ so that $\tilde{\omega} \in \mathcal{W}^{d,p}(\wedge^l \mathcal{R})$. We then consider the pullback of $\chi_i \tilde{\omega}$ to $\mathbb{R}^n$ via $\kappa_i^{-1} \colon \mathbb{R}^n \to U_i$. This pullback, denoted $\tilde{\omega}_i^\#$, has compact support and belongs to $\mathcal{W}^{d,p}(\wedge^l \mathbb{R}^n)$. Using the fact that in $\mathbb{R}^n$, the differential operator $d$ has constant coefficients, we see that the convolution of $\tilde{\omega}_i^\#$ with standard mollifiers provides us with a $C^\infty$ approximation of $\tilde{\omega}_i^\#$. We now return to the manifold $\mathcal{R}$ by pulling back via the map $\kappa_i \colon U_i \to \mathbb{R}^n$. The details are left to the reader. ■

Using the Hodge stars $* \colon \Gamma(\wedge^{n-l} Y) \to \Gamma(\wedge^l Y)$ and $* \colon \Gamma(\wedge^l X) \to \Gamma(\wedge^{n-l} X)$, we may introduce another pullback operation, denoted by $f_\# \colon \Gamma(\wedge^l Y) \to \Gamma(\wedge^l X)$ and given by

$$f_\#(\omega) = (-1)^{nl-l} * f^\#(*\,\omega) \tag{3.15}$$

for $\omega \in \Gamma(\wedge^l Y)$. We leave it to the reader to verify that $f_\#$ commutes with codifferentiation

$$f_\#(d^*\omega) = d^*(f_\#\omega) \tag{3.16}$$

As before, we see that

$$f_\#\colon \mathcal{W}^{d^*,p}(\wedge^l Y) \to \mathcal{W}^{d^*,p}(\wedge^l X) \tag{3.17}$$

is a Banach space isomorphism. We now have a corollary dual to Corollary 3.6.

COROLLARY 3.7. – *$C^\infty(\wedge^l \mathcal{R})$ is dense in $\mathcal{W}^{d^*,p}(\wedge^l M)$, for all $1 \leq p < \infty$.*

A proper subspace of $\mathcal{W}^{d,p}(\wedge^l M)$ ($\mathcal{W}^{d^*,p}(\wedge^l M)$, respectively), well adapted to our boundary value problems, is the class $\mathcal{W}_T^{d,p}(\wedge^l M)$ ($\mathcal{W}_N^{d^*,p}(\wedge^l M)$) of differential forms with vanishing tangential (normal) part on $\partial M$. The following corollary provides a natural and intrinsic characterization of $\mathcal{W}_T^{d,p}(\wedge^l M)$ and $\mathcal{W}_N^{d^*,p}(\wedge^l M)$ for all $1 \leq p < \infty$.

COROLLARY 3.8. – *A differential form belongs to $\mathcal{W}_T^{d,p}(\wedge^l M)$ ($\mathcal{W}_N^{d^*,p}(\wedge^l M)$, respectively) if and only if its zero extension to the reference manifold belongs to $\mathcal{W}^{d,p}(\wedge^l \mathcal{R})$, ($\mathcal{W}^{d^*,p}(\wedge^l \mathcal{R})$). The space $C_0^\infty(\wedge^l M)$ is dense in both $\mathcal{W}_T^{d,p}(\wedge^l M)$ and $\mathcal{W}_N^{d^*,p}(\wedge^l M)$.*

PROOF. – We only give proof for the case of $\mathcal{W}_T^{d,p}(\wedge^l M)$ and comment that the $\mathcal{W}_N^{d^*,p}(\wedge^l M)$ case is Hodge star dual. It follows from Definition 3.1 that the zero extension of $\omega$ belongs to $\mathcal{W}^{d,p}(\wedge^l \mathcal{R})$ with $d\omega = 0$ outside of $M$.

We shall have established the converse if we prove that $C_0^\infty(\wedge^l M)$ is dense in $\mathcal{W}_T^{d,p}(\wedge^l M)$. Suppose that $\omega \in \mathcal{W}^{d,p}(\wedge^l \mathcal{R})$ vanishes outside of $M$. To make its support slightly smaller, we recall the perturbation of identity $F_t\colon \mathcal{R} \to \mathcal{R}$, $0 < t < 1$ as given by (2.26) and (2.27). Since $\overline{M} \subset F_t(M)$, pulling back via the diffeomorphism $F_t$, we obtain a form $F_t^\#(\omega) \in \mathcal{W}^{d,p}(\wedge^l \mathcal{R})$ which is supported in a compact subset of $M$. We also have

$$\lim_{t\to 0} F_t^\#(\omega) = F_0^\#(\omega) = \omega \quad \text{in } \mathcal{W}^{d,p}(\wedge^l \mathcal{R})$$

In view of these observations, we only need to approximate each $F_t^\#(\omega)$ by forms of class $C_0^\infty(\wedge^l M)$. This is done the same way as in the proof of Corollary 3.6. The only point remaining to be mentioned is that by using mollifiers, we have actually obtained forms supported in a slightly larger set than that of $F_t^\#(\omega)$, but still in $M$. ■

DEFINITION 3.9. – *Let $d_T$ and $d_N^*$ denote the closures of the differential operators*

$$d\colon C_0^\infty(\wedge^l M) \subset \mathcal{L}^p(\wedge^l M) \to \mathcal{L}^p(\wedge^{l+1} M)$$

*and*

$$d^*\colon C_0^\infty(\wedge^l M) \subset \mathcal{L}^p(\wedge^l M) \to \mathcal{L}^p(\wedge^{l-1} M), \qquad 1 \leq p < \infty$$

*respectively.*

Corollary (3.8) simply means that the domains of these operators are $\mathcal{W}_T^{d,p}(\wedge^l M)$

and $\mathcal{W}_N^{d^*,p}(\wedge^l M)$, respectively. Formula (2.34) shows that $d_T$: $\mathcal{W}_T^{d,p}(\wedge^l M) \to \mathcal{L}^p(\wedge^{l+1} M)$ is the Banach space adjoint of $d^*$: $\mathcal{W}^{d^*,q}(\wedge^{l+1} M) \to \mathcal{L}^q(\wedge^l M)$. Similarly, $d_N^*$ is the Banach space adjoint of $d$: $\mathcal{W}^{d,q}(\wedge^{l-1} M) \to \mathcal{L}^q(\wedge^l M)$, $1 < p, q < \infty$, $p+q = pq$.

3.4. *Harmonic fields.* – A form $h \in \mathcal{L}^1_{\text{loc}}(\wedge^l M)$ which is both closed and co-closed (i.e. $dh = d^* h = 0$) will be called a harmonic field of degree $l$. We denote by $\mathcal{H}(\wedge^l M)$ the space of all harmonic fields on $M$ and regard it as well known that such forms are $C^\infty$-smooth. Notice that $\mathcal{H}(\wedge^l M)$ serves as a natural extension of holomorphic functions to Riemannian manifolds. Indeed, a 1-form $h = u(x, y)\, dx + v(x, y)\, dy$ on $\mathbb{R}^2$ is a harmonic field if and only if the complex function $f = u - iv$ is holomorphic. The three basic Banach spaces of harmonic fields of concern to us are

$$\mathcal{H}^p(\wedge^l M) = \mathcal{W}^{1,p}(\wedge^l M) \cap \mathcal{H}(\wedge^l M) \tag{3.18}$$

$$\mathcal{H}_T^p(\wedge^l M) = \{h \in \mathcal{H}^p(\wedge^l M) \colon h_T = 0\} \tag{3.19}$$

$$\mathcal{H}_N^p(\wedge^l M) = \{h \in \mathcal{H}^p(\wedge^l M) \colon h_N = 0\} \tag{3.20}$$

where $1 \leq p \leq \infty$. Clearly, the Hodge star operator preserves harmonic fields. Precisely, we have $* \mathcal{H}(\wedge^l M) = \mathcal{H}(\wedge^{n-l} M)$ and $* \mathcal{H}_T^p(\wedge^l M) = \mathcal{H}_N^p(\wedge^{n-l} M)$.

3.5. *Partly Sobolev classes of second order.* – A form $\gamma \in \mathcal{W}^{1,p}(\wedge^l M)$ is said to belong to $\mathcal{L}^{2,p}(\wedge^l M)$ if $d\gamma \in \mathcal{W}^{1,p}(\wedge^{l+1} M)$ and $d^* \gamma \in \mathcal{W}^{1,p}(\wedge^{l-1} M)$. The norm for this space is given by

$$\|\gamma\|_{2,p} = \|\gamma\|_{1,p} + \|d\gamma\|_{1,p} + \|d^* \gamma\|_{1,p} \tag{3.21}$$

REMARK 3.10. – *We should point out that $\mathcal{L}^{2,p}(\wedge^l M)$ is a proper complete subspace of $\mathcal{W}^{2,p}(\wedge^l M)$. For example, harmonic fields of class $\mathcal{W}^{1,p}(\wedge^l M)$ are members of $\mathcal{L}^{2,p}(\wedge^l M)$, but need not belong to $\mathcal{W}^{2,p}(\wedge^l M)$.*

The following closed subspaces of $\mathcal{L}^{2,p}(\wedge^l M)$ will be useful

$$\begin{cases} \mathcal{L}_T^{2,p}(\wedge^l M) = \{\gamma \in \mathcal{L}^{2,p}(\wedge^l M) \colon \gamma_T = (d^* \gamma)_T = 0\} \\ \mathcal{L}_N^{2,p}(\wedge^l M) = \{\gamma \in \mathcal{L}^{2,p}(\wedge^l M) \colon \gamma_N = (d\gamma)_N = 0\} \\ \mathcal{L}_0^{2,p}(\wedge^l M) = \{\gamma \in \mathcal{L}^{2,p}(\wedge^l M) \colon (d\gamma)_N = (d^* \gamma)_T = 0\} \end{cases} \tag{3.22}$$

Density of smooth forms in these spaces is a delicate problem. A characterization of higher order Sobolev spaces via the exterior and coexterior derivatives is pursued in [BS96].

## 4. – Gaffney type inequalities.

The inequalities we study in this section represent critical estimates for the operators $d$ and $d^*$.

4.1. *Gradient estimates in* $\mathbb{R}^n_+$. – We prepare for the more general results (i.e. on manifolds) by presenting a variant in Euclidean space. At this point, we deviate slightly from the rest of the text by using $\wedge^l$ to denote $\wedge^l \mathbb{R}^n$. This convention is only in force for this subsection and thereafter we return to $\wedge^l = \wedge^l \mathcal{R}$.

PROPOSITION 4.1. – *There exists a constant* $C = C(p, n)$, $1 < p < \infty$, *so that for any* $\alpha \in C_T^\infty(\overline{\mathbb{R}}^n_+, \wedge^l)$ *with compact support, we have*

$$\|\nabla \alpha\|_p \leq C(n, p)(\|d\alpha\|_p + \|d^* \alpha\|_p) \tag{4.1}$$

In the statement of this proposition, we are using

$$\nabla \alpha = \left( \frac{\partial \alpha}{\partial x_1}, \ldots, \frac{\partial \alpha}{\partial x_n} \right)$$

where $\dfrac{\partial \alpha}{\partial x_i} = \sum_I \dfrac{\partial \alpha^I}{\partial x_i} dx_I$. For the proof, when $\alpha = \sum \alpha^I dx_I$ and $\beta = \sum \beta^J dx_J$ are $l$-forms in a domain of $\mathbb{R}^n$ whose partials exist, we write

$$\langle \nabla \alpha, \nabla \beta \rangle = \sum_{i=1}^n \left\langle \frac{\partial \alpha}{\partial x_i}, \frac{\partial \beta}{\partial x_i} \right\rangle = \sum_{i, I} \frac{\partial \alpha^I}{\partial x_i} \frac{\partial \beta^I}{\partial x_i}$$

Our arguments are based on certain identities and $\mathcal{L}^p$-estimates for the Riesz transforms in $\mathbb{R}^n$. We have divided the proof into a sequence of lemmas.

LEMMA 4.2. – *We have the following identity*

$$\langle d\alpha, d\beta \rangle + \langle d^* \alpha, d^* \beta \rangle - \langle \nabla \alpha, \nabla \beta \rangle = \sum_{\#(I \cap J) = l-1} \pm (\alpha^I_{I-J} \beta^J_{J-I} - \alpha^I_{J-I} \beta^J_{I-J}) \tag{4.2}$$

Here and subsequently, $f_{I-J}$ stands for the partial derivative $\partial f / \partial x_i$, where $\{i\} = I - J$. The signs in the summation depend on $I$ and $J$ but we do not specify them because they will play no role in the sequel.

PROOF OF LEMMA 4.2. – Let us denote by $\mathcal{B}(\alpha, \beta)$ the bilinear form in the left hand side of 4.2. Because of bilinearity, it is sufficient to verify formula 4.2 for the forms

$$\alpha = \mathrm{A}\, dx_I, \qquad \beta = \mathrm{B}\, dx_J$$

where A and B are differentiable functions. We then have

$$* \alpha = \mathrm{A} * dx_I, \qquad * \beta = \mathrm{B} * dx_J$$
$$d\alpha = \mathrm{A}_i\, dx_i \wedge dx_I, \qquad d\beta = \mathrm{B}_j\, dx_j \wedge dx_J$$

Hereafter, $\mathrm{A}_i$ stands for the partial derivative $\partial \mathrm{A} / \partial x_i$ and we use the Einstein summation convention. Notice also that

$$d * \alpha = \mathrm{A}_i\, dx_i \wedge * dx_I, \qquad d * \beta = \mathrm{B}_j\, dx_j \wedge * dx_J$$

Hence

$$\begin{aligned}
\langle d\alpha, d\beta\rangle &= A_i B_j \langle dx_i \wedge dx_I, dx_j \wedge dx_J\rangle &:= X\\
\langle d^*\alpha, d^*\beta\rangle &= A_i B_j \langle dx_i \wedge * dx_I, dx_j \wedge * dx_J\rangle &:= X^*\\
\langle \nabla\alpha, \nabla\beta\rangle &= A_k B_k \langle dx_I, dx_J\rangle &:= Y
\end{aligned}$$

It is clear that if $\#(I \cap J) < l-1$ then each of the expressions $X$, $X^*$ and $Y$ vanish. Thus, (4.2) holds trivially. Now, two cases are possible. Either $\#(I \cap J) = l$ or $\#(I \cap J) = l-1$. In the first case, we see that $I = J$. Hence, $X = \sum_{k \notin I} A_k B_k$, $X^* = \sum_{k \in I} A_k B_k$ and $Y = \sum_{k=1}^{n} A_k B_k$. Thus $\mathcal{B}(\alpha, \beta) = 0$, as desired. In the second case, $I = \{p\} \cup K$ and $J = \{q\} \cup K$ for some multiindex $K = I \cap J$ with $\#K = l-1$, $p \neq q$ and $p$, $q \notin K$. This gives

$$\begin{aligned}
X &= A_q B_p \langle dx_q \wedge dx_I, dx_p \wedge dx_J\rangle\\
X^* &= A_p B_q \langle dx_p \wedge * dx_I, dx_q \wedge * dx_J\rangle
\end{aligned} \qquad \text{(no summation)}$$

and $Y = 0$. By elementary combinatorial arguments, we find that

$$\langle dx_p \wedge * dx_I, dx_q \wedge * dx_J\rangle = -\langle dx_q \wedge dx_I, dx_p \wedge dx_J\rangle := \varepsilon = \pm 1$$

In all, we obtain

$$\mathcal{B}(\alpha, \beta) = \varepsilon(A_p B_q - A_q B_p) = \varepsilon(A_{I-J} B_{J-I} - A_{J-I} B_{I-J})$$

as claimed. ■

REMARK 4.3. – *Our proof reveals that* $\langle dx_{J-I} \wedge dx_I, dx_{I-J} \wedge dx_J\rangle$ *is the sign in* (4.2).

Let us rewrite identity 4.2 as

$$\mathcal{B}(\alpha, \beta) = \sum_{\#(I \cap J) = l-1} \pm [(\alpha^I_{I-J} \beta^J)_{J-I} - (\alpha^I_{J-I} \beta^J)_{I-J}] \tag{4.3}$$

We denote by $\mathcal{L}^{1,p}(\mathbb{R}^n_+)$ ($\mathcal{L}^{1,p}(\mathbb{R}^n)$, respectively) the space of locally integrable functions on $\mathbb{R}^n_+$ ($\mathbb{R}^n$) whose distributional gradient belongs to $\mathcal{L}^p(\mathbb{R}^n_+, \mathbb{R}^n)$ ($\mathcal{L}^p(\mathbb{R}^n, \mathbb{R}^n)$, respectively), $1 \leq p \leq \infty$. Let $\mathcal{L}^{1,p}_0(\mathbb{R}^n_+)$ denote the space of functions from $\mathcal{L}^{1,p}(\mathbb{R}^n_+)$ whose zero extension to $\mathbb{R}^n$ belongs to $\mathcal{L}^{1,p}(\mathbb{R}^n)$.

LEMMA 4.4. – *If* $\alpha \in C^\infty_T(\overline{\mathbb{R}^n_+}, \wedge^l)$ *has compact support and* $\beta \in \mathcal{L}^{1,q}(\wedge^l \mathbb{R}^n_+)$ *with* $\beta_T = 0$ *and* $q \geq 1$, *then*

$$\int_{\mathbb{R}^n_+} \mathcal{B}(\alpha, \beta) = 0 \tag{4.4}$$

PROOF. – In view of (4.2), it suffices to show that

$$\int_{\mathbb{R}^n_+} (\alpha^I_{I-J}\beta^J)_{J-I} = \int_{\mathbb{R}^n_+} (\alpha^I_{J-I}\beta^J)_{I-J} = 0 \tag{4.5}$$

for all $l$-tuples $I$ and $J$ with $\#(I \cap J) = l-1$. First notice that both integrands in (4.5) belong to $\mathcal{W}^{1,p}(\mathbb{R}^n_+)$. Furthermore, for each $k = 1, 2, \ldots, n$ and any $\phi \in \mathcal{W}^{1,q}(\mathbb{R}^n_+)$, Fubini's theorem yields

$$\int_{\mathbb{R}^n_+} \phi_{x_k} = \int \ldots \int \left( \int_a^\infty \phi_{x_k}\, dx_k \right) dx_1 \ldots d\widehat{x}_k \ldots dx_n = 0$$

where $a = 0$ if $k = n$ and $a = -\infty$ otherwise. In particular, (4.5) holds if $J - I \neq \{n\} \neq I - J$. In the case $\{n\} = I - J$, $\beta^J$ is a coefficient of the tangential part of $\beta$. Thus, $\beta^J \in \mathcal{W}^{1,q}_0(\mathbb{R}^n_+)$ and (4.5) follows by applying (4.6). Moving to the case when $\{n\} = J - I$, we see that $\alpha^I$ is a coefficient of the tangential part of $\alpha$. Consequently, $\alpha^I$ vanishes on $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. Since $\{n\} \neq I - J$, the partial derivative $\alpha^I_{I-J}$ vanishes on $\mathbb{R}^{n-1}$ as well. Identity (4.6) once again implies (4.5). ■

LEMMA 4.5. – *For each $f \in \mathcal{L}^{1,p}_0(\mathbb{R}^n_+)$ $(\mathcal{L}^{1,p}(\mathbb{R}^n_+))$, there exists $g \in \mathcal{L}^{1,q}_0(\mathbb{R}^n_+)$ $(\mathcal{L}^{1,q}(\mathbb{R}^n_+))$, $1 < p,\ q < \infty$, $p + q = pq$, such that*

$$\|\nabla f\|_p \|\nabla g\|_q \leq C(n, q)(\nabla f, \nabla g) \tag{4.6}$$

*where the constant $C(n, p)$ depends only on $q$ and the dimension. The norms and the scalar product are in $\mathbb{R}^n_+$.*

PROOF. – Denote by $\overline{x}$ the reflection of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ through $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. That is, $\overline{x} = (x_1, \ldots, x_{n-1}, -x_n)$. Using this reflection map, we extend $f$ to $\mathbb{R}^n$ by requiring that

$$f(\overline{x}) = -f(x) \qquad \text{in case } f \in \mathcal{L}^{1,p}_0(\mathbb{R}^n_+) \tag{4.7}$$

$$f(\overline{x}) = f(x) \qquad \text{in case } f \in \mathcal{L}^{1,p}(\mathbb{R}^n_+) \tag{4.8}$$

The extended function (still denoted by $f$) belongs to $\mathcal{L}^{1,p}(\mathbb{R}^n)$. We now consider the case of $f \in \mathcal{L}^{1,p}_0(\mathbb{R}^n_+)$ and comment only that the case $f \in \mathcal{L}^{1,p}(\mathbb{R}^n_+)$ follows similarly.

First notice that the gradient of the extended function satisfies $\nabla f(\overline{x}) = -\overline{\nabla f(x)}$. Observe that the vector field $F = |\nabla f|^{p-2}\nabla f = (F_1, \ldots, F_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|F|^q = |\nabla f|^p$ and so $F \in \mathcal{L}^q(\mathbb{R}^n, \mathbb{R}^n)$. We also have

$$F(\overline{x}) = -\overline{F(x)} \tag{4.9}$$

Now we define $g\colon \mathbb{R}^n \to \mathbb{R}$ as the Riesz potential of $F$. Precisely,

$$g(x) = c_n \int_{\mathbb{R}^n} \frac{\langle x-y, F(y)\rangle}{|x-y|^n}\, dy \tag{4.10}$$

The gradient of $g$ can then be expressed using the Riesz transforms in $\mathbb{R}^n$ according to

$$G = \nabla g = -R\langle R, F\rangle = -\left\{ R_i\left(\sum_{j=1}^{n} R_j F_j\right)\right\}_{i=1,\dots,n}$$

This yields the estimate

$$\|\nabla g\|_{q,\mathbb{R}^n} = \|G\|_{q,\mathbb{R}^n} \leq C(n,q)\|F\|_{q,\mathbb{R}^n} = C(n,q)\|\nabla f\|_{p,\mathbb{R}^n}^{p-1}$$

Thus $g \in \mathcal{L}^{1,q}(\mathbb{R}^n)$.

Using familiar identities for the Riesz transforms, we obtain

$$(\nabla f, \nabla g) = -(\nabla f, R\langle R, F\rangle)_{\mathbb{R}^n} = -(R\langle R, \nabla f\rangle, F)_{\mathbb{R}^n} = (\nabla f, F)_{\mathbb{R}^n} = \|\nabla f\|_{p,\mathbb{R}^n}^{p-1}\|\nabla f\|_{p,\mathbb{R}^n}$$

Combining this with the previous estimate for $\nabla g$, gives

$$\|\nabla f\|_{p,\mathbb{R}^n}\|\nabla g\|_{q,\mathbb{R}^n} \leq C(n,q)(\nabla f, \nabla g)_{\mathbb{R}^n} \tag{4.11}$$

In order to reduce this estimate to $\mathbb{R}^n_+$, we observe that

$$g(\bar{x}) = -g(x), \tag{4.12}$$

which follows from (4.9) and an easy change of variables in the integral (4.10). We now see that $G(\bar{x}) = -\overline{G(x)}$ which in turn reveals $\|\nabla g\|_{q,\mathbb{R}^n} = 2^{1/q}\|\nabla g\|_q$, $\|\nabla f\|_{p,\mathbb{R}^n} = 2^{1/p}\|\nabla f\|_p$ and $(\nabla f, \nabla g)_{\mathbb{R}^n} = 2(\nabla f, \nabla g)$. Therefore, inequality (4.11) is equivalent to (4.6). Finally, relation (4.12) gives that $g$ restricted to $\mathbb{R}^n_+$ belongs to $\mathcal{W}^{1,q}_0(\mathbb{R}^n_+)$. ∎

REMARK 4.6. – *Essentially the same proof applies when $f$ and $g$ are functions on $\mathbb{R}^n_+$ with values in a finite dimensional inner product space. In particular, for differential forms $f, g\colon \mathbb{R}^n_+ \to \wedge^l \mathbb{R}^n$.*

PROOF OF PROPOSITION 4.1. – It will be convenient to regard $\alpha$ as a function on $\overline{\mathbb{R}^n_+}$ with values in the $\binom{n}{l}$-dimensional inner product space $\wedge^l = \wedge^l \mathbb{R}^n$. Recall the orthogonal decomposition (2.9), $\wedge^l = (\wedge^l_T \mathbb{R}^n) \oplus (\wedge^l_N \mathbb{R}^n)$, with respect to $\mathbb{R}^{n-1} = \partial\mathbb{R}^n_+ \subset \mathbb{R}^n$. Accordingly, $\alpha = u + v$ where $u\colon \overline{\mathbb{R}^n_+} \to \wedge^l_T \mathbb{R}^n$ and $v\colon \overline{\mathbb{R}^n_+} \to \wedge^l_N \mathbb{R}^n$ are orthogonal pointwise, $u \in C^\infty_T(\overline{\mathbb{R}^n_+}, \wedge^l)$ and $v \in C^\infty_N(\overline{\mathbb{R}^n_+}, \wedge^l)$. This, in view of $v_T = \alpha_T - u_T = 0$, yields $v \in \mathcal{L}^{1,p}_0(\mathbb{R}^n_+, \wedge^l_N \mathbb{R}^n)$ while $u \in \mathcal{L}^{1,p}(\mathbb{R}^n_+, \wedge^l_T \mathbb{R}^n)$.

By Lemma 4.5, there exist forms $\mathrm{u} \in \mathcal{L}^{1,q}(\mathbb{R}^n_+, \wedge^l_T \mathbb{R}^n)$ and $\mathrm{v} \in \mathcal{L}^{1,q}_0(\mathbb{R}^n_+, \wedge^l_N \mathbb{R}^n)$, $p + q = pq$, such that

$$\|\nabla u\|_p \leq C(n,q)(\nabla u, \nabla \mathrm{u}) \tag{4.13}$$

$$\|\nabla v\|_p \leq C(n,q)(\nabla v, \nabla \mathrm{v}) \tag{4.14}$$

Here we have assumed that $\|\nabla u\|_q = \|\nabla v\|_q = 1$. This introduces no loss in generalilty since we may normalize u and v. These are orthogonal components of an $l$-form $\beta = u + v \in \mathcal{W}_T^{1,q}(\wedge^l \mathbb{R}^n_+)$. Therefore

$$\|\nabla \alpha\|_p \leq \|\nabla u\|_p + \|\nabla v\|_p \leq C(n, q)[(\nabla u, \nabla u) + (\nabla v, \nabla v)] = C(n, q)(\nabla \alpha, \nabla \beta)$$

On the other hand, by Lemma 4.4 we have

$$(\nabla \alpha, \nabla \beta) = (d\alpha, d\beta) + (d^* \alpha, d^* \beta) \leq$$

$$\leq C(n)\|\nabla \beta\|_q (\|d\alpha\|_p + \|d^* \alpha\|_p) \leq 2C(n)(\|d\alpha\|_p + \|d^* \alpha\|_p) \qquad \blacksquare$$

4.2. *Gaffney's inequality for* $\mathcal{W}^{1,p}(\wedge^l M)$. – Before stating the global result, we use the groundwork just developed to give the following

LEMMA 4.7. – *For each point* $a \in \overline{M}$ *there exists a constant* $C = C(a)$ *such that*

$$\|\omega\|_{1,p,M} \leq C(\|\omega\|_{p,M} + \|d\omega\|_{p,M} + \|d^* \omega\|_{p,M}) \tag{4.15}$$

*for all* $\omega \in C_T^\infty(\overline{M})$ *vanishing outside a sufficiently small neighborhood of* $a$. *This neighborhood may depend on the point* $a \in \overline{M}$, *but not on* $\omega$.

PROOF. – For $a \in \partial M$, choose a coordinate system $\{U, \kappa = (x_1, \ldots, x_n)\}$, $\kappa(U \cap M) = \mathbb{R}^n_+$ so that the tangent vector $\partial/\partial x_n$ is orthogonal to $T_b(\partial M)$ at every point $b \in \partial M \cap U$. In addition, we may assume that $\kappa(a) = 0$ and that the components of the metric tensor (w.r.t. $\kappa$) satisfy $g_{ij}(0) = \delta_{ij}$. The latter condition is always guaranteed by requiring that the tangent vectors $\partial/\partial x_1, \ldots, \partial/\partial x_n$ are orthonormal at the point $a \in \partial M$.

Next, let $\alpha = \sum_I \alpha^I(x)\, dx_I$ denote the pullback of $\omega$ via the mapping $\kappa^{-1}: \overline{\mathbb{R}^n_+} \to U \cap \overline{M}$. Obviously, $\alpha \in C_T^\infty(\overline{\mathbb{R}^n_+}, \wedge^l)$ and has compact support. Of course, pulling back $d\omega$ gives $d\alpha$. However, since $d^* \omega$ involves the Hodge star, its pullback is

$$\sum A_{i,I}^J(x) \frac{\partial \alpha^I}{\partial x_i}\, dx_J + \sum B_I^J(x)\, \alpha^I\, dx_J \tag{4.16}$$

where $A_{i,I}^J$ and $B_I^J$ are $C^\infty$ functions on $\overline{\mathbb{R}^n_+}$ whose explicit nature depends on the representation of the metric tensor $g$ (i.e. on $\kappa$). Further, our choice of coordinate system $\kappa$ guarantees that at $0 \in \mathbb{R}^n$, the first term of (4.16) is $d^* \alpha$. Thus we see that inequality (4.15) can be rephrased, equivalently, as

$$\|\alpha\|_{1,p} \leq C_\kappa(\|\alpha\|_p + \|d\alpha\|_p + \|\mathcal{P}\alpha\|_p) \tag{4.17}$$

in $\mathbb{R}^n_+$, where $\mathcal{P} = \mathcal{P}(x, D)$ is a first order linear differential operator with $C^\infty$-coefficients for which

$$\mathcal{P}(0, D) = d^* \tag{4.18}$$

Now, to prove (4.15), we use Proposition 4.1 and the triangle inequality to obtain

$$\begin{aligned}\|\alpha\|_{1,p} &\leq C(n,p)(\|\alpha\|_p + \|d\alpha\|_p + \|d^*\alpha\|_p)\\ &\leq C(n,p)(\|\alpha\|_p + \|d\alpha\|_p + \|\mathcal{P}\alpha\|_p + \|(\mathcal{P}-d^*)\,\alpha\|_p)\end{aligned}$$

Next observe that for given $\varepsilon > 0$ and when $\alpha$ is supported in a sufficiently small neighborhood of $0 \in \mathbb{R}^n$, we have the pointwise estimate

$$|(\mathcal{P}-d^*)\,\alpha| \leq \varepsilon\left(|\alpha| + \sum_{i,I}\left|\frac{\partial\alpha^I}{\partial x_i}\right|\right)$$

These two estimates apparently imply (4.17).

If $a$ is an interior point of $M$, we may repeat these arguments with $\mathbb{R}^n$ in place of $\mathbb{R}^n_+$. ■

THEOREM 4.8. – *There exists a constant $C_p = C_p(M)$ such that*

$$\|\omega\|_{1,p} \leq C_p(\|\omega\|_p + \|d\omega\|_p + \|d^*\omega\|_p) \tag{4.19}$$

*for every $\omega \in \mathcal{W}_T^{1,p}(\wedge^l M) \cup \mathcal{W}_N^{1,p}(\wedge^l M)$, $1 < p < \infty$.*

PROOF. – We will be concerned only with $\omega \in \mathcal{W}_T^{1,p}(\wedge^l M)$ which, in view of Corollary 3.4, we may assume to be in $C_T^\infty(\overline{M}, \wedge^l)$. For each point $a \in \overline{M}$, choose a neighborhood according to Lemma 4.7.

Since $\overline{M}$ is compact, we can select finitely many of these neighborhoods to cover $\overline{M}$, say $\{\Omega_i : i = 1, \dots, m\}$. Let $\{\chi_i \in C_0^\infty(\Omega_i) : \sum \chi_i = 1\}$ be a partition of unity subordinate to this covering. To each $\Omega_i$ there corresponds a constant $C_i$ so that (4.15) holds with $\omega_i = \chi_i \omega$ in place of $\omega$. Now $\omega = \sum \chi_i \omega$ and since $d\omega_i = \chi_i d\omega + d\chi_i \wedge \omega$ and $d^*\omega_i = \chi_i d^*\omega - (-1)^{nl+n} * (d\chi_i \wedge *\omega)$, we may conclude with the desired estimate

$$\begin{aligned}\|\omega\|_{1,p} &\leq \sum_{i=1}^m \|\omega_i\|_{1,p}\\ &\leq \sum_{i=1}^m C_i(\|\omega_i\|_p + \|d\omega_i\|_p + \|d^*\omega_i\|_p) \leq C(\|\omega\|_p + \|d\omega\|_p + \|d^*\omega\|_p)\end{aligned}$$ ■

REMARK 4.9. – *Gaffney's inequality quickly implies that both $\mathcal{H}_T^p(\wedge^l M)$ and $\mathcal{H}_N^p(\wedge^l M)$ are finite dimensional.*

Indeed, inequality (4.19) applied to $h \in \mathcal{H}_T^p(\wedge^l M) \cup \mathcal{H}_N^p(\wedge^l M)$, $1 \leq p \leq \infty$ reads as $\|h\|_{1,p} \leq C_p \|h\|_p$. By compactness of the imbedding $\mathcal{W}^{1,p}(\wedge^l M) \to \mathcal{L}^p(\wedge^l M)$, it follows that the unit ball of the subspace $\mathcal{H}_T^p(\wedge^l M) \subset \mathcal{L}^p(\wedge^l M)$ is relatively compact. Thus, $\dim \mathcal{H}_T^p(\wedge^l M) < \infty$. Similarly, $\dim \mathcal{H}_N^p(\wedge^l M) < \infty$.

It is a consequence of the regularity theory of C.B. Morrey that

REMARK 4.10. – *Harmonic fields with vanishing tangential or normal component are actually $C^\infty$-smooth up to the boundary. That is*

$$\mathcal{H}_T^p(\wedge^l M) \subset C_T^\infty(\overline{M}, \wedge^l) \quad \textit{and} \quad \mathcal{H}_N^p(\wedge^l M) \subset C_N^\infty(\overline{M}, \wedge^l) \tag{4.20}$$

For this reason, we shall omit the superscript $p$ in the notation for these two spaces. $L^p$-bounds of a harmonic field in terms of its tangential and normal part can be found in [IMS95].

We will find estimates for a form $\omega$ exclusively in terms of the $p$-norms of $d\omega$ and $d^*\omega$ to be very useful. Our next theorem is dedicated to demonstrating one such result.

Before stating this result, notice that given $\omega \in \mathcal{L}^1(\wedge^l M)$, the orthogonal projections of $\omega$ to $\mathcal{H}_T(\wedge^l M)$ or $\mathcal{H}_N(\wedge^l M)$ are well defined. Recall that both are finite dimensional subspaces of $C^\infty(\overline{M}, \wedge^l) \subset \mathcal{L}^2(\wedge^l M)$. Thus, if $h_1, \ldots, h_m$ is an orthonormal basis for $\mathcal{H}_T(\wedge^l M)$, then the projection $H_T\colon \mathcal{L}^1(\wedge^l M) \to \mathcal{H}_T(\wedge^l M)$ is defined by

$$H_T(\omega) = \sum_{i=1}^{m} c_i h_i\,, \qquad c_i = (\omega, h_i)_M \tag{4.21}$$

Analogously defined is the projection $H_N\colon \mathcal{L}^1(\wedge^l M) \to \mathcal{H}_N(\wedge^l M)$. We denote by $\mathcal{H}_T^{\mathrm{per}}(\wedge^l M)$ and $\mathcal{H}_N^{\mathrm{per}}(\wedge^l M)$, the subspaces of $\mathcal{L}^1(\wedge^l M)$ consisting of forms which are orthogonal to $\mathcal{H}_T(\wedge^l M)$ and $\mathcal{H}_N(\wedge^l M)$ respectively.

THEOREM 4.11. – *For each $1 < p < \infty$ and $\omega \in \mathcal{W}_T^{1,p}(\wedge^l M) \cap \mathcal{H}_T^{\mathrm{per}}(\wedge^l M)$ or $\omega \in \mathcal{W}_N^{1,p}(\wedge^l M) \cap \mathcal{H}_N^{\mathrm{per}}(\wedge^l M)$, we have*

$$\|\omega\|_{1,p} \leqslant C_p(M)(\|d\omega\|_p + \|d^*\omega\|_p) \tag{4.22}$$

PROOF. – Suppose to the contrary that (4.22) fails. This means that for each positive integer $k$, one can find $\omega_k \in \mathcal{W}_T^{1,p}(\wedge^l M)$ which is orthogonal to $\mathcal{H}_T(\wedge^l M)$ satisfying the inequality

$$\|d\omega_k\|_p + \|d^*\omega_k\|_p \leqslant \frac{1}{k}\,\|\omega_k\|_{1,p}$$

Because of homogeneity, there is no loss of generality in assuming that $\|\omega_k\|_p = 1$. Combining these observations with Gaffney's Inequality (4.19) yields

$$\|\omega_k\|_{1,p} \leqslant \frac{k\|\omega_k\|_p}{k - C_p} \leqslant 2$$

and

$$\|d\omega_k\|_p + \|d^*\omega_k\|_p \leqslant \frac{C_p\|\omega_k\|_p}{k - C_p} \leqslant \frac{2}{k}$$

for $k > 2C_p$. The first estimate implies that there exists a subsequence of $\{\omega_k\}$, converging weakly to some $\omega \in \mathcal{W}_T^{1,p}(\wedge^l M)$. As such, $\omega$ is also orthogonal to $\mathcal{H}_T(\wedge^l M)$. By compactness of the imbedding $\mathcal{W}_T^{1,p}(\wedge^l M) \subset \mathcal{L}^p(\wedge^l M)$, we see that $\|\omega\|_p = 1$. On the other hand, the second estimate above shows that $\omega$ is actually a harmonic field of class $\mathcal{H}_T^p(\wedge^l M) \subset C_T^\infty(\overline{M}, \wedge^l) \subset \mathcal{L}^2(\overline{M}, \wedge^l)$ and as such, it is orthogonal to itself. Hence, $\omega = 0$. This disagreement invalidates our assumption that (4.22) fails. ∎

We shall see that Theorem 4.11 is part of a more general spectrum of results concerning Poincaré type inequalities. All of which will be formulated after the Hodge decompositions have been established.

## 5. – Hodge decompositions.

Decompositions of a differential form into exact, coexact and harmonic components play an essential role in the potential theory on Riemannian manifolds. In fact, our paper systematically exploits these decompositions. For smooth forms, Hodge decompositions were motivated by the theory of deRham cohomology. However, we will not develop this point here. As a consequence of our study of nonlinear PDEs, it became clear that Hodge decompositions for $\mathcal{L}^p$-forms are a major prerequisite. It is apparent, however, that the extremely rich literature on this subject, does not cover all required details.

In this section, we formulate basic decompositions in Sobolev classes and outline the proofs which are necessary for completeness of the arguments in the sequel.

5.1. *A brief historical account.* – It is the essence of linear Hodge theory that each smooth form $\omega \in C^\infty(\wedge^l \mathcal{R})$ on a closed manifold $\mathcal{R}$ splits according to

$$\omega = d\alpha + d^*\beta + h \tag{5.1}$$

where $\alpha \in C^\infty(\wedge^{l-1}\mathcal{R})$, $\beta \in C^\infty(\wedge^{l+1}\mathcal{R})$ and $h \in \mathcal{H}(\wedge^l \mathcal{R})$. The forms $d\alpha$, $d^*\beta$ and $h$ are unique and mutually orthogonal with respect to the inner product $(\,,)$ on $\mathcal{L}^2(\wedge^l \mathcal{R})$. This yields the following orthogonal direct sum decompostion

$$C^\infty(\wedge^l \mathcal{R}) = dC^\infty(\wedge^{l-1}\mathcal{R}) \oplus d^* C^\infty(\wedge^{l+1}\mathcal{R}) \oplus \mathcal{H}(\wedge^l \mathcal{R}) \tag{5.2}$$

where $dC^\infty(\wedge^{l-1}\mathcal{R})$ and $d^*C^\infty(\wedge^{l+1}\mathcal{R})$ are the spaces of exact and coexact $l$-forms on $\mathcal{R}$, respectively.

Historically, this decomposition has been used to conclude that each deRham cohomology class of $\mathcal{R}$ is uniquely represented by a harmonic field. Indeed, $d^*\beta = 0$ when $\omega$ is closed and consequently, decomposition (5.1) reduces to

$$\omega = d\alpha + h$$

Concerning a regular open region $M \subset \mathcal{R}$, we have three types of Hodge decomposition

$$\begin{cases} C^\infty(\overline{M}, \wedge^l) = dC_T^\infty(\overline{M}, \wedge^{l-1}) \oplus d^* C_N^\infty(\overline{M}, \wedge^{l+1}) \oplus \mathcal{H}(\overline{M}, \wedge^l) \\ C^\infty(\overline{M}, \wedge^l) = dC_T^\infty(\overline{M}, \wedge^{l-1}) \oplus d^* C^\infty(\overline{M}, \wedge^{l+1}) \oplus \mathcal{H}_T(\overline{M}, \wedge^l) \\ C^\infty(\overline{M}, \wedge^l) = dC^\infty(\overline{M}, \wedge^{l-1}) \oplus d^* C_N^\infty(\overline{M}, \wedge^{l+1}) \oplus \mathcal{H}_N(\overline{M}, \wedge^l) \end{cases} \tag{5.3}$$

We trust that all of the notation used above is self explanatory. The summands occuring in the right hand side of these formulas are mutually orthogonal.

Applications of these decompositions pertain largely to the relative deRham cohomology of a manifold with boundary. This idea goes back at least as far as the work of A. W. Tucker [Tuc41] and G. F. D. Duff and D. C. Spencer [DS52].

Let us rephrase the decompositions (5.13) as follows. Given $\phi \in C^\infty(\overline{M}, \wedge^l)$, there exists $\alpha \in C^\infty(\overline{M}, \wedge^{l-1})$, $\beta \in C^\infty(\overline{M}, \wedge^{l+1})$ and $h \in \mathcal{H}(\overline{M}, \wedge^l)$ such that

$$\phi = d\alpha + d^*\beta + h \tag{5.4}$$

where the boundary conditions for $\alpha$, $\beta$ and $h$ are described by (5.3). Because of orthogonality, the terms $d\alpha$, $d^*\beta$ and $h$ are uniquely determined. However, the forms $\alpha$ and $\beta$ are not unique. Using the decompositions (5.3), but with $\alpha$ and $\beta$ in place of $\omega$, it follows that we can also require one of the following sets of conditions be satisfied

$$\alpha_T = 0, \quad \beta_N = 0, \quad \alpha \in d^* C^\infty(\overline{M}, \wedge^l) \text{ and } \beta \in dC^\infty(\overline{M}, \wedge^l) \tag{5.5}$$

$$\alpha_T = 0, \quad h_T = 0, \quad \alpha \in d^* C^\infty(\overline{M}, \wedge^l) \text{ and } \beta \in dC_T^\infty(\overline{M}, \wedge^l) \tag{5.6}$$

$$\beta_N = 0, \quad h_N = 0, \quad \alpha \in d^* C_N^\infty(\overline{M}, \wedge^l) \text{ and } \beta \in dC^\infty(\overline{M}, \wedge^l) \tag{5.7}$$

Under any one of these sets of conditions, the forms $\alpha$ and $\beta$ are uniquely determined. We may now express each $\omega \in C^\infty(\wedge^l \overline{M})$ uniquely as

$$\omega = \phi + \psi \tag{5.8}$$

where $\phi$ and $\psi$ are subject to one of the following constraints

$$\begin{array}{ll} \phi \in dC_T^\infty(\wedge^{l-1}\overline{M}) \text{ and } d^*\psi = 0 & \\ \phi \in dC^\infty(\wedge^{l-1} M) \text{ and } d^*\psi = 0\,, & \psi_N = 0 \\ d\phi = 0 \text{ and } \psi \in d^* C_N^\infty(\wedge^{l+1}\overline{M}) & \\ d\phi = 0 \text{ and } \psi \in d^* C^\infty(\wedge^{l+1}\overline{M}), & \phi_T = 0 \end{array}$$

Of course, the latter two constraints are Hodge star dual to the first two. We shall use these examples as a guide to formulate boundary value problems for nonlinear equations.

REMARK 5.1. – *Forms $\alpha$ and $\beta$ as in* (5.5), (5.6) *and* (5.7) *are orthogonal to their cor-*

*responding spaces of harmonic fields. In particular, using Theorem* 4.11, *we obtain*

$$\|\alpha\|_{1,p} \leq C_p(M)\|d\alpha\|_p \quad \text{and} \quad \|\beta\|_{1,p} \leq C_p(M)\|d^*\beta\|_p \tag{5.9}$$

Abstract boundary value problems for differential forms are discussed by P. E. Conner [Con56]. In Subsect. 5.3, we will touch on a few aspects of Conner's approach to these problems. For more theory, we refer the reader to the work of M. Gaffney (e.g. [Gaf54]).

The crucial step in generalizing decompositions (5.3) to Sobolev classes is due to J. Eells and C. B. Morrey [Mor66] who proved the differentiability and $\mathcal{L}^p$-integrability of $L^2$-decompositions. Our formualtion of the Hodge decompositions rely on these results of Morrey.

5.2. *The spaces of exact and coexact forms.* – In connection with the formulas of (5.3), we introduce the following subspaces of $\mathcal{L}^p(\wedge^l M)$, $1 \leq p \leq \infty$

$$d\mathcal{W}^{1,p}(\wedge^{l-1}M) = \{d\alpha \colon \alpha \in \mathcal{W}^{1,p}(\wedge^{l-1}M)\}$$

$$d\mathcal{W}^{1,p}_T(\wedge^{l-1}M) = \{d\alpha \colon \alpha \in \mathcal{W}^{1,p}_T(\wedge^{l-1}M)\}$$

Analogously defined are the spaces of coexact forms;

$$d^*\mathcal{W}^{1,p}(\wedge^{l+1}M) = \{d^*\alpha \colon \alpha \in \mathcal{W}^{1,p}(\wedge^{l+1}M)\}$$

$$d^*\mathcal{W}^{1,p}_N(\wedge^{l+1}M) = \{d^*\alpha \colon \alpha \in \mathcal{W}^{1,p}_N(\wedge^{l+1}M)\}$$

It is an immediate consequence of Corollaries 3.7 and 3.8 that the above spaces are completions of the corresponding classes of smooth exact and coexact $l$-forms occuring in (5.3). However, it is far from being evident that

COROLLARY 5.2. – *For* $1 < p < \infty$, *all four of these classes are complete subspaces of* $\mathcal{L}^p(\wedge^l M)$.

PROOF. – Suppose $d\omega_j \to \phi$ in $\mathcal{L}^p(\wedge^l M)$ for some $\omega_j \in \mathcal{W}^{1,p}(\wedge^{l-1}M)$ ($\mathcal{W}^{1,p}_T(\wedge^{l-1}M)$, respectively), $j = 1, 2, \ldots$. Our goal is to show that $\phi = d\omega$ for some $\omega \in \mathcal{W}^{1,p}(\wedge^{l-1}M)$ ($\mathcal{W}^{1,p}_T(\wedge^{l-1}M)$). Using Corollary 3.2, we may certainly assume that $\omega_j \in C^\infty(\overline{M}, \wedge^{l-1})$ ($C^\infty_T(\overline{M}, \wedge^{l-1})$, respectively). We need to modify the forms $\omega_j$ so that the sequence $\{\omega_j\}$ will stay bounded in the Sobolev norm. For this, we decompose each $\omega_j$ according to (5.4)

$$\omega_j = d\alpha_j + d^*\beta_j + h_j$$

We then replace $\omega_j$ by coexact forms $\omega_j' = d^*\beta_j = \omega_j - d\alpha_j - h_j$. Clearly, $d^*\omega_j' = 0$ and $d\omega_j' = d\omega_j \to \phi$ in $\mathcal{L}^p(\wedge^l M)$. In case of no boundary constraints for $\omega_j$, we only require that $\beta_j \in C^\infty_N(\overline{M}, \wedge^l)$. This guarantees that $\omega_j' \in C^\infty_N(\overline{M}, \wedge^{l-1}) \cap \mathcal{H}^{\text{per}}_N(\wedge^{l-1}M)$ and so by Theorem 4.11 we obtain the estimate

$$\|\omega_j'\|_{1,p} \leq C_p(M)\|d\omega_j\|_p \leq C$$

In case $\omega_j \in C_T^\infty(\overline{M}, \wedge^{l-1})$, we take $\alpha_j \in C_T^\infty(\overline{M}, \wedge^{l-2})$ and $h_j \in \mathcal{H}_T(\wedge^{l-1} M)$ which implies that $\omega_j' \in C_T^\infty(\overline{M}, \wedge^{l-1}) \cap \mathcal{H}_T^{\mathrm{per}}(\wedge^{l-1} M)$ so again the estimate above holds.

We are now in position to define $\omega \in \mathcal{W}^{1,p}(\wedge^{l-1} M)$ $(\mathcal{W}_T^{1,p}(\wedge^{l-1} M))$ as the weak limit of a subsequence of $\{\omega_j'\}$ in $\mathcal{W}^{1,p}(\wedge^{l-1} M)$. Thus $d\omega = \phi$, as desired. The coexact cases are handled analogously. ■

As a consequence of the Hodge decompositions, we obtain

COROLLARY 5.3. – *For $1 < p < \infty$, we have*

$$d\mathcal{W}^{1,p}(\wedge^{l-1} M) = d\mathcal{W}^{d,p}(\wedge^{l-1} M)$$

$$d\mathcal{W}_T^{1,p}(\wedge^{l-1} M) = d\mathcal{W}_T^{d,p}(\wedge^{l-1} M)$$

$$d^*\mathcal{W}^{1,p}(\wedge^{l+1} M) = d^*\mathcal{W}_N^{d^*,p}(\wedge^{l+1} M)$$

$$d^*\mathcal{W}_N^{1,p}(\wedge^{l+1} M) = d^*\mathcal{W}_N^{d^*,p}(\wedge^{l-1} M)$$

The proof is similar to that of Corollary 5.2.

To shorten notation, we write $\omega \in \operatorname{im} d_T$ or $\omega \in \operatorname{im} d_N^*$ if, for some $1 \leq p < \infty$, $\omega$ belongs to $d\mathcal{W}_T^{1,p}$ or $d^*\mathcal{W}_N^{1,p}$, respectively. In view of Remark 3.5, we see that

$$d\mathcal{W}_T^{1,p} \subset \{\omega \in d\mathcal{W}^{1,p} \colon \omega_T = 0\} \equiv (d\mathcal{W}^{1,p})_T$$

Equality in this inclusion does not occur in general and the quotient space $(d\mathcal{W}^{1,p})_T / d\mathcal{W}_T^{1,p}$ becomes an interesting cohomological object in its own right (e.g. Consider an annulus in the plane. Then a smooth radial function can represent a nonzero element of this space). Of course, the Hodge star dual spaces $(d^*\mathcal{W}^{1,p})_N / d^*\mathcal{W}_N^{1,p}$ are not necessarily trivial either.

The Hodge decompositions (5.3) may now be formulated precisely for the $\mathcal{L}^p$-space.

THEOREM 5.7. – *For $1 < p < \infty$, we have the following direct sum decompositions*

$$\mathcal{L}^p(\wedge^l M) = d\mathcal{W}_T^{1,p}(\wedge^{l-1} M) \oplus d^*\mathcal{W}_N^{1,p}(\wedge^{l+1} M) \oplus \mathcal{H}^p(\wedge^l M) \tag{5.10}$$

$$\mathcal{L}^p(\wedge^l M) = d\mathcal{W}_T^{1,p}(\wedge^{l-1} M) \oplus d^*\mathcal{W}^{1,p}(\wedge^{l+1} M) \oplus \mathcal{H}_T(\wedge^l M) \tag{5.11}$$

$$\mathcal{L}^p(\wedge^l M) = d\mathcal{W}^{1,p}(\wedge^{l-1} M) \oplus d^*\mathcal{W}_N^{1,p}(\wedge^{l+1} M) \oplus \mathcal{H}_N(\wedge^l M) \tag{5.12}$$

*Moreover, if a differential form $\omega \in \mathcal{L}^p(\wedge^l M)$ is written $\omega = d\alpha + d^*\beta + h$ in accordance with one of the above three decompositions, then*

$$\|d\alpha\|_p + \|d^*\beta\|_p + \|h\|_p \leq C_p(M)\|\omega\|_p \tag{5.13}$$

We restrain ourselves to only a few comments about the proof of this theorem. The $L^2$-decompositions are rather special and follow by variational principles. The case $p \geq 2$ was thoroughly examined by C. B. Morrey. We then notice that the uniform estimate (5.13) follows from the closed graph theorem. Further, Morrey's results can be extended to $1 < p < 2$ by duality arguments as demonstrated in [Sco95].

Associated with the Hodge decompositions are the following orthogonal projections onto respective subspaces:

*Harmonic Projections*

$$H, \quad H_T, \quad H_N: \mathcal{L}^2(\wedge^l M) \to \mathcal{H}^2(\wedge^l M), \quad H_T^2(\wedge^l M), \quad \mathcal{H}_N^2(\wedge^l M)$$

*Exact Projections*

$$E, \quad E_T: \mathcal{L}^2(\wedge^l M) \to d\mathcal{W}^{1,2}(\wedge^{l-1} M), \quad d\mathcal{W}_T^{1,2}(\wedge^{l-1} M)$$

*Coexact Projections*

$$E^*, \quad E_N^*: \mathcal{L}^2(\wedge^l M) \to d^*\mathcal{W}^{1,2}(\wedge^{l+1} M), \quad d^*\mathcal{W}_N^{1,2}(\wedge^{l+1} M)$$

PROPOSITION 5.5. – *The harmonic, exact and coexact projections extend to bounded linear operators of $\mathcal{L}^p(\wedge^l M)$, $1 < p < \infty$ onto the corresponding spaces with Sobolev exponent $p$ in place of 2.*

Although it is certainly possible to present verification of this result via the Calderón-Zygmund theory, we simply observe that this fact is equivalent with Theorem 5.4, with $C_p(M)$ serving as an upper bound for the norms of these operators. Let us denote by $I: \mathcal{L}^p(\wedge^l M) \to \mathcal{L}^p(\wedge^l M)$ the identity operator, $1 < p < \infty$. Theorem 5.4 can be rephrased as

$$\begin{cases} I = E_T + E_N^* + H \\ I = E_T + E^* + H_T \\ I = E + E_N^* + H_N \end{cases} \tag{5.14}$$

Repeated application of Theorem 5.4 enables us to write

COROLLARY 5.6. – *Each $\omega \in \mathcal{L}^p(\wedge^l M)$, $1 < p < \infty$, decomposes according to*

$$\omega = d\alpha + d^*\beta + h$$

*with one of the following sets of boundary conditions*

$$\begin{cases} \alpha \in \mathcal{W}_T^{1,p}(\wedge^{l-1} M) \cap d^*\mathcal{W}^{1,p}(\wedge^l M) \\ \beta \in \mathcal{W}_N^{1,p}(\wedge^{l+1} M) \cap d\mathcal{W}^{1,p}(\wedge^l M) \\ h \in \mathcal{H}^p(\wedge^l M) \end{cases} \tag{5.15}$$

$$\begin{cases} \alpha \in \mathcal{W}_T^{1,p}(\wedge^{l-1} M) \cap d^* \mathcal{W}^{1,p}(\wedge^l M) \\ \beta \in \mathcal{W}^{1,p}(\wedge^{l+1} M) \cap d\, \mathcal{W}_T^{1,p}(\wedge^l M) \\ h \in \mathcal{H}_T^p(\wedge^l M) \end{cases} \tag{5.16}$$

$$\begin{cases} \alpha \in \mathcal{W}^{1,p}(\wedge^{l-1} M) \cap d_N^* \mathcal{W}^{1,p}(\wedge^l M) \\ \beta \in \mathcal{W}_N^{1,p}(\wedge^{l+1} M) \cap d\, \mathcal{W}^{1,p}(\wedge^l M) \\ h \in \mathcal{H}_N^p(\wedge^l M) \end{cases} \tag{5.17}$$

*Under these conditions, the forms $\alpha$ and $\beta$ are uniquely determined and satisfy*

$$\|\alpha\|_{1,p} + \|\beta\|_{1,p} \le C_p(M)\|\omega\|_p \tag{5.18}$$

PROOF. – We need only establish inequality (5.18). For this, we observe that both $\alpha$ and $\beta$ are orthogonal to corresponding spaces of harmonic fields. In particular, Theorem 4.11 applies to $\alpha$ and $\beta$ giving $\|\alpha\|_{1,p} \le C\|d\alpha\|_p$ and $\|\beta\|_{1,p} \le C\|d^*\beta\|_p$. This, combined with (5.13) gives (5.18). ■

5.3. *Green's operators.* – Hodge decompositions are closely linked with the study of the Laplace-Beltrami operator (Laplacian)

$$\Delta = dd^* + d^*d \colon C^\infty(\overline{M}, \wedge^l) \to C^\infty(\overline{M}, \wedge^l) \tag{5.19}$$

It is not difficult to see that the second order Sobolev spaces, as introduced in Subsect. 3.5, are the domains of the following closures of the Laplacian

$$\begin{cases} \Delta_0 = d_N^* d + d_T d^* \colon \mathcal{L}_0^{2,p}(\wedge^l M) \to \mathcal{L}^p(\wedge^l M) \\ \Delta_T = d^* d_T + d_T d^* \colon \mathcal{L}_T^{2,p}(\wedge^l M) \to \mathcal{L}^p(\wedge^l M) \\ \Delta_N = d_N^* d + dd_N^* \colon \mathcal{L}_0^{2,p}(\wedge^l M) \to \mathcal{L}^p(\wedge^l M) \end{cases} \tag{5.20}$$

These operators, in view of Corollary 5.6, give rise to three types of Green's operators defined on $\mathcal{L}^p(\wedge^l M)$;

$$\begin{cases} G \colon \mathcal{L}^p(\wedge^l M) \to \mathcal{L}_0^{2,p}(\wedge^l M) \\ I = \Delta_0 G + H \end{cases} \tag{5.21}$$

$$\begin{cases} G_T \colon \mathcal{L}^p(\wedge^l M) \to \mathcal{L}_T^{2,p}(\wedge^l M) \\ I = \Delta_T G_T + H_T \end{cases} \tag{5.22}$$

$$\begin{cases} G_N \colon \mathcal{L}^p(\wedge^l M) \to \mathcal{L}_N^{2,p}(\wedge^l M) \\ I = \Delta_N G_N + H_N \end{cases} \tag{5.23}$$

These are the strongest possible forms of the Hodge decompositions. The projections are simply expressed by means of Green's operators as

$$\begin{cases} E_T = d_T d^* G\,, & E_N^* = d_N^* dG \\ E = dd_N^* G_N & E^* = d^* d_T G_T \end{cases} \tag{5.24}$$

However, for the purposes of this paper, Green's operators will play no essential role. Consequently, we do not take the space here to prove them. The boundedness of the exact and coexact projections into $\mathcal{L}^p(\wedge^l M)$ will suffice. Finally, we would like to comment that in the case $p = 2$, the Laplace operators $\Delta_0$, $\Delta_T$ and $\Delta_N$ are positive and self-adjoint. This is of fundamental importance in developing an abstract $\mathcal{L}^2$-potential theory on manifolds with boundary; see [Con56].

5.4. *Duals and orthogonal complements.* – In the sections to follow, we will address the classical questions of existence, uniqueness and regularity of solutions to a wide class of nonlinear PDEs on manifolds. To deal with the nonlinear character of these equations, we will apply Browder's theory of monotone operators to the existence question. This being the case, we need to select a critical list of spaces of $l$-forms defined in the previous sections and carefully identify their corresponding duals. Further, in deriving the Euler-Lagrange equations for the $\mathcal{L}^p$-projections, we will need the orthogonal complements of these spaces as well. In this regard, we give the following Theorem.

THEOREM 5.7. – *Drawing on the notation developed in the previous sections, we have*

| *Space* | *Dual space* | *Identification* | *Orthogonal complement* |
|---|---|---|---|
| $d\mathcal{W}^{1,p}$ | $d\mathcal{W}^{1,q}$ | $dd_N^* G_N$ | $\mathcal{L}^q \cap \ker d_N^*$ |
| $d^*\mathcal{W}^{1,p}$ | $d^*\mathcal{W}^{1,q}$ | $d^* d_T G_T$ | $\mathcal{L}^q \cap \ker d_T$ |
| $d\mathcal{W}_T^{1,p}$ | $d\mathcal{W}_T^{1,q}$ | $d_T d^* G$ | $\mathcal{L}^q \cap \ker d^*$ |
| $d^*\mathcal{W}_N^{1,p}$ | $d^*\mathcal{W}_N^{1,q}$ | $d_N^* dG$ | $\mathcal{L}^q \cap \ker d$ |
| $\mathcal{L}^p \cap \ker d$ | $\mathcal{L}^q \cap \ker d$ | $dd^* G_N + H_N$ | $d^*\mathcal{W}_N^{1,q}$ |
| $\mathcal{L}^p \cap \ker d^*$ | $\mathcal{L}^q \cap \ker d^*$ | $d^* dG_T + H_T$ | $d\mathcal{W}_T^{1,q}$ |
| $\mathcal{L}^p \cap \ker d_T$ | $\mathcal{L}^q \cap \ker d_T$ | $dd^* G + H$ | $d^*\mathcal{W}^{1,q}$ |
| $\mathcal{L}^p \cap \ker d_N^*$ | $\mathcal{L}^q \cap \ker d_N^*$ | $d^* dG + H$ | $d\mathcal{W}^{1,q}$ |

Before proceeding to the proof of this theorem, perhaps a few comments about how to read the table are in order. The first two entries, $d\mathcal{W}^{1,p}$ and $d\mathcal{W}^{1,q}$, indicate that under the familiar Riesz integral representation of $\mathcal{L}^p$-duals, $d\mathcal{W}^{1,q}$ is dual to $d\mathcal{W}^{1,p}$. More accurately, given $\omega \in d\mathcal{W}^{1,q}$, the representation $(\omega, \alpha) = \int_M \langle \omega, \alpha \rangle$ for all $\alpha \in d\mathcal{W}^{1,p}$, is a Banach space isomorphism between $d\mathcal{W}^{1,q}$ as a subspace of $\mathcal{L}^q$ and the dual to $d\mathcal{W}^{1,p}$ when $d\mathcal{W}^{1,p}$ is considered as a subspace of $\mathcal{L}^p$. The operator $T = dd_N^* G_N$ is defined on all $\mathcal{L}^q$ with image equal to $d\mathcal{W}^{1,q}$. Thus, if $R\colon \mathcal{L}^p \to \mathcal{L}^q$ denotes the usual Riesz identification, then $TR$ is a natural way to uniquely identify an element of $d\mathcal{W}^{1,p}$ with an element in its dual. The space $\mathcal{L}^q \cap \ker d_N^*$ is the subspace consisting of all

$\alpha \in \mathcal{L}^q$ which are orthogonal to $d\mathcal{W}^{1,p}$ in the sense that $0 = (\omega, \alpha) = \int_M \langle \omega, \alpha \rangle$ for every $\omega \in d\mathcal{W}^{1,p}$. Of course, this discussion applies analogously to the other rows of the table above.

PROOF OF THEOREM 5.7. – Since verification of each row follows similar reasoning via different $\mathcal{L}^p$-Hodge decompositions, we deal only with the first row. Let

$$\Phi : d\mathcal{W}^{1,p}(\wedge^{l-1} M) \to \mathbb{R}$$

be an element from the dual of $d\mathcal{W}^{1,p}(\wedge^{l-1} M)$. Since $\Phi$ is bounded linear on $d\mathcal{W}^{1,p}$ as a subspace of $\mathcal{L}^p$, we may apply the Hahn-Banach Theorem to extend $\Phi$ continuously to all of $\mathcal{L}^p(\wedge^l M)$. In this setting, there is $\omega \in \mathcal{L}^q(\wedge^l M)$ representing $\Phi$ according to

$$\Phi(\alpha) = (\omega, \alpha)$$

for all $\alpha \in \mathcal{L}^p(\wedge^l M)$. Decomposing $\omega$ according to (5.23) gives

$$\omega = \Delta_N G_N(\omega) + H_N(\omega) = dd_N^* G_N(\omega) + d_N^* dG_N(\omega) + H_N(\omega)$$

Integration by parts then reveals

$$\Phi(\alpha) = (dd_N^* G_N(\omega), \alpha)$$

for all $\alpha \in d\mathcal{W}^{1,p}(\wedge^{l-1} M)$. Of course, $dd_N^* G_N(\omega) \in d\mathcal{W}^{1,q}$ as desired.

To see that such a representation is unique, suppose that $\eta \in d\mathcal{W}^{1,q}$ satisfies $(\eta, \alpha) = 0$ for all $\alpha \in d\mathcal{W}^{1,p}(\wedge^{l-1} M)$. We will argue that $\eta = 0$. Let $\tau$ be an arbitrary $l$-form of class $\mathcal{L}^q$. Decompose $\tau$ according to (5.17) and apply integration by parts to get $(\eta, \tau) = 0$. This is enough to conclude that $\eta = 0$.

Notice that in the course of proving that $d\mathcal{W}^{1,q}$ is dual to $d\mathcal{W}^{1,p}$, we have observed that $d\mathcal{W}^{1,q} = dd_N^* G_N \mathcal{L}^q$.

Next, let's uncover the orthogonal complement of $d\mathcal{W}^{1,p}$. For this, suppose $\omega \in d\mathcal{W}^{1,p}$ and $\phi \in \mathcal{L}^q \cap \ker d_N^*$. Thus $\omega = d\beta$ and according to Definition 3.9, there is a sequence $\{\phi_k\} \subset C_0^\infty$ with $\phi_k \to \phi$ and $d^* \phi_k \to d^* \phi = 0$ in $\mathcal{L}^q$. We can now estimate $(\omega, \phi)$ with

$$\begin{aligned} |(\omega, \phi)| &\leq |(\omega, \phi - \phi_k)| + |(\omega, \phi_k)| = |(\omega, \phi - \phi_k)| + |(\beta, d^* \phi_k)| \\ &\leq \|\omega\|_p \|\phi - \phi_k\|_q + \|\beta\|_p \|d^* \phi_k\|_q \to 0 \text{ as } k \to \infty \end{aligned}$$

The equality here follows from the compact support of the smooth forms $\phi_k$ and integration by parts. Thus $\omega$ belongs to the orthogonal complement of $\mathcal{L}^q \cap \ker d_N^*$. For the reverse inclusion, let $\phi \in \mathcal{L}^q$ be orthogonal to all of $d\mathcal{W}^{1,p}$. Decomposition (5.12) says that $\phi$ may be expressed as

$$\phi = d^* \beta + h, \qquad \beta_N = h_N = 0$$

Since $\phi_N = d^* \beta_N + h_N = 0$, we have $\phi \in \mathcal{W}_N^{d^*, p}$ with $d^* \phi = 0$. According to the comments following Definition 3.9, we may conclude that $\phi \in \ker d_N^*$. ■

## 6. – Poincaré inequalities for partly Sobolev classes.

Having disposed of Gaffney's Inequality and the Hodge decompositions, we can now establish new estimates and relations between the Sobolev spaces.

6.1. *Intersections of partly Sobolev classes.* – For $1 < p < \infty$, we introduce the space $\mathcal{L}^{1,p}(\wedge^l M) = \mathcal{W}^{d,p}(\wedge^l M) \cap \mathcal{W}^{d^*,p}(\wedge^l M)$ and equip it with the norm

$$\|\omega\|_{\mathcal{L}^{1,p}(M)} = \|\omega\|_{p,M} + \|d\omega\|_{p,M} + \|d^*\omega\|_{p,M} \tag{6.1}$$

Before continuing, we give a characterization of the subspaces of $\mathcal{L}^{1,p}(\wedge^l M)$ with vanishing tangential and normal parts.

THEOREM 6.1. – *For each* $1 < p < \infty$, *we have*

$$\mathcal{L}^{1,p}(\wedge^l \mathcal{R}) = W^{1,p}(\wedge^l \mathcal{R}) \tag{6.2}$$

$$\mathcal{L}^{1,p}(\wedge^l M) \subset \mathcal{W}^{1,p}_{\mathrm{loc}}(\wedge^l M) \tag{6.3}$$

$$\mathcal{W}^{1,p}_0(\wedge^l M) = \mathcal{W}^{d,p}_T(\wedge^l M) \cap \mathcal{W}^{d^*,p}_N(\wedge^l M) \tag{6.4}$$

$$\mathcal{W}^{1,p}_T(\wedge^l M) = \mathcal{W}^{d,p}_T(\wedge^l M) \cap \mathcal{W}^{d^*,p}(\wedge^l M) \tag{6.5}$$

$$\mathcal{W}^{1,p}_N(\wedge^l M) = \mathcal{W}^{d,p}(\wedge^l M) \cap \mathcal{W}^{d^*,p}_N(\wedge^l M) \tag{6.6}$$

PROOF. – We begin by proving (6.2). Let $\omega$ be an arbitrary element of $\mathcal{L}^{1,p}(\wedge^l \mathcal{R})$. As usual, using a partition of unity, we may assume that $\omega$ is supported in a coordinate neighborhood, say $(U, \kappa\colon U \to \mathbb{R}^n)$. Let $\omega^\#$ denote the pullback of $\omega$ via the map $\kappa^{-1}\colon \mathbb{R}^n \to U$. Obviously, $\omega^\#$ belongs to $\mathcal{L}^p(\wedge^l \mathbb{R}^n)$ and has compact support. Moreover, the hypothesis that $d\omega$ and $d^*\omega$ are $\mathcal{L}^p$-integrable on $\mathcal{R}$ can be stated equivalently as

$$\mathcal{P}\omega^\# \in \mathcal{L}^p(\wedge^{l+1}\mathbb{R}^n) \quad \text{and} \quad \mathcal{Q}\omega^\# \in \mathcal{L}^p(\wedge^{l-1}\mathbb{R}^n)$$

where $\mathcal{P}(x, D)\colon \Gamma(\wedge^l \mathbb{R}^n) \to \Gamma(\wedge^{l+1}\mathbb{R}^n)$ and $\mathcal{Q}(x, D)\colon \Gamma(\wedge^l \mathbb{R}^n) \to \Gamma(\wedge^{l-1}\mathbb{R}^n)$ are first order differential operators on $\mathbb{R}^n$ with $C_0^\infty$-coefficients. We then apply Friedrichs theory on the equivalence of weak and strong extensions of a first order differential operator [Fri44] (see also [Hor41]). Accordingly, there exists a sequence of forms, say $\omega_j^\# \in C_0^\infty(\wedge^l \mathbb{R}^n)$ such that $\omega_j^\# \to \omega^\#$, $\mathcal{P}\omega_j^\# \to \mathcal{P}\omega^\#$ and $\mathcal{Q}\omega_j^\# \to \mathcal{Q}\omega^\#$ in $\mathcal{L}^p$.

Pulling back the $\omega_j^\#$'s to the reference manifold via the map $\kappa\colon U \to \mathbb{R}^n$, we obtain a sequence, denoted $\omega_j \in C_0^\infty(\wedge^l U)$, such that $\omega_j \to \omega$ in $\mathcal{L}^p(\wedge^l \mathcal{R})$, $d\omega_j \to d\omega$ in $\mathcal{L}^p(\wedge^{l+1}\mathcal{R})$ and $d^*\omega_j \to d^*\omega$ in $\mathcal{L}^p(\wedge^{l-1}\mathcal{R})$. Finally, with the aid of Gaffney's Inequality, we conclude that $\{\omega_j\}$ actually converges to $\omega$ in the norm of $\mathcal{W}^{1,p}(\wedge^l \mathcal{R})$, proving (6.2).

Now, (6.4) is straightforward because the zero extension of a form from $\mathcal{W}^{d,p}_T(\wedge^l M) \cap \mathcal{W}^{d^*,p}_N(\wedge^l M)$ belongs to $\mathcal{W}^{d,p}(\wedge^l \mathcal{R}) \cap \mathcal{W}^{d^*,p}(\wedge^l \mathcal{R})$.

Similar reasoning applies to the inclusion (6.3). For this, we simply notice that $\chi\omega \in \mathcal{W}^{d,p}(\wedge^l \mathcal{R}) \cap \mathcal{W}^{d^*,p}(\wedge^l \mathcal{R})$ whenever $\chi \in C_0^\infty(M)$ and $\omega \in \mathcal{L}^{1,p}(\wedge^l M)$.

Unfortunately, Friedrichs theory does not apply to (6.5) and (6.6). The trouble is that the boundary constraints make the problem of approximation by smooth forms more delicate. From what has already been proven, it follows that each $\omega \in \mathcal{W}_T^{d,p}(\bigwedge^l M) \cap \mathcal{W}^{d^*,p}(\bigwedge^l M)$ has generalized first order derivatives which are locally $\mathcal{L}^p$-integrable on $M$. The task is now to prove the $\mathcal{L}^p$-integrability of these derivatives near an arbitrary boundary point, say $b \in \partial M$.

Let $(U, \kappa = (x_1, \ldots, x_n): U \to \mathbb{R}^n)$ be a coordinate neighborhood at $b$ so that the vectors $\partial/\partial x_n$ are orthogonal to $\partial M$ at each point of $U \cap \partial M$. In addition, we require that $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ form an orthonormal basis for $T_b\mathcal{R}$. Using this coordinate system in $U$, we write

$$\omega = \sum_I \omega^I dx_I \tag{6.7}$$

where the $\omega^I$ are functions on $U \cap M$ of class $\mathcal{W}_{\mathrm{loc}}^{1,p}(U \cap M)$. We notice that the $l$-forms $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_l}$ give an orthonormal basis for $\bigwedge^l(T_b\mathcal{R})$ but they are not necessarily orthogonal at other points. Next, we consider the partials $\partial\omega/\partial x_i \in \mathcal{L}_{\mathrm{loc}}^p(U \cap M, \bigwedge^l)$ defined by

$$\frac{\partial \omega}{\partial x_i} = \sum \frac{\partial \omega^I}{\partial x_i} dx_I, \qquad i = 1, \ldots, n \tag{6.8}$$

and recall the formulas

$$\begin{cases} d\omega = \displaystyle\sum_{i,I} \frac{\partial \omega^I}{\partial x_i} dx_i \wedge dx_I \\ d(*\omega) = \displaystyle\sum_{i,I} \frac{\partial \omega^I}{\partial x_i} dx_i \wedge *dx_I + \sum \omega^I d(*dx_I) \end{cases} \tag{6.9}$$

We look at (6.9) as a linear system of equations with $\partial\omega^I/\partial x_n$ as unknowns. At the point $b$, this system takes the form

$$\begin{cases} d\omega = \displaystyle\sum_{n \notin I} \frac{\partial \omega^I}{\partial x_n} dx_n \wedge dx_I + \sum_{i=1}^{n-1} \sum_I \frac{\partial \omega^I}{\partial x_i} dx_i \wedge dx_I \\ d(*\omega) = \displaystyle\sum_{n \in I} \frac{\partial \omega^I}{\partial x_n} dx_n \wedge *dx_I + \sum_{i=1}^{n-1} \sum_I \frac{\partial \omega^I}{\partial x_i} dx_i \wedge *dx_I + \sum \omega^I d(*dx_I) \end{cases} \tag{6.10}$$

This purely algebraic calculation does not involve any assumptions about the regularity of $\partial\omega^I/\partial x_i$. It shows, in particular, that at $b$ the variables $\partial\omega^I/\partial x_n$ depend linearly on $\omega$, $d\omega$, $d^*\omega$ and $\partial\omega/\partial x_i$ with $i = 1, 2, \ldots, n-1$.

By the implicit function theorem, the system (6.9) is solvable for $\partial\omega/\partial x_n$ near the point $b$. More precisely

$$\frac{\partial \omega}{\partial x_n} = \mathcal{F}\left(\omega, \frac{\partial \omega}{\partial x_1}, \ldots, \frac{\partial \omega}{\partial x_{n-1}}, d\omega, d^*\omega\right)$$

where $\mathcal{F}$ is a linear form in the indicated variables whose coefficients depend only on the coordinate system $\kappa = (x_1, \ldots, x_n)$ and are $C^\infty$-smooth. This gives us a pointwise estimate

$$\left| \frac{\partial \omega}{\partial x_n} \right| \leqslant C_\kappa \left( |\omega| + \left| \frac{\partial \omega}{\partial x_1} \right| + \ldots + \left| \frac{\partial \omega}{\partial x_{n-1}} \right| + |d\omega| + |d^* \omega| \right) \tag{6.11}$$

Thus, the $p$-norm of $\partial\omega/\partial x_n$ (near the point $b$) is controlled by the $p$-norms of $\omega$, $d\omega$, $d^*\omega$ and $\partial\omega/\partial x_i$ with $i = 1, 2, \ldots, n-1$. We are, therefore, reduced to showing that $\partial\omega/\partial x_i \in \mathcal{L}^p(\wedge^l V)$ for $i = 1, 2, \ldots, n-1$, where $V \subset U \cap M$ is a regular open region whose boundary coincides with $\partial M$ near the point $b$. Poincarés lemma guarantees that having chosen $V$ small and regular enough, we may assume that $\mathcal{H}_T(\wedge^l V) = 0$. A final convenient assumption we are allowed to make is that $\omega$ vanishes outside a small neighborhood of the point $b \in \partial M$. For, if necessary, we replace $\omega$ by $\chi\omega$, where $\chi$ is a suitable $C^\infty$-function on $\mathcal{R}$ equal to 1 in a neighborhood of $b$.

Since $\omega$ was assumed to have vanishing tangential part on $\partial M$, we obtain $\omega_T = 0$ on $\partial V$. We also have

$$(\omega, d\eta) = (d^*\omega, \eta) \tag{6.12}$$

for all $\eta \in C^\infty(\overline{V}, \wedge^{l-1})$ with $\eta_T = 0$ on $\partial M$. Compare this with formula (2.34).

We shall have established the $\mathcal{L}^p$-integrability of $\partial\omega/\partial x_i$ if we prove that

$$\left( \frac{\partial \omega}{\partial x_i}, \phi \right) \leqslant C \|\omega\|_{\mathcal{L}^{1,p}(M)} \|\phi\|_q, \qquad p + q = pq \tag{6.13}$$

for every test form $\phi \in C_0^\infty(\wedge^l V)$ and $i = 1, 2, \ldots, n-1$, where $C$ depends only on the coordinate system $\kappa = (x_1, \ldots, x_n)$: $U \to \mathbb{R}^n$ and the region $V$. Integration by parts reveals

$$\left( \frac{\partial \omega}{\partial x_i}, \phi \right) = -\left( \omega, \frac{\partial \phi}{\partial x_i} \right) + \int_V \mathcal{B}(\omega, \phi) \tag{6.14}$$

where $\mathcal{B}$ is a bilinear form with $C^\infty(\overline{V})$-coefficients. In particular, we only need to show that

$$\left( \omega, \frac{\partial \phi}{\partial x_i} \right) \leqslant C \|\omega\| \mathcal{L}^{1,p}(M) \|\phi\|_q \tag{6.15}$$

for $i = 1, 2, \ldots, n-1$. With all of this in mind, we proceed as follows. First, we split $\phi$ on $V$ according to Hodge decompositions (5.6) and (5.9).

$$\phi = d\alpha + d^*\beta + h, \qquad \alpha_T = h_T = 0 \tag{6.16}$$

where $\alpha \in d^* C^\infty(\overline{V}, \wedge^l)$, $\beta \in dC_T^\infty(\overline{V}, \wedge^l)$, and $h \in \mathcal{H}_T(\wedge^l V)$. The latter implies $h = 0$. By Corollary 5.6, we have

$$\|\alpha\|_{1,q} + \|\beta\|_{1,q} \leqslant C_q \|\phi\|_q \tag{6.17}$$

It is crucial to observe that the partials $\partial\alpha/\partial x_i$ have vanishing tangential part on $\partial M$ for all $i = 1, \ldots, n-1$. To see this, we write

$$\alpha = \sum_{n \notin I} \alpha^I \, dx_I + \sum_{n \in I} \alpha^I \, dx_I \tag{6.18}$$

Since $\partial/\partial x_n$ was orthogonal to the vectors $\partial/\partial x_1, \ldots, \partial/\partial x_{n-1}$, the condition $\alpha_T = 0$ is the same as $\alpha^I = 0$ on $\partial M$ whenever $n \notin I$. The latter is obviously preserved under the partial differentiations $\partial/\partial x_1, \ldots, \partial/\partial x_{n-1}$. Thus the tangential part of $\partial \alpha/\partial x_i$ vanishes on $\partial M$ as well.

We write

$$\frac{\partial \phi}{\partial x_i} = d \frac{\partial \alpha}{\partial x_i} + d^* \frac{\partial \beta}{\partial x_i} + \left( \frac{\partial}{\partial x_i} d - d \frac{\partial}{\partial x_i} \right) \alpha + \left( \frac{\partial}{\partial x_i} d^* - d^* \frac{\partial}{\partial x_i} \right) \beta \tag{6.19}$$

where the commutators in parentheses are first order differential operators with $C^\infty$-coefficients in $\overline{U}$. In particular

$$\left\| \left( \frac{\partial}{\partial x_i} d - d \frac{\partial}{\partial x_i} \right) \alpha \right\|_q + \left\| \left( \frac{\partial}{\partial x_i} d^* - d^* \frac{\partial}{\partial x_i} \right) \beta \right\|_q \leqslant C(\|\alpha\|_{1,q} + \|\beta\|_{1,q}) \tag{6.20}$$

Since $\omega_T = 0$ on $\partial V$, we have

$$\left( \omega, d^* \frac{\partial \beta}{\partial x_i} \right) = \left( d\omega, \frac{\partial \beta}{\partial x_i} \right) \leqslant \|d\omega\|_p \|\beta\|_{1,q} \tag{6.21}$$

Similarly,

$$\left( \omega, d^* \frac{\partial \alpha}{\partial x_i} \right) = \left( d\omega, \frac{\partial \alpha}{\partial x_i} \right) \leqslant \|d^* \omega\|_p \|\beta\|_{1,q} \tag{6.22}$$

because $(\partial \alpha/\partial x_i)_T = 0$ on $\partial M$. Finally, combining (6.19), (6.20), (6.21) and (6.22) yields

$$\left( \omega, \frac{\partial \phi}{\partial x_i} \right) \leqslant C(\|\omega\|_p + \|d\omega\|_p + \|d^*\omega\|_p)(\|\alpha\|_{1,q} + \|\beta\|_{1,q}) \leqslant C \|\omega\|_{\mathcal{L}^{1,p}(M)} \|\phi\|_q$$

which completes the proof of (6.5). The dual identity (6.6) holds as well. ■

We can now strengthen formulas (3.19) and (3.20) with

LEMMA 6.2. – *If $\omega \in \mathcal{L}^p(\wedge^l M)$, $1 < p < \infty$, is both closed and coclosed, then $\omega$ is in $\mathcal{H}_T(\wedge^l M)$ or $\mathcal{H}_N(\wedge^l M)$ if and only if $\omega_T = 0$ or $\omega_N = 0$ respectively.*

6.2. *Poincaré inequalities.* – For the purpose of studying Sobolev spaces of differential forms in greater detail, estimates of Poincaré type are often useful. The following result can be regarded as a refinement of Theorem 4.11.

THEOREM 6.3. – *Given $\omega \in \mathcal{L}^{1,p}(\wedge^l M)$, $1 < p < \infty$, there exists a harmonic field*

*$h \in \mathcal{H}^p(\wedge^l M)$ such that $\omega - h \in \mathcal{W}^{1,p}(\wedge^l M)$ and*

$$\|\omega - h\|_{1,p} \leq C_p(M)(\|d\omega\|_p + \|d^*\omega\|_p) \tag{6.33}$$

*Moreover, if $\omega_T = 0$ or $\omega_N = 0$, then $h_T = 0$ or $h_N = 0$ respectively.*

THEOREM 6.4. – *Given $\omega \in \mathcal{W}^{d,p}(\wedge^l M)$, $1 < p < \infty$, ($\omega \in \mathcal{W}^{d^*,p}(\wedge^l M)$, respectively) there exists a closed (coclosed) form $\omega_0 \in \mathcal{L}^p(\wedge^l M)$ such that $\omega - \omega_0 \in \mathcal{W}^{1,p}(\wedge^l M)$ and*

$$\|\omega - \omega_0\|_{1,p} \leq C_p(M)\|d\omega\|_p \quad (C_p(M)\|d^*\omega\|_p) \tag{6.24}$$

*Moreover, if $\omega_T = 0$ ($\omega_N = 0$, respectively), then $(\omega_0)_T = 0$ ($(\omega_0)_N = 0$).*

The proofs of these theorems are based on the Hodge decompositions for Sobolev spaces and Theorem 6.1.

PROOF OF THEOREM 6.3. – We split $\omega \in \mathcal{L}^{1,p}(\wedge^l M)$ into parts $\omega = d\alpha + d^*\beta + h$ according to decomposition (5.10). Note that both $d\alpha$ and $d^*\beta$ are members of the class $\mathcal{L}^{1,p}(\wedge^l M)$ because $d^*(d\alpha) = d^*\omega \in \mathcal{L}^p(\wedge^{l-1} M)$ and $d(d^*\beta) = d\omega \in \mathcal{L}^p(\wedge^{l+1} M)$. Moreover, $(d\alpha)_T = 0$ and $(d^*\beta)_N = 0$ (see Remark 3.5). By Theorem 6.1, these forms actually belong to $\mathcal{W}^{1,p}(\wedge^l M)$. They are also orthogonal to all harmonic fields of class $\mathcal{H}^1(\wedge^l M)$ as is easily confirmed. Theorem 4.11 yields inequality (6.23) as follows

$$\begin{aligned} \|\omega - h\|_{1,p} &\leq \|d\alpha\|_{1,p} + \|d^*\beta\|_{1,p} \\ &\leq C_p(M)(\|d^*d\alpha\|_p + \|dd^*\beta\|_p) = C_p(M)(\|d\omega\|_p + \|d^*\omega\|_p) \end{aligned} \tag{6.25}$$

Moreover, if $\omega_T = 0$, then we split $\omega$ according to decomposition (5.11). In this case, we obtain $h \in \mathcal{H}_T(\wedge^l M)$, $d\alpha \in \mathcal{W}_T^{1,p}(\wedge^l M)$ and $d^*\beta \in \mathcal{W}_T^{1,p}(\wedge^l M)$. Although $d\alpha$ is orthogonal to all $\mathcal{H}^1(\wedge^l M)$, the form $d^*\beta$ is only orthogonal to $\mathcal{H}_T(\wedge^l M)$, which is still sufficient to apply Theorem 4.11. It follows that (6.23) holds in this case as well. The case of $\omega_N = 0$ is handled similarly by using decomposition (5.12). ■

PROOF OF THEOREM 6.4. – For $\omega \in \mathcal{W}^{d,p}(\wedge^l M)$, $1 < p < \infty$, and using the same decompositions as in the previous proof, we define the closed form $\omega_0 = d\alpha + h$. We then notice that $\omega - \omega_0 = d^*\beta \in \mathcal{W}^{1,p}(\wedge^l M)$ and the required estimate follows as before. The details are left to the reader. ■

REMARK 6.5. – *Our proof shows that $\omega - \omega_0$ belongs to*

$$\mathcal{W}^{1,p}(\wedge^l M) \cap d^*\mathcal{W}_N^{1,p}(\wedge^{l+1} M), \qquad \textit{if } \omega \in \mathcal{W}^{d,p}(\wedge^l M)$$

$$\mathcal{W}_T^{1,p}(\wedge^l M) \cap d^*\mathcal{W}^{1,p}(\wedge^{l+1} M), \qquad \textit{if } \omega \in \mathcal{W}_T^{d,p}(\wedge^l M)$$

## 7. – Some nonlinear problems.

This section is intended to motivate our future investigation of a particular class of nonlinear PDEs for differential forms on Riemannian manifolds (Hodge systems). The first subsection touches on a few nonlinear aspects of the Hodge decomposition and is closely linked with Poincaré type estimates.

7.1. *$\mathcal{L}^p$-Projections.* – Throughout this subsection, $\mathcal{V} = \mathcal{V}^p(\wedge^l M)$ stands for one of the following subspaces of $\mathcal{L}^p(\wedge^l M)$, $1 < p < \infty$:

$$\begin{array}{llll} d\mathcal{W}^{1,p}(\wedge^{l-1} M) & \text{or} & d\mathcal{W}_T^{1,p}(\wedge^{l-1} M) & \text{(Exact forms)} \\ d^*\mathcal{W}^{1,p}(\wedge^{l+1} M) & \text{or} & d^*\mathcal{W}_N^{1,p}(\wedge^{l+1} M) & \text{(Coexact forms)} \\ \mathcal{L}^p(\wedge^l M) \cap \ker d & \text{or} & \mathcal{L}^p(\wedge^l M) \cap \ker d_T & \text{(Closed forms)} \\ \mathcal{L}^p(\wedge^l M) \cap \ker d^* & \text{or} & \mathcal{L}^p(\wedge^l M) \cap \ker d_N^* & \text{(Closed forms)} \end{array}$$

Note that all of these spaces are complete. In order to obtain a precise approximation of a given form $\omega \in \mathcal{L}^p(\wedge^l M)$ by an element from $\mathcal{V}^p(\wedge^l M)$, it is natural to examine the nonlinear projection

$$\Pi = \Pi_p : \mathcal{L}^p(\wedge^l M) \to \mathcal{V}^p(\wedge^l M) \tag{7.1}$$

which carries $\omega \in \mathcal{L}^p(\wedge^l M)$ to the nearest element of $\mathcal{V}$ as measured by the $\mathcal{L}^p$-metric. This image is called the $\mathcal{L}^p$-projection of $\omega$ into $\mathcal{V}$. The existence and uniqueness of such a projection is easily established by convexity arguments. The element $\Pi\omega$ just defined solves the minimization problem

$$\|\omega - \Pi\omega\|_p^p = \min \int_M |\omega - \eta|^p \tag{7.2}$$

subject to all test forms $\eta \in \mathcal{V}^p(\wedge^l M)$. In this way, we obtain what is known as the integral form of the Euler-Lagrange equation

$$\int_M \langle \eta, |\omega - \Pi\omega|^{p-2}(\omega - \Pi\omega)\rangle = 0 \tag{7.3}$$

for all $\eta \in \mathcal{V}(\wedge^l M)$. Given $\xi$, an element of a normed linear space, we will make frequent use of its $s$ power, $s \geq 0$,

$$\xi^s = |\xi|^{s-1}\xi, \qquad 0^s = 0 \tag{7.4}$$

Clearly, $(\omega - \Pi\omega)^{p-1}$ belongs to the dual space $\mathcal{L}^q(\wedge^l M)$, $p + q = pq$. Moreover, equation (7.3) says that this form is orthogonal to $\mathcal{V} = \mathcal{V}^p(\wedge^l M)$. Because of its geometric meaning, the $\mathcal{L}^p$-projection is obviously a bounded operator. Namely,

$$\|\omega - \Pi_p\omega\|_p \leq \|\omega\|_p \qquad \text{Hence,} \quad \|\Pi_p\omega\|_p \leq 2\|\omega\|_p \tag{7.5}$$

For nonlinear operators such as $\Pi_p$, $p \neq 2$, this estimate does not necessarily imply continuity. However, with the aid of the Euler-Lagrange equation, we can prove continuity of $\Pi_p$ and give a uniform estimate for the modulus of continuity of $\Pi$.

PROPOSITION 7.1. – *For each* $1 < p < \infty$, *there exists a constant* $C_p$ *such that*

$$\|\Pi\alpha - \Pi\beta\|_p \leq C_p(\|\alpha\|_p + \|\beta\|_p)^{1-t}\|\alpha - \beta\|_p^t \tag{7.6}$$

*for all* $\alpha, \beta \in \mathcal{L}^p(\wedge^l M)$, *where* $t = 1/(1 + |p-2|)$.

PROOF. – Set $\xi = \alpha - \Pi\alpha$ and $\zeta = \beta - \Pi\beta$. The Euler-Lagrange equation reads $(\xi^{p-1} - \zeta^{p-1}, \eta) = 0$, for all $\eta \in \mathcal{V}$. Applying this to $\eta = \Pi\beta - \Pi\alpha$, yields

$$(\xi^{p-1} - \zeta^{p-1}, \xi - \zeta) = (\xi^{p-1} - \zeta^{p-1}, \alpha - \beta)$$

Hence, by Hölder's inequality

$$(\xi^{p-1} - \zeta^{p-1}, \xi - \zeta) \leq \|\xi^{p-1} - \zeta^{p-1}\|_q \|\alpha - \beta\|_p \tag{7.7}$$

The following two inequalities, which hold in an arbitrary inner product space, will be useful both here and in the future.

$$(|\xi| + |\zeta|)^{p-2}|x - \zeta|^2 \leq A\langle \xi^{p-1} - \zeta^{p-1}, \xi - \zeta\rangle \tag{7.8}$$

$$|\xi^{p-1} - \zeta^{p-1}| \leq B(|\xi| + |\zeta|)^{p-2}|x - \zeta| \tag{7.9}$$

whre $A = A(p)$ and $B = B(p)$. For notational convenience, we introduce the exponent $a = (1/2)\min(p, q) \leq 1$. Combining (7.7), (7.8), (7.9) and Hölder's inequality gives

$$\int_M (|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2 \leq A\|\xi^{p-1} - \zeta^{p-1}\|_q \|\alpha - \beta\|_p$$

$$\leq AB\left[\int_M (|\xi| + |\zeta|)^{ap-2a}|\xi - \zeta|^{2a}(|\xi| + |\zeta|)^{p-pa}\right]^{1/q}\|\alpha - \beta\|_p$$

$$\leq AB\left(\int_M (|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2\right)^{a/q}\left(\int_M (|\xi| + |\zeta|)^p\right)^{(1-a)/q}\|\alpha - \beta\|_p$$

Hence,

$$\int_M (|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2 \leq (AB\|\alpha - \beta\|_p)^{q/(q-a)}(\|\xi\|_p + \|\zeta\|_p)^{(p-ap)/(q-a)} \tag{7.10}$$

On the other hand, using Hölder's inequality again, we see that

$$\|\xi - \zeta\|_p \leq \left[\int_M (|\xi| + |\zeta|)^{ap-2a}|\xi - \zeta|^{2a}(|\xi| + |\zeta|)^{p-ap}\right]^{1/p} \tag{7.11}$$

$$\leq \left(\int_M (|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2\right)^{a/p}\left(\int_M (|\xi| + |\zeta|)^p\right)^{(1-a)/p}$$

which combined with (7.10) yields

$$\|\xi - \zeta\|_p \leqslant (AB\|\alpha - \beta\|_p)^{(aq-a)/(q-a)}(\|\xi\|_p + \|\zeta\|_p)^{(q-qa)/(q-a)}$$

where $(aq - a)/(q - a) = t$, as is easily verified. For the final conclusion, we observe $\|\Pi\alpha - \Pi\beta\|_p \leqslant \|\xi - \zeta\|_p + \|\alpha - \beta\|_p$. By (7.5), we also have $\|\xi\|_p + \|\zeta\|_p \leqslant \|\alpha\|_p + \|\beta\|_p$ Together, these give the desired estimate

$$\|\Pi\alpha - \Pi\beta\|_p \leqslant [1 + (AB)^t](\|\alpha\|_p + \|\beta\|_p)^{1-t}\|\alpha - \beta\|_p^t \qquad \blacksquare$$

As we have seen, the orthogonal projections extend continuously from $\mathcal{L}^2(\wedge^l M)$ to $\mathcal{L}^s(\wedge^l M)$ for $1 < s < \infty$. It is of interest to know whether or not one can extend the nonlinear operators $\Pi_p \colon \mathcal{L}^p(\wedge^l M) \to \mathcal{V}^p(\wedge^l M)$ to $\mathcal{L}^s(\wedge^l M)$ for $s$ different from the natural exponent $p$. One of our primary results in this direction is

THEOREM 7.2. – *For $1 < p \leqslant s < \infty$, there exists $C_s = C_s(p, M)$ such that*

$$\|\Pi_p \omega\|_s \leqslant C_s \|\omega\|_s \tag{7.12}$$

*for all $\omega \in \mathcal{L}^s(\wedge^l M)$.*

We shall prove this theorem only for $s$ sufficiently close to $p$, see Remark 9.6. Analysis similar to that in [Iwa83] shows that estimate (7.12) holds for all $s \geqslant p$. Critical to this proof is the $C^\alpha$-regularity theorem of K. Uhlenbeck [Uh77] (see also [Ham92]) and the $L^p$-theory of the sharp maximal operator as developed by C. Fefferman and E. Stein [FS72]. It would exceed the scope of this paper to discuss all of these advances here (see [Str 99]). It is worth mentioning, however, that (7.12) is also valid for $s$ slightly smaller than the natural exponent $p$. We put off discussing these estimates below the natural exponent until Subsect. 9.3, where such estimates will be treated in greater generality; see Theorem 9.5 and Remark 9.6. Precisely how small the exponent $s$ can be is not known. However, there are enough arguments to safely conjecture that the operator $\Pi_p \colon \mathcal{L}^p(\wedge^l M) \to \mathcal{V}^p(\wedge^l M)$ extends as a bounded operator $\Pi_p \colon \mathcal{L}^s(\wedge^l M) \to \to \mathcal{V}^s(\wedge^l M)$ for all $s > \max\{1, p-1\}$. It is also of interest to know whether $\Pi_p$ is a continuous operator in $\mathcal{L}^s(\wedge^l M)$, as it is for the natural exponent $s = p$.

We now examine the Euler-Lagrange equation (7.3). For this, recall the orthogonal complements of $\mathcal{V}^p(\wedge^l M)$ as listed in Theorem 5.7. Our goal is to show that all eight cases for the space $\mathcal{V}(\wedge^l M)$ reduce equivalently to the same type of first order differential equation. Namely,

$$\psi = \mathfrak{A}(\phi) \tag{7.13}$$

for $\phi \in d\mathcal{W}^{1,p}(\wedge^{l-1} M)$ and $\psi \in \mathcal{L}^q(\wedge^l M) \cap \ker d^*$, where

$$\mathfrak{A} : \wedge^l M \to \wedge^l M \tag{7.14}$$

is a given bundle map. Thus, $\mathfrak{A}$ also acts on sections $\Gamma(\wedge^l M)$ with values in $\Gamma(\wedge^l M)$. Concerning boundary conditions, we will have either

$$\phi \in \operatorname{im} d_T \quad \text{(the Dirichlet boundary condition)} \tag{7.15}$$

or

$$\psi \in \ker d_N^* \quad \text{(the Neumann boundary condition)} \tag{7.16}$$

Let us illustrate the situation in case of the projection $\Pi: \mathcal{L}^p(\wedge^l M) \to d\mathcal{W}^{1,p}(\wedge^{l-1} M)$. Thus $\mathcal{V}^{\mathrm{per}} = \mathcal{L}^q(\wedge^l M) \cap \ker d_N^*$ and the Euler-Lagrange equation reads

$$(\omega - \phi)^{p-1} = \psi \tag{7.17}$$

or equivalently

$$\omega = \phi + \psi^{q-1} \tag{7.18}$$

where $\phi = \Pi\omega \in d\mathcal{W}^{1,p}(\wedge^{l-1} M)$ and $\psi \in \mathcal{L}^q(\wedge^l M) \cap \ker d_N^*$. If one projects $\omega$ to $\mathcal{V} = d\mathcal{W}_T^{1,p}(\wedge^{l-1} M)$, then $\mathcal{V}^{\mathrm{per}} = \mathcal{L}^q(\wedge^l M) \cap \ker d^*$, so the equation remains the same but the new boundary condition is $\phi \in \operatorname{im} d_T$, while $\psi \in \ker d^*$.

Other cases lead to equations similar to (7.18), in which $l$ may possibly be replaced by $n - l$ and the exponent $p$ by its Hölder conjugate $q$. It is worth noting that equation (7.18) gives exactly the $\mathcal{L}^2$-Hodge decomposition when $p = q = 2$. See formula (5.8) and the boundary constraints which follow it.

7.2. *Quasiregular mappings on Riemannian manifolds.* – Let $M$ and $N$ be arbitrary $C^\infty$-smooth oriented Riemannian manifolds of dimension $n$. We shall consider mappings $f: M \to N$ of Sobolev class $\mathcal{W}^{1,s}_{\mathrm{loc}}(M, N)$, $1 \leq s \leq \infty$. The differential $Df(x)$: $T_x M \to T_y N$, $y = f(x)$, is defined at almost every point $x \in M$. We assume that the Jacobian determinant $J(x, f) = \det Df(x)$ is non-negative ($f$ preserves orientation).

DEFINITION 7.3. – *An orientation preserving mapping $f \in \mathcal{W}^{1,s}_{\mathrm{loc}}(M, N)$ is said to have finite dilatation if*

$$\max_{|\xi|=1} |Df(x)\,\xi| = \mathcal{K}(x) \min_{|\xi|=1} |Df(x)\,\xi| \qquad a.e. \quad x \in M \tag{7.19}$$

*where $1 \leq \mathcal{K}(x) < \infty$ is called the scalar dilatation at x. If $\mathcal{K} \in \mathcal{L}^\infty(M)$, we introduce the maximal dilatation $K = \|\mathcal{K}(x)\|_\infty$ and call such f, K-quasiregular. Finally, a homeomorphism of class $\mathcal{W}^{1,n}_{\mathrm{loc}}(M, N)$ which is K-quasiregular is called K-quasiconformal.*

Of course, in formula (7.19), the norm of the tangent vectors $\xi$ is the one given by the inner product on $T_x M$ while the norm of $Df(x)\xi$ is from $T_y N$, where $y = f(x)$.

The theory of quasiregular mappings successfully extends both geometric and analytic aspects of holomorphic functions of one complex variable. In the plane, these aspects are well understood due to the work of L. Ahlfors, A. Beurling, B. Bojarski, I. N. Vekua and many others. See for instance, [AB50], [Ahl66], [Boj55], and [Vek62]. Largely as a consequence of these studies, the theory has been greatly expanded to higher dimensions whose fundamental principles were set in the pioneering work of F. W. Gehring, J. Väisälä, Y. Reshetniak, O. Martio and S. Rickman. See [Geh62], [Res69] and [MRV69, MRV70, MRV71]. For a thorough treatment of analytic properties of quasiregular mappings, we refer the reader to [Res89], [BI83] and [Ric93]. A Quasiconformal mapping $f\colon M \to N$ is, in fact, conformal with respect to a new metric on $M$, called the *dilatation tensor* of $f$. In general, this tensor is only measurable but still uniformly elliptic with respect to the original metric on $M$. Continuing the analogy with the complex case, we shall sketch how the conceptual foundations of quasiregular mappings lead to the governing PDEs. In many respects, these equations generalize the familiar Cauchy-Riemann system. We shall discuss here just a few of these equations. See [IM93] and [Iwa92] for the Euclidean case.

Let us begin with a linear mapping $\mathcal{L}\colon E \to F$ between inner product spaces. It follows from Definition 7.3 that $\mathcal{L}$ is $K$-quasiconformal in case $\det \mathcal{L} > 0$ and the following dilatation condition holds

$$|\mathcal{L}\xi|_F \leq K |\mathcal{L}\zeta|_F \tag{7.20}$$

for all unit vectors $\xi, \zeta \in E$. We then define a symmetric positive definite mapping $G\colon E \to E$

$$\mathcal{L}^t \mathcal{L} = (\det \mathcal{L})^{2/n} G \tag{7.21}$$

Observe that $\det G = 1$ and

$$K^{-2} |\xi|_E^2 \leq \langle G\xi, \xi\rangle_E \leq K^2 |\xi|_E^2 \tag{7.22}$$

Indeed, if $0 < \lambda_1^2 \leq \ldots \leq \lambda_n^2$ denote the eigenvalues of $G$, then inequality (7.20) means that

$$\lambda_n^2 = \max \langle G\xi, \xi\rangle_E \leq K^2 \min \langle G\zeta, \zeta\rangle_E = K^2 \lambda_1^2$$

Since $\lambda_1 \lambda_2 \ldots \lambda_n = 1$, we obtain $\lambda_n^n \leq K^n \lambda_1^n \leq K^n \lambda_1 \ldots \lambda_n = K^n$ and $\lambda_1^n \geq K^{-n} \lambda_n^n \geq K^{-n} \lambda_n \ldots \lambda_1 = K^{-n}$ which is really just (7.22).

Of course, $G$ defines a new inner product on the space $E$ so that the mapping $\mathcal{L}\colon E \to F$ becomes conformal with respect to this new metric. Note that $G$ is simply a scalar multiple of the pullback via $\mathcal{L}$ of the metric on $F$.

It will be convenient to introduce the symmetric square root of $G$

$$L = \sqrt{G}\colon E \to E\,, \qquad L \circ L = G\,, \qquad \det L = 1 \tag{7.23}$$

The eigenvalues of $L$ are $0 < \lambda_1 \leq \ldots \leq \lambda_n$ and inequality (7.22) reads

$$K^{-1} |\xi|_E \leq |L\xi|_E \leq K |\xi|_E \tag{7.24}$$

for all $\xi \in E$.

We now recall the $l$-th exterior powers $G_\#\colon \wedge^l E \to \wedge^l E$ and $L_\#\colon \wedge^l E \to \wedge^l E$ where we observe $L_\# \circ L_\# = G_\#$. The eigenvalues of $L_\#$ are the products $\lambda_{i_1} \ldots \lambda_{i_l}$ corresponding to all ordered $l$-tuples $1 \leq i_1 < \ldots < i_l \leq n$. For unit $l$-covectors $\xi \in \wedge^l E$, we then obtain

$$\langle G_\# \xi, \xi\rangle_{\wedge^l E} = |L_\# \xi|^2_{\wedge^l E} \geq \lambda_1^2 \ldots \lambda_l^2 \geq K^{-2l}$$

and

$$\langle G_\# \xi, \xi \rangle_{\wedge^l E} = |L_\# \xi|^2_{\wedge^l E} \leqslant \lambda_n^2 \dots \lambda_{n-l+1}^2 = \frac{1}{\lambda_1^2 \dots \lambda_l^2} \leqslant K^{2l}$$

Hence

$$K^{-2l} |\xi|^2_{\wedge^l E} \leqslant \langle G_\# \xi, \xi \rangle_{\wedge^l E} \leqslant K^{2l} |\xi|^2_{\wedge^l E}$$

or equivalently,

$$K^{-l} |\xi|_{\wedge^l E} \leqslant |L_\# \xi|_{\wedge^l E} \leqslant K^l |\xi|_{\wedge^l E}$$

for all $\xi \in \wedge^l E$.

We now return to the general $K$-quasiregular mapping $f\colon M \to N$. Its differential $Df(x)\colon T_x M \to T_{f(x)} N$ is defined at almost every point $x \in M$ and is a linear $K$-quasiconformal transformation. The first differential equation of particular relevance is the so-called Beltrami system

$$D^t f(x)\, Df(x) = J(x, f)^{2/n} G(x), \qquad x \in M \tag{7.25}$$

where $D^t f$ stands for the transpose of $Df$. As we have previously remarked, the bundle map $G\colon TM \to TM$ is symmetric with determinant equal to 1 and, more importantly, is uniformly elliptic

$$K^{-2} |\xi|^2 \leqslant \langle G\xi, \xi \rangle \leqslant K^2 |\xi|^2 \tag{7.26}$$

for all vector fields $\xi \in \Gamma(TM)$. $G$ is a measurable section of the endomorphism bundle $\mathrm{End}(TM)$. The essence of our approach is to regard $G$ as a new metric tensor on $M$, conformally equivalent to the pullback via $f$ of the metric on $N$. For this reason, we call $G$ the *dilatation tensor*. It is a tautology that $f$ is conformal with respect to this metric.

Not much is known about Riemannian measurable structures in dimensions greater than 2. No doubt this is due to the difficulty in defining the curvature tensor of $G$. On the other hand, intrinsic topological properties of quasiregular mappings can be observed only in the presence of measurable dilatation. It is fortunate that certain results of smooth conformal geometry can be carried over to measurable Riemannian structures without the necessity of differentiating the metric tensor.

Fix an arbitrary harmonic field on $N$, say

$$\xi \in \mathcal{H}(\wedge^l N), \qquad d\xi = d^* \xi = 0 \tag{7.27}$$

It should be noted that on some manifolds, harmonic fields must necessarily vanish at some points. However, because of the local nature of the equations in question, we may confine ourselves (if necessary) to a small region of $N$ so that we may always assume

$$0 < \inf |\xi| \leqslant \sup |\xi| < \infty \tag{7.28}$$

This assumption is essential for ellipticity of the forthcoming equations. See (7.42) and (7.43). The unknowns of our equations will be the pullbacks of $\xi$ via the mapping

$f: M \to N$. Namely,

$$\phi = f^{\#}(\xi) \in \Gamma(\wedge^l M) \quad \text{and } \psi = f_{\#}(\xi) \in \Gamma(\wedge^l M) \tag{7.29}$$

From here on, we assume that the Sobolev exponent for $f$ satisfies $s \geq \max\{l, n-l\}$. This makes it legitimate to apply exterior differentiation. Accordingly, $d\phi = f^{\#}(d\xi) = 0$ and $d^*\psi = f_{\#}(d^*\xi) = 0$. We then observe that

$$\begin{cases} \phi \in \mathcal{L}^{\lambda p}_{\mathrm{loc}}(\wedge^l M) \\ d\phi = 0 \end{cases} \quad \text{and} \quad \begin{cases} \psi \in \mathcal{L}^{\lambda q}_{\mathrm{loc}}(\wedge^l M) \\ d^*\psi = 0 \end{cases} \tag{7.30}$$

where $p = n/l$, $q = n/(n-l)$ and $\lambda = s/n$. It has been recognized in quasiconformal analysis that the most natural Sobolev exponent $s$ in which to consider quasiregular mappings is the dimension of the manifolds. In which case, we have $\lambda = 1$.

Our immediate goal is to show that the forms $\phi$ and $\psi$ are coupled by equations similar to (7.13). Before stating these equations, we need to recall the linear map $\mathcal{L}: E \to F$, its $l$-th exterior power $\mathcal{L}_{\#}: \wedge^l F \to \wedge^l E$ and the Hodge star operator $*: \wedge^l E \to \wedge^{n-l} E$.

LEMMA 7.4. – *For a given $l$-covector $\xi \in \wedge^l F$, define $\phi = \mathcal{L}^{\#}(\xi) \in \wedge^l E$ and $\psi = \mathcal{L}_{\#}(\xi) \in \wedge^l E$. Then,*

$$\psi = |\xi|^{2-q} \langle G_{\#}\phi, \phi\rangle^{(p-2)/2} G_{\#}\phi, \quad p = \frac{n}{l} \tag{7.31}$$

*or equivalently*

$$\phi = |\xi|^{2-q} \langle G^{\#}\psi, \psi\rangle^{(q-2)/2} G^{\#}\psi, \quad q = \frac{n}{n-l} \tag{7.32}$$

REMARK 7.5. – *Using $L = \sqrt{G}$, formulas* (7.31) *and* (7.32) *can be given the more symmetric form*

$$\left(\frac{L_{\#}\phi}{|\xi|}\right)^p = \left(\frac{L^{\#}\psi}{|\xi|}\right)^q \tag{7.33}$$

PROOF. – First notice that equation (7.31) is invariant under multiplication of both $\xi \in \wedge^l F$ and $\mathcal{L} \in \mathrm{Hom}(E, F)$ by a scalar. Therefore, there is no loss of generality in assuming that $|\xi| = 1$ and $\det \mathcal{L} = 1$. Applying formulas from Subsect. 2.1, we obtain

$$G = \mathcal{L}^t\mathcal{L}, \quad G_{\#} = \mathcal{L}_{\#}(\mathcal{L}_{\#})^t \quad \text{and } G^{\#} = \mathcal{L}^{\#}(\mathcal{L}^{\#})^t \tag{7.34}$$

Hence, again following Subsect. 2.1, $G_{\#}\phi = \mathcal{L}_{\#}(\mathcal{L}_{\#})^t\mathcal{L}^{\#}\xi = \mathcal{L}_{\#}\xi = \psi$, which is equivalent to $G^{\#}\psi = \phi$. Next, we compute $\langle G_{\#}\phi, \phi\rangle = \langle G^{\#}\psi, \psi\rangle = \langle \phi, \psi\rangle = \langle \mathcal{L}_{\#}\xi, \mathcal{L}^{\#}\xi\rangle = \langle (\mathcal{L}^{\#})^t\mathcal{L}_{\#}\xi, \xi\rangle = \langle \xi, \xi\rangle \det \mathcal{L} = 1$. These identities imply (7.31) and (7.32) at once. ■

We remark that in the conformal case we have $G_\# = id$ and therefore our formulas are concisely expressed as

$$* \mathcal{L}^\# = (\det \mathcal{L})^{(n-2l)/n} \mathcal{L}^\# * : \wedge^l F \to \wedge^{n-l} E \tag{7.35}$$

In particular, in even dimensions, say $n = 2l$, the Hodge star commutes with the pull-backs on $l$-forms

$$* \mathcal{L}_\# = \mathcal{L}_\# * \quad \text{and} \quad * \mathcal{L}^\# = \mathcal{L}^\# * \tag{7.36}$$

This commutation law substantially simplifies equations for conformal mappings in even dimensions; see [DS89] for $l = 2$ and [IM93] for arbitrary $l$.

PROPOSITION 7.6. – *The differential forms $\phi$ and $\psi$ defined in (7.29) are coupled by the nonlinear equation*

$$\psi = \mathfrak{A}_p(\phi), \qquad d\phi = d^* \psi = 0 \tag{7.37}$$

*where $\mathfrak{A}_p \colon \wedge^l M \to \wedge^l M$ is a bundle map defined by*

$$\mathfrak{A}_p(\phi) = |\xi|^{2-p} \langle \phi, G_\# \phi \rangle^{(p-2)/2} G_\# \phi, \qquad p = \frac{n}{l} \tag{7.38}$$

*In this equation, we understand that the norm $|\xi|$ is evaluated at $y = f(x)$, making $|\xi|$ a positive function on $M$.*

For the conformal case, $G$ is the identity, so equation (7.37) takes the form

$$\psi = |\xi|^{2-p} \phi^{p-1} \tag{7.39}$$

Or, equivalently

$$\left( \frac{\psi}{|\xi|} \right)^q = \left( \frac{\phi}{|\xi|} \right)^p$$

Of special interest in even dimensions is the case $l = n/2$, since it leads to a linear system. More precisely, $p = q = 2$ and we only need to assume that $f \in \mathcal{W}^{1,l}_{\mathrm{loc}}(M, N)$ to obtain

$$\psi = G_\# \phi \quad \text{or equivalently } \phi = G^\# \psi \tag{7.40}$$

where $d\phi = d^* \psi = 0$. Further, when $G = id$, we obtain Cauchy-Riemann type equations

$$\phi = \psi, \qquad d\phi = d^* \psi = 0 \tag{7.41}$$

Clearly, $\phi$ and $\psi$ are harmonic fields and consequently $C^\infty$-smooth. Forms in a pair of this kind will be called harmonic conjugate fields. It is worth noting that equations (7.40) and (7.41) do not contain $\xi$ and, therefore, are elliptic even if $\xi$ vanishes at some points of $N$.

Applying exterior differentiation (in the distributional sense), we eliminate $\psi$ from

(7.37) to obtain second order equations for $\phi$ and $\psi$

$$d^*[|\xi|^{2-p}\langle\phi, G_\#\phi\rangle^{(p-2)/2} G_\#\phi] = 0 \tag{7.42}$$

Similarly, we have

$$d[|\xi|^{2-q}\langle\psi, G^\#\psi\rangle^{(q-2)/2} G^\#\psi] = 0 \tag{7.43}$$

These equations are easily recognized as Euler-Lagrange equations for the variational integrals:

$$\int_M |L_\#\phi|^p |\xi|^{2-p} \tag{7.44}$$

and

$$\int_M |L^\#\psi|^q |\xi|^{2-q} \tag{7.45}$$

Our next section deals with examples of nonvariational type equations.

7.3. *A nonlinear Hodge decomposition.* – Another kind of nonlinear problem involving differential forms concerns generalizations of the Hodge decompositions.

Given a bundle map $\mathcal{B}$: $\wedge^l M \times \wedge^l M \to \wedge^l M$, we wish to express each $\omega \in \mathcal{L}^p(\wedge^l M)$, $1 < p < \infty$, as

$$\omega = \mathcal{B}(\phi, \psi), \qquad d\phi = d^*\psi = 0 \tag{7.46}$$

Via the linear case, $\mathcal{B}(\phi, \psi) = \phi + \psi$, the Hodge decompositions suggest each of the following four sets of boundary constraints

$$\begin{cases} \phi \in d\mathcal{W}_T^{1,p}(\wedge^{l-1}M), & \text{and} \quad \psi \in \mathcal{L}^p(\wedge^l M) \cap \ker d^* \\ \phi \in \mathcal{L}^p(\wedge^l M) \cap \ker d_T & \text{and} \quad \psi \in d^*\mathcal{W}^{1,p}(\wedge^{l+1}M) \\ \phi \in d\mathcal{W}^{1,p}(\wedge^{l-1}M) & \text{and} \quad \psi \in \mathcal{L}^p(\wedge^l M) \cap \ker d_N^* \\ \phi \in \mathcal{L}^p(\wedge^l M) \cap \ker d & \text{and} \quad \psi \in d^*\mathcal{W}_N^{1,p}(\wedge^{l+1}M) \end{cases} \tag{7.47}$$

Under these conditions, the decomposition $\omega = \phi + \psi$ exists and is unique. We will impose similar boundary constraints for the nonlinear case. For example, for $\omega \in \mathcal{L}^p(\wedge^l M)$, we may consider a closed form $\phi$ which is obtained as the $\mathcal{L}^p$-projection of $\omega$ into the corresponding space as listed in the first column of formulas (7.47). This leads to a nonlinear decomposition

$$\omega = \phi + \psi^{q-1}, \qquad \phi \in \mathcal{L}^p(\wedge^l M) \cap \ker d, \qquad \psi \in \mathcal{L}^q(\wedge^l M) \cap \ker d^* \tag{7.48}$$

with $p + q = pq$.

Another example, which does not come from a variational problem, is the nonlinear decomposition

$$\omega = \phi^a + \psi^b, \qquad a, b > \frac{1}{p} \tag{7.49}$$

with $\phi \in \mathcal{L}^{ap}(\wedge^l M)$, $\psi \in \mathcal{L}^{bp}(\wedge^l M)$, $d\phi = d^* \psi = 0$ and a set of boundary conditions from (7.47). In this case, however, the Sobolev exponents for $\phi$ and $\gamma$ must be replaced by $ap$ and $bp$, respectively. The problem which must be overcome is that variational principles no longer apply to (7.49), except for the cases $(a, b) = (1, 1/(p-1))$ and $(a, b) = (1/(p-1), 1)$. Instead, we shall use the Browder-Minty theory of monotone operators. For reasons which will be clear in the proof of Corollary 8.7, this method requires that $p = 1/a + 1/b$. From here on, we assume that this relation holds. We first eliminate $\psi$ from equation (7.49)

$$d^*(\phi^a - \omega)^{1/b} = 0 \tag{7.50}$$

For $\omega \in \mathcal{L}^p(\wedge^l M)$ fixed, we define a nonlinear mapping $A\colon \Gamma(\wedge^l M) \to \Gamma(\wedge^l M)$ by the rule

$$A\xi = (\xi^a - \omega)^{1/b}, \qquad \xi \in \Gamma(\wedge^l M)$$

We impose a final condition on the numbers $a$ and $b$ in order to insure the following monotonicity property of the mapping $A$.

PROPOSITION 7.7. – *If $a$ and $b$ satisfy*

$$K := \min\{a^2, a^{-2}\} + \min\{b^2, b^{-2}\} > 1 \tag{7.51}$$

*then*

$$(K-1)\,|\xi - \zeta|\,|A\xi - A\zeta| \leqslant \langle \xi - \zeta, A\xi - A\zeta \rangle \tag{7.52}$$

*for all $\xi, \zeta \in \Gamma(\wedge^l M)$.*

PROOF. – We give only the main idea of the proof which consists of three steps.

*Step* 1. For each exponent $0 < a < \infty$ there exists a constant $\lambda_a$ such that

$$1 \geqslant \lambda_a \geqslant \min\{a, a^{-1}\}$$

and the following inequality

$$\langle \xi^a - \zeta^a, \xi - \zeta \rangle \geqslant \lambda_a\,|\xi^a - \zeta^a|\,|\xi - \zeta| \tag{7.53}$$

holds for vectors $\xi, \zeta$ from an inner product space.

REMARK 7.8. – *From here on, we assume that $\lambda_a$ denotes the largest constant for which* (7.53) *remains valid. Clearly, $\lambda_a = \lambda_b$ if $ab = 1$ and $\lambda_1 = 1$.*

To see (7.53), we note that by the remark just given, we may assume that $a \geqslant 1$. In view of homogeneity, we may also assume that $|\xi| = 1 > |\zeta| = x > 0$. Therefore,

$\langle \xi, \zeta \rangle = tx$ for some $-1 \leqslant t \leqslant 1$. After squaring, inequality (7.53) takes the form

$$a^2[1 + x^{a+1} - tx(1 + x^{a-1})]^2 \geqslant (1 + x^{2a} - 2tx^a)(1 + x^2 - 2tx)$$

For this inequality, it suffices to show that

$$\frac{1 + x^{a+1} - tx(1 + x^{a-1})}{1 + x^2 - 2tx} \geqslant \frac{1}{a} \tag{7.54}$$

and

$$\frac{1 + x^{a+1} - tx(1 + x^{a-1})}{1 + x^{2a} - 2tx^a} \geqslant \frac{1}{a} \tag{7.55}$$

We look at these expressions as functions (homographies) with respect to $t$. Clearly, their extremal values are attained at one of the endpoints $t = 1$ or $t = -1$. To complete the verification of (7.53), we need only prove that

$$\frac{1}{a} \leqslant \frac{1 + x^a}{1 + x} \leqslant \frac{1 - x^a}{1 - x} \leqslant a \tag{7.56}$$

The inequality in the middle follows from $x \geqslant x^a$ while those on the left and right are equivalent to $ax^a - x \leqslant a - 1$ and $x^a - ax \leqslant a - 1$, respectively. Both functions are convex, thus their maximum occurs at either $x = 0$ or $x = 1$, which is obviously less than $a - 1$. This completes the proof of (7.53).

*Step* 2. The sharp constant for inequality (7.52) in place of $K - 1$ is given by

$$\lambda(a, b) = \lambda_a \lambda_b - \sqrt{1 - \lambda_a^2}\sqrt{1 - \lambda_b^2} \tag{7.57}$$

With the aid of a rotation, we need only consider vectors $\xi$, $\zeta$ and $\omega$ from a three dimensional space, say the Euclidean space $\mathbb{R}^3$. Denote by $X\xi = \xi^a - \omega$ and

$$\frac{\langle X\xi - X\zeta, \xi - \zeta \rangle}{|X\xi - X\zeta|\,|\xi - \zeta|} = \cos\alpha \geqslant \lambda_a, \qquad 0 \leqslant \alpha < \frac{\pi}{2}$$

$$\frac{\langle A\xi - A\zeta, X\xi - X\zeta \rangle}{|A\xi - A\zeta|\,|X\xi - X\zeta|} = \cos\beta \geqslant \lambda_b, \qquad 0 \leqslant \beta < \frac{\pi}{2}$$

$$\frac{\langle A\xi - A\zeta, \xi - \zeta \rangle}{|A\xi - A\zeta|\,|\xi - \zeta|} = \cos\gamma, \qquad 0 \leqslant \gamma < \pi$$

By an elementary geometric argument, we find that $\gamma \leqslant \alpha + \beta$. Thus,

$$\cos\gamma \geqslant \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \geqslant \lambda_a\lambda_b - \sqrt{1 - \lambda_a^2}\sqrt{1 - \lambda_b^2}$$

We see that our estimate is sharp since equality may occur.

*Step* 3. Using the estimate $\lambda_a \geqslant \min\{a, a^{-1}\}$, we want to know under what conditions for $a$ and $b$ the constant $\lambda(a, b)$ is positive. This happens if the constant $K$ defined by (7.51) is greater than 1. In fact, for $K > 1$ we have $\lambda(a, b) \geq K - 1$. ∎

## 8. – Hodge systems.

In Section 7, we looked at examples of nonlinear problems and related differential equations. There are, of course, more equations for differential forms on manifolds which play an important role in mathematics and physics. In this section, we describe a general setting in which the equations (7.17), (7.37), (7.40), (7.41) and (7.49) occur naturally as particular cases of the so called *Hodge system.*

Suppose we are given a bundle map $\mathfrak{A} : \wedge^l M \to \wedge^l M$. By a *solution* for $\mathfrak{A}$, we mean a pair $(\phi, \psi) \in \Gamma(\wedge^l M) \times \Gamma(\wedge^l M)$ of $l$-forms on $M$ which are coupled by the equations

$$\psi = \mathfrak{A}(\phi), \qquad d\phi = 0, \qquad d^* \psi = 0 \tag{8.1}$$

Without getting into the technicalities of the definition, let us assume in advance that $\mathfrak{A}$ is a homeomorphism on each fibre $\wedge_x^l M$, $x \in M$. We then refer to such equations as a *Hodge system.* A more rigorous definition will follow shortly.

System (8.1) is still underdetermined. As it was confirmed by the Hodge decompositions and well exemplified by the $\mathcal{L}^p$-projections, one may look for $\phi$ to be an exact form or for $\psi$ to be a coexact form. However, the system would be overdetermined if one required that both of these conditions hold simultaneously.

Next observe that it suffices to examine only the case

$$\phi = d\alpha, \qquad \psi \in \ker d^* \tag{8.2}$$

Since, if $\phi \in \ker d$ and $\psi = d^* \beta$, we simply replace equation (8.1) by its *Hodge dual* equation

$$\psi^* = \mathfrak{A}^*(\phi^*) \tag{8.3}$$

where

$$\mathfrak{A}^* = (-1)^{l(n-l)} * \mathfrak{A}^{-1} * : \wedge^{n-l} M \to \wedge^{n-l} M$$

with $\mathfrak{A}^{-1}$ denoting the inverse of $\mathfrak{A}$. The new unknowns are expressed in terms of $\phi$ and $\psi$ by the rules

$$\phi^* = (-1)^{l(n-l)} * \psi \in \operatorname{im} d \tag{8.4}$$

$$\psi^* = (-1)^{l(n-l)} * \phi \in \ker d^* \tag{8.5}$$

The question now arises as to how we should formulate the boundary conditions for the Hodge system (8.1)-(8.2). Of course, we are only entitled to formulate such conditions in terms of the tangential part of $\alpha$ or the normal part of $\psi$ on $\partial M$. However, a more careful analysis reveals that, just as with holomorphic functions, where one can prescribe the real or imaginary part on the boundary, one can prescribe $\alpha_T$ or $\psi_N$ on $\partial M$, but not both. Without too much detail, we are now in a position to state two *well posed* homogeneous boundary value problems for the Hodge system

$$\psi = \mathfrak{A}(\phi) \tag{8.6}$$

Specifically, the *Dirichlet Problem*:

$$\phi \in \operatorname{im} d_T \quad \text{and} \quad \psi \in \ker d^* \tag{8.7}$$

and the *Neumann Problem*:

$$\phi \in \operatorname{im} d \quad \text{and} \quad \psi \in \ker d_N^* \tag{8.8}$$

8.1. *Homogeneous systems and Hodge conjugate fields.* – Obviously, the natural exponent of the Lebesgue space in which to consider the solutions of (8.6) will depend on the growth of the bundle map $\mathfrak{A} : \wedge^l M \to \wedge^l M$. We shall first discuss homogeneous systems of linear growth, for which the $\mathcal{L}^2$-space is most natural. Suppose we are given a bundle map

$$\mathfrak{H}: \wedge^l M \to \wedge^l M \tag{8.9}$$

satisfying the *Lipschitz Condition*:

$$|\mathfrak{H}\xi - \mathfrak{H}\zeta| \leqslant K|\xi - \zeta| \tag{8.10}$$

the *Monotonicity Condition*:

$$\langle \xi - \zeta, \mathfrak{H}\xi - \mathfrak{H}\zeta \rangle \geqslant K^{-1}|\xi - \zeta|^2 \tag{8.11}$$

and the *Homogeneity Condition*:

$$\mathfrak{H}(t\xi) = t\mathfrak{H}\xi \tag{8.12}$$

for all $\xi, \zeta \in \Gamma(\wedge^l M)$, $t \in \mathbb{R}$, where $K \geqslant 1$ is a constant. Notice that conditions (8.10) and (8.11) can be concisely rephrased

$$|\xi - \zeta - \mathfrak{H}\xi + \mathfrak{H}\zeta| \leqslant k|\xi - \zeta + \mathfrak{H}\xi - \mathfrak{H}\zeta|$$

where $0 \leqslant k \leqslant 1$ depends only on $K \geqslant 1$.

In general, $\mathfrak{H}$ need not be linear except when $K = 1 (k = 0)$. In which case, $\mathfrak{H}$ is the identity on $\wedge^l M$. The Hodge system is then particularly simple

$$\phi = \psi, \quad \phi \in \ker d, \quad \psi \in \ker d^* \tag{8.13}$$

Now a harmonic field can be viewed as a pair $(\phi, \psi) \in \mathcal{L}^2_{\text{loc}}(\wedge^l M) \times \mathcal{L}^2_{\text{loc}}(\wedge^l M)$ of solutions to the system (8.13).

In analogy with harmonic conjugate functions on the complex plane, we refer to a pair $(\phi, \psi) \in \mathcal{L}^2_{\text{loc}}(\wedge^l M) \times \mathcal{L}^2_{\text{loc}}(\wedge^l M)$ of solutions to the homogeneous system

$$\psi = \mathfrak{H}(\phi), \quad d\phi = 0, \quad d^*\psi = 0 \tag{8.14}$$

as *$\mathfrak{H}$-conjugate fields*, or simply an *$\mathfrak{H}$-couple*.

Following the lead of the familiar $p$-harmonic operator, $1 < p < \infty$, we now consider a bundle map

$$\mathfrak{H}_p \colon \wedge^l M \to \wedge^l M$$

such that

$$|\mathfrak{H}_p \xi - \mathfrak{H}_p \zeta| \leq K(|\xi| + |\zeta|)^{p-2} |\xi - \zeta| \tag{8.15}$$

$$\langle \mathfrak{H}_p \xi - \mathfrak{H}_p \zeta, \xi - \zeta \rangle \geq K^{-1}(|\xi| + |\zeta|)^{p-2} |\xi - \zeta|^2 \tag{8.16}$$

$$\mathfrak{H}_p(t\xi) = t|t|^{p-2} \mathfrak{H}_p(\xi) \tag{8.17}$$

for all $\xi$, $\zeta \in \Gamma(\wedge^l M)$ and $t \in \mathbb{R}$, where $K \geq 1$ is a constant.

This apparent generality is easily managed in view of the fact that for every such $\mathfrak{H}_p$, there is a unique bundle map (8.9), so that

$$\mathfrak{H}_p \xi = \langle \xi, \mathfrak{H}\xi \rangle^{(p-2)/2} \mathfrak{H}\xi \tag{8.18}$$

Indeed, (8.18) defines a bijection between the class of Hodge systems of linear growth and the class of Hodge systems of $p$-th power growth. It is inverted according to the formula

$$\mathfrak{H}\xi = \langle \xi, \mathfrak{H}_p \xi \rangle^{(2-p)/p} \mathfrak{H}_p \xi \tag{8.19}$$

Hence, $\mathfrak{H}$ satisfies conditions (8.10), (8.11) and (8.12), as is easily verified.

The dual to $\mathfrak{H}_p$ is defined by $\mathfrak{H}_q = (-1)^{ln-l} * \mathfrak{H}_p^{-1} * : \wedge^{n-l} M \to \wedge^{n-l} M$. It is important to note that $\mathfrak{H}_q$ satisfies conditions similar to (8.15), (8.16) and (8.17) but with the Hölder conjugate exponent $q$ in place of $p$.

DEFINITION 8.1. – *A pair* $(\phi, \psi) \in \mathcal{L}^p_{\mathrm{loc}}(\wedge^l M) \times \mathcal{L}^q_{\mathrm{loc}}(\wedge^l M)$, $1/p + 1/q = 1$ *is called an* $\mathfrak{H}_p$-conjugate couple *in case*

$$\psi = \mathfrak{H}_p(\phi), \quad d\phi = 0 \ \textit{and} \ d^* \psi = 0 \tag{8.20}$$

REMARK 8.2. – *The Hodge systems will also be studied for pairs* $(\phi, \psi)$ *of class* $\mathcal{L}^{\lambda p}_{\mathrm{loc}}(\wedge^l M) \times \mathcal{L}^{\lambda q}_{\mathrm{loc}}(\wedge^l M)$ *with some* $\lambda \geq \max\{1/p, 1/q\}$ *and such pairs will be called* $\mathfrak{H}_p$*-couples as well.*

8.2. *Nonhomogeneous systems.* – The study of Hodge systems under homogeneous boundary conditions (8.7) and (8.8) involves no loss of generality as long as we do not require the bundle map $\mathfrak{A} : \wedge^l M \to \wedge^l M$ in (8.6) be homogeneous. For this reason, we focus our attention on nonhomogeneous Hodge systems of the form

$$\psi + \psi_0 = \mathfrak{H}_p(\phi + \phi_0) \tag{8.21}$$

where $(\phi_0, \psi_0)$ is a given pair from $\mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$, while the unknown pair $(\phi, \psi)$ satisfies either the conditions of the <u>*Dirichlet Problem*</u>:

$$\phi \in d\mathcal{W}^{d,p}_T(\wedge^l M) \quad \text{and} \quad \psi \in \mathcal{L}^q(\wedge^l M) \cap \ker d^* \tag{8.22}$$

or those of the *Neumann Problem*:

(8.23) $$\phi \in d\mathcal{W}^{d,p}(\wedge^l M) \quad \text{and} \quad \psi \in \mathcal{L}^q(\wedge^l M) \cap \ker d_N^*$$

REMARK 8.3. – *Note that* $dW_T^{1,p} = dW_T^{d,p}$ *and* $dW^{d,p} = dW^{d,p}$. *See Corollary* 5.3.

The existence and uniqueness of such solutions will be established by the method of monotone operators. Ultimately, we will examine the nonhomogeneous equation (8.21) with both the data $(\phi_0, \psi_0)$ and the solution $(\phi, \psi)$ belonging to $\mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L}^{\lambda q}(\wedge^l M)$. However, for $\lambda \neq 1$, the monotone operators argument breaks down.

We begin with some estimates for the case of the natural exponents $p$ and $q$ (i.e. $\lambda = 1$).

THEOREM 8.4. – *For each data* $(\phi_0, \psi_0) \in \mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$, *there exists a unique solution* $(\phi, \psi) \in \mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$ *to* (8.21) *subject to one of the conditions* (8.22) *or* (8.23). *In either case, we have a uniform estimate*

(8.24) $$\int_M (|\phi|^p + |\psi|^q) \leq C_p(K) \int_M (|\phi_0|^p + |\psi_0|^q)$$

In this way, we are led to two nonlinear operators

(8.25) $$\mathfrak{D}, \mathfrak{N}: \mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M) \to \mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$$

defined by the rule $(\phi_0, \psi_0) \to (\phi, \psi)$, with Dirichlet condition (8.22) for the operator $\mathfrak{D}$ and the Neumann condition (8.23) for $\mathfrak{N}$. These operators will play the same role in our nonlinear theory as the Riesz transforms do in the more familiar linear case. Our goal is to extend the Calderón-Zygmund theory of singular integrals to the operators $\mathfrak{D}$ and $\mathfrak{N}$. Their natural space is, of course, $\mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$ in which the boundedness of $\mathfrak{D}$ and $\mathfrak{N}$ is ensured by inequality (8.24). Notice that the constant $C_p(K)$, as might be expected, is independent of the manifold $M$. However, because of nonlinearity, boundedness does not necessarily mean continuity. Our next estimate is similar to (7.6) and establishes continuity of $\mathfrak{D}$ and $\mathfrak{N}$ in their natural space $\mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$.

THEOREM 8.5. – *Under the hypothesis of Theorem* 8.4, *if moreover,* $\alpha_0, \alpha \in \mathcal{L}^p(\wedge^l M)$ *and* $\beta_0, \beta \in \mathcal{L}^q(\wedge^l M)$ *solve the equation* $\mathfrak{H}_p(\alpha + \alpha_0) = \beta + \beta_0$ *and* $(\alpha, \beta)$ *verifies the same boundary conditions as* $(\phi, \psi)$, *then*

(8.26) $$\int_M (|\phi - \alpha|^p + |\psi - \beta|^q)$$
$$\leq C_p(K) \left[ \int_M (|\phi_0|^p + |\alpha_0|^p + |\psi_0|^q + |b_0|^q) \right]^{1-t} \left[ \int_M (|\phi_0 - \alpha_0|^p + |\psi_0 - \beta_0|^q) \right]^t$$

*where* $t = 1/(1 + |p - 2|)$.

The proof of Theorem 8.5 involves lengthy but rather elementary estimates. The essential technical details are the same as in the proof of Proposition 7.1, so we leave them to the reader.

PROOF OF THEOREM 8.4. – We give the proof only for the case of Dirichlet boundary conditions (8.22). The Neumann problem is similar.

We begin by recalling the exact projection

$$E_T : \mathcal{L}^q(\wedge^l M) \to d\mathcal{W}_T^{1,q}(\wedge^{l-1} M), \qquad \frac{1}{p} + \frac{1}{q} = 1$$

See Proposition 5.5 and formula (5.24). It follows from the Hodge decomposition that the kernel of $E_T$ consists of coclosed forms. More precisely,

$$\ker E_T = \mathcal{L}^q(\wedge^l M) \cap \ker d^* \tag{8.27}$$

Accordingly, equation (8.21) is equivalent to

$$E_T[\mathfrak{H}_p(\phi + \phi_0)] = E_T[\psi_0] \tag{8.28}$$

Next, we define a nonlinear operator

$$\mathfrak{F} : d\mathcal{W}_T^{1,p}(\wedge^{l-1} M) \to d\mathcal{W}_T^{1,q}(\wedge^{l-1} M)$$

according to

$$\mathfrak{F}\xi = E_T[\mathfrak{H}_p(\xi + \phi_0)] \tag{8.29}$$

for $\xi \in d\mathcal{W}_T^{1,p}(\wedge^{l-1} M)$.

It is important to observe that $\mathfrak{F}$ maps $d\mathcal{W}_T^{1,p}(\wedge^{l-1} M)$ into its dual space $d\mathcal{W}_T^{1,q}(\wedge^{l-1} M)$ (see Theorem 5.7), which makes it legitimate to apply the Browder-Minty theory of monotone operators. In this light, our proof falls naturally into three parts.

(i) $\underline{\mathfrak{F} \textit{ is continuous}}$ Indeed, by Proposition 5.5, $E_T$ is a continuous operator with norm $\leqslant C_q(M)$. See (5.13)

$$\begin{aligned}
\|\mathfrak{F}\xi - \mathfrak{F}\zeta\|_q &\leqslant C_q(M)\,\|\mathfrak{H}_p(\xi + \phi_0) - \mathfrak{H}_p(\zeta + \phi_0)\|_q \\
&\leqslant KC_q(M)\,\|(|\xi + \phi_0| + |\zeta + \phi_0|)^{p-2}\,|\xi - \zeta|\|_q \\
&\leqslant KC_q(M)\left[\int_M (|\xi + \phi_0| + |\zeta + \phi_0|)^{p-1}\,|\xi - \zeta|\right]^{1/q} \\
&\leqslant KC_q(M)(\|\xi\|_p + \|\zeta\|_p + 2\|\phi_0\|_p)^{(p-1)/q}\,\|\xi - \zeta\|_p^{1/q}
\end{aligned}$$

(ii) $\underline{\mathfrak{F} \textit{ is strictly monotone}}$ For $\xi,\ \zeta \in d\mathcal{W}_T^{1,p}(\wedge^{l-1} M)$ we identify $\mathfrak{F}\xi - \mathfrak{F}\zeta \in$

$d\mathcal{W}_T^{1,q}(\wedge^{l-1}M)$ with a linear functional on $d\mathcal{W}_T^{1,p}(\wedge^{l-1}M)$ (see Subsect. 5.4). It follows that

$$(\mathfrak{F}\xi - \mathfrak{F}\zeta)(\xi - \zeta) =$$

$$= \int_M \langle E_T[\mathfrak{H}_p(\xi + \phi_0) - \mathfrak{H}_p(\zeta + \phi_0)],\ \xi - \zeta\rangle = \int_M \langle \mathfrak{H}_p(\xi + \phi_0) - \mathfrak{H}_p(\zeta + \phi_0),\ \xi - \zeta\rangle$$

The latter equality follows from the observation that $E_T = I - E^* - H_T\colon \mathcal{L}^q(\wedge^l M) \to \mathcal{L}^q(\wedge^l M)$ and the form $\xi - \zeta$, being a member of $d\mathcal{W}_T^{1,p}(\wedge^{l-1}M)$, is orthogonal to the range of both $E^*$ and $H_T$. Now, thanks to monotonicity condition (8.16), we obtain

$$(\mathfrak{F}\xi - \mathfrak{F}\zeta)(\xi - \zeta) \geq K^{-1}\int_M (|\xi + \phi_0| + |\zeta + \phi_0|)^{p-2}|\xi - \zeta|^2 \geq 0 \tag{8.30}$$

Equality occurs if and only if $\xi = \zeta$ a.e.

(iii) $\mathfrak{F}$ *is coercive* Indeed, letting $\zeta = -\phi_0$ in (8.30) gives

$$(\mathfrak{F}\xi)(\xi + \phi_0) \geq K^{-1}\int_M |\xi + \phi_0|^p$$

since $\mathfrak{F}(-\phi_0) = 0$. In particular, $((\mathfrak{F}\xi)\,\xi)/\|\xi\|_p$ goes to infinity as $\|\xi\|_p \to \infty$.

Having verified these three conditions, we may apply the Browder-Minty theorem [Bro63] to see that the mapping

$$\mathfrak{F}\colon d\mathcal{W}_T^{1,p}(\wedge^{l-1}M) \to d\mathcal{W}_T^{1,q}(\wedge^{l-1}M) \tag{8.31}$$

is both one to one and surjective. It follows that equation (8.28) is uniquely solvable for $\phi \in d\mathcal{W}_T^{1,p}(\wedge^{l-1}M)$. Finally, we define $\psi \in \mathcal{L}^q(\wedge^{l-1}M)$ to satisfy (8.21). Equation (8.28) clearly forces $E_T\psi = 0$ and consequently $d^*\psi = 0$, as desired. Estimate (8.24) is then established by applying Theorem 8.5 to $\alpha = \beta = \alpha_0 = \beta_0 = 0$. ■

REMARK 8.6. – *It is important to observe that in the proof of Theorem* 8.4, *we did not use the homogeneity hypothesis* (8.17). *Indeed, we used only that* $\mathfrak{H}_p(0) = 0$, *which guarantees that* $\mathfrak{H}_p$ *maps* $\mathcal{L}^p(\wedge^l M)$ *into* $\mathcal{L}^q(\wedge^l M)$.

We are now in a position to prove the existence and uniqueness of decomposition (7.49).

COROLLARY 8.7. – *For* $1 < p < \infty$ *and* $a$, $b$ *satisfying both* $1/a + 1/b = p$ *and* $\min(a^2, a^{-2}) + \min(b^2, b^{-2}) > 1$, *we have that each* $\omega \in \mathcal{L}^p(\wedge^l M)$ *can be uniqely decomposed according to*

$$\omega = \phi^a + \psi^b \tag{8.32}$$

*where* $\phi \in \mathcal{L}^{ap}(\wedge^l M)$ *and* $\psi \in \mathcal{L}^{bp}(\wedge^l M)$ *are subject to one of the following boundary constraints*

(i) $\phi \in \operatorname{im} d_T$ *and* $\psi \in \ker d^*$

(ii) $\phi \in \operatorname{im} d$ *and* $\psi \in \ker d_N^*$

(iii) $\phi \in \ker d$ *and* $\psi \in \operatorname{im} d_N^*$

(iv) $\phi \in \ker d_T$ *and* $\psi \in \operatorname{im} d^*$

PROOF. – It suffices to establish existence and uniqueness of the decomposition under one of the constraints (i) or (ii), since the remaining cases are Hodge star dual to these.

Let us first compute $\psi$ from (8.32)

$$\psi = (\omega - \phi^a)^{1/b}$$

Recall the bundle map $A\colon \wedge^l M \to \wedge^l M$ given by $A\xi = (\xi^a - \omega)^{1/b}$ (see (7.50)). Hence,

$$A\phi + \omega^{1/b} = -\psi + \omega^{1/b}$$

We may express this as a Hodge system

$$\mathfrak{H}_s \phi = -\psi + \psi_0$$

where $s = 1 + a/b = ap$ and the bundle map $\mathfrak{H}_s\colon \mathcal{L}^s(\wedge^l M) \to \mathcal{L}^r(\wedge^{n-l} M)$, $1/s + 1/r = 1$ is defined by

$$\mathfrak{H}_s \xi = A\xi + \omega^{1/b}$$

Here, the given form $\psi_0 = \omega^{1/b}$ belongs to $\mathcal{L}^r(\wedge^l M)$ while the unknown form $\phi$ belongs to $\mathcal{L}^s(\wedge^l M)$. We use Proposition 7.7 to verify that $\mathfrak{H}_s$ satisfies hypothesis (8.15) and (8.16) with $s$ in place of $p$. Finally it is clear that $\mathfrak{H}_s(0) = 0$ and so Corollary 8.7 follows quickly from Theorem 8.4 and Remark (8.6). ■

8.3. *Estimates beyond the natural exponent.* – Up to now, we have considered the Hodge system (8.21) only in its natural space $\mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$. This led us to a definition of two nonlinear operators $\mathfrak{D}$ and $\mathfrak{N}$ corresponding to the Dirichlet and Neumann boundary conditions, see (8.25). In order to provide a baseline for the analysis of these operators, we first look at the nonhomogeneous Cauchy-Riemann system on $M$. That is, $\mathfrak{H}_p = \mathrm{Id}\colon \wedge^l M \to \wedge^l M$.

$$\psi + \psi_0 = \phi + \phi_0 \tag{8.33}$$

The solutions of the corresponding boundary value problems are easily found from decompositions (5.14) by means of singular integrals. Precisely,

$$\phi = E_T(\psi_0 - \phi_0), \qquad \psi = (E^* + H_T)(\phi - \psi_0) \tag{8.34}$$

for the Dirichlet problem, and

$$\phi = E(\psi_0 - \phi_0), \qquad \psi = (E_N^* + H_N)(\phi - \psi_0) \tag{8.35}$$

for the Neumann problem. In either case, due to the boundedness of the above projections in $\mathcal{L}^{2\lambda}(\wedge^l M)$, we have

$$\int_M (|\phi|^2 + |\psi|^2)^\lambda \leq C(\lambda, M) \int_M (|\phi_0|^2 + |\psi_0|^2)^\lambda, \qquad \lambda > \frac{1}{2} \tag{8.36}$$

Now, one may ask whether the nonlinear operators $\mathfrak{D}$ and $\mathfrak{N}$, originally defined on $\mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$ by (8.25), actually extend to $\mathcal{L}^{p'}(\wedge^l M) \times \mathcal{L}^{q'}(\wedge^l M)$, with some $p'$ and $q'$ different from $p$ and $q$. Because of the homogeneity hypothesis (8.17), the correct spaces in which such extensions might take place are of the type $\mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L}^{\lambda q}(\wedge^l M)$ for $\lambda > \max\{1/p, 1/q\}$. Naturally, the factor $\lambda = p'/p = q'/q$ will depend on the ellipticity constant $K$.

We come now to the central estimate for Hodge systems.

THEOREM 8.8. – *Given any nonhomogeneous Hodge system* (8.21), *there exist positive numbers*

$$a_p(M, K) < 1 < b_p(M, K) \tag{8.37}$$

*and a constant* $C_p(M, K)$ *such that if* $(\phi_0, \psi_0)$ *and* $(\phi, \psi)$ *belong to* $\mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L}^{\lambda q}(\wedge^l M)$ *with*

$$a_p(M, K) \leq \lambda \leq b_p(M, K) \tag{8.38}$$

*and solve either the Dirichlet or Neumann problem for* (8.21), *then*

$$\int_M (|\phi|^p + |\psi|^q)^\lambda \leq C_p(M, K) \int_M (|\phi_0|^p + |\psi_0|^q)^\lambda \tag{8.39}$$

The following retrospective comments about inequality (8.39) are in order. One well known approach to estimates above the natural exponent for nonlinear PDEs ($\lambda > 1$ in our case) is to use Gehring's Lemma [Geh73] on reverse Holder inequalities. We shall not follow this idea here since it fails for $\lambda < 1$. Another interesting method, this time for estimating «very weak solutions» ($\lambda < 1$) of nonlinear PDEs has been proposed by J. Lewis [Lew93]. His method involves looking at maximal functions of the gradient. In our case, however, the partials of $\phi$ and $\psi$ are involved only via $d\phi$ and $d^*\psi$. Thus, we have no control of the full gradient of $\phi$ and $\psi$. The methods most useful for such equations have been developed in [Iwa92] and [IS93]; see also [Str95]. It is our intent to demonstrate these ideas here.

PROOF. – Suppose $\phi_0$, $\phi \in \mathcal{L}^{\lambda p}(\wedge^l M)$ and $\psi_0$, $\psi \in \mathcal{L}^{\lambda q}(\wedge^l M)$ verify the Hodge system

$$\mathfrak{H}_p(\phi + \phi_0) = \psi + \psi_0, \qquad d\phi = d^*\psi = 0 \tag{8.40}$$

where $\phi$ is subject to the Dirichlet type boundary condition

$$\phi \in d\,\mathcal{W}_T^{d,\lambda p}(\wedge^{l-1} M) \tag{8.41}$$

We may argue similarly for the Neumann problem. Although the precise values of the numbers $a_p(M, K)$ and $b_p(M, K)$ will be specified later, we assume in advance that

$$\frac{1}{2} \leqslant \max\left\{\frac{1}{p}, \frac{1}{q}\right\} \leqslant a_p(M, K) < 1 < b_p(M, K) \leqslant 2$$

If $\lambda = 1$, one can compute the inner product of both sides of (8.40) with the form $\phi$. Since $d\mathcal{W}_T^{d,p}(\wedge^{l-1} M)$ is orthogonal to $\mathcal{L}^q(\wedge^l M) \cap \ker d^*$, we obtain

$$(8.42) \qquad (\phi, \mathfrak{H}_p(\phi + \phi_0)) = (\phi, \psi_0)$$

This, in view of conditions (8.15) and (8.16) leads immediately to estimate (8.39) with $\lambda = 1$. Disappointingly, for $\lambda \neq 1$, this simple argument no longer works. In order to arrive at integrals of the type $\int_M |\phi|^{\lambda p}$ and $\int_M |\psi|^{\lambda q}$, we multiply (8.40) by the form

$$|\phi|^{\lambda p - p}\phi \in \mathcal{L}^{\lambda q/(\lambda q - 1)}(\wedge^l M)$$

Unfortunately, integrating over $M$ will not annihilate the unknown form $\psi$. An easy computation leads to the following pointwise inequality

$$\frac{|\phi|^{\lambda p}}{2K} - C(p, K)|\phi_0|^{\lambda p} \leqslant \langle \mathfrak{H}_p(\phi + \phi_0), |\phi|^{\lambda p - p}\phi\rangle =$$

$$= \langle \psi_0, |\phi|^{\lambda p - p}\phi\rangle + \langle \psi, |\phi|^{\lambda p - p}\phi\rangle \leqslant C(p, K)|\psi_0|^{\lambda q} + \frac{|\phi|^{\lambda p}}{4K} + \langle \psi, |\phi|^{\lambda p - p}\phi\rangle.$$

Hence, integrating over $M$ yields

$$(8.43) \qquad \|\phi\|_{\lambda p}^{\lambda p} \leqslant 4KC(p, K)(\|\phi_0\|_{\lambda p}^{\lambda p} + \|\psi_0\|_{\lambda q}^{\lambda q}) + 4K(\psi, |\phi|^{\lambda p - p}\phi)$$

Notice that the last term was not present in the $\lambda = 1$ case because $\phi$ and $\psi$ were orthogonal. Fortunately, we do not actually need this term to vanish. As we shall see, this term will be absorbed by the left hand side if $\lambda$ is sufficiently close to 1. The key is the following inequality

$$(8.44) \qquad (\psi, |\phi|^{\lambda p - p}\phi) \leqslant C_p(M)|\lambda - 1|\,\|\phi\|_{\lambda p}^{\lambda p - p + 1}\|\psi\|_{\lambda q}$$

which holds for arbitrary forms $\phi \in d\mathcal{W}_T^{d,\lambda p}(\wedge^l M)$ and $\psi \in \mathcal{L}^{\lambda q}(\wedge^l M) \cap \ker d^*$. Having this inequality, we procede as follows. From (8.40) and conditions (8.15) and (8.16), we estimate $\|\psi\|_{\lambda q}$ in terms of $\psi_0$, $\phi$ and $\phi_0$. This, combined with (8.43) and (8.44) yields

$$(8.45) \qquad \int_M (|\phi|^{\lambda p} + |\psi|^{\lambda q}) \leqslant |\lambda - 1|C(p, K, M)\int_M |\phi|^{\lambda p} + C(p, K, M)\int_M (|\phi_0|^p + |\psi_0|^q))^\lambda$$

We shall have established inequality (8.39) if we set the numbers $a_p(M, K) < 1 < b_p(M, K)$ close enough to 1 so that

$$(8.46) \qquad |\lambda - 1|C(p, K, M) < 1$$

for all $\lambda \in [a_p(M, K), b_p(M, K)]$.

Thus, we are left only with the task of proving (8.44). Note that Hölder's inequality gives only the rough estimate

$$(\psi, |\phi|^{\lambda p - p}\phi) \leq \|\phi\|_{\lambda p}^{\lambda p - p + 1} \|\psi\|_{\lambda q} \tag{8.47}$$

regardless of the assumptions that $\phi \in \operatorname{im} d_T$ and $\psi \in \ker d^*$. However, for $\lambda = 1$ these assumptions ensure that $(\psi, \phi) = 0$, proving (8.44). As might be expected, the general case will follow by an interpolation between (8.47) and $(\psi, \phi) = 0$. To this effect, we decompose

$$|\phi|^{\lambda p - p}\phi = d\alpha + \gamma, \qquad \alpha \in \mathcal{W}^{1, \lambda q/(\lambda q - 1)}(\wedge^{l-1} M), \quad d^*\gamma = 0 \tag{8.48}$$

where $d\alpha = E_T(|\phi|^{\lambda p - p}\phi)$ and $\gamma = (E^* + H_T)(|\phi|^{\lambda p - p}\phi)$; see decomposition (5.14). For abbreviation we introduce a bounded linear operator

$$T = E^* + H_T \colon \mathcal{L}^r(\wedge^l M) \to \mathcal{L}^r(\wedge^l M) \cap \ker d^* \tag{8.49}$$

where $r$ can be any number greater than 1. It is important to notice that

$$T\phi = 0, \quad (\text{since } \phi \in d\,\mathcal{W}_T^{d, \lambda p} \text{ with } \lambda p > 1) \tag{8.50}$$

Another useful observation is that

$$(\psi, d\alpha) = 0, \quad (\text{since } \psi \in \mathcal{L}^{\lambda q}(\wedge^l M) \cap \ker d^* \text{ and } \alpha \in \mathcal{W}_T^{1, \lambda q/(\lambda q - 1)}(\wedge^{l-1} M)) \tag{8.51}$$

Hence,

$$(\psi, |\phi|^{\lambda p - p}\phi) = (\psi, \gamma) \leq \|\psi\|_{\lambda q} \|\gamma\|_{\lambda q/(\lambda q - 1)} = \|\psi\|_{\lambda q} \|T(|\phi|^{\lambda p - p}\phi)\|_{\lambda q/(\lambda q - 1)} \tag{8.52}$$

It is a simple matter of boundedness of the operator $T \colon \mathcal{L}^r(\wedge^l M) \to \mathcal{L}^r(\wedge^l M)$, $r = \lambda q/(\lambda q - 1)$, that

$$\|T(|\phi|^{\lambda p - p}\phi)\|_{\lambda q/(\lambda q - 1)} \leq C_p(M) \|\phi\|_{\lambda p}^{\lambda p - p + 1}$$

But this estimate is not sufficient to prove inequality (8.39). We need to improve the constant to show that

$$\|T(|\phi|^{\lambda p - p}\phi)\|_{\lambda q/(\lambda q - 1)} \leq |\lambda - 1| C(p, M) \|\phi\|_{\lambda p}^{\lambda p - p + 1} \tag{8.53}$$

for $\lambda$ sufficiently close to 1. This refined estimate holds only when $T\phi = 0$ and can be obtained by using the following stability property of the kernel of the operator $T$.

PROPOSITION 8.9. – *Suppose $T \colon \mathcal{L}^r(\wedge^l M) \to \mathcal{L}^r(\wedge^l M)$ is a bounded linear operator for all $1 < r < \infty$. Denote its norm by $\|T\|_r$. Then for every $\phi \in \mathcal{L}^s(\wedge^l M)$ with $T\phi = 0$, we have*

$$\|T(|\phi|^{\varepsilon}\phi)\|_{s/(1+\varepsilon)} \leq |\varepsilon| C \|\phi\|_s^{1+\varepsilon} \tag{8.54}$$

*where*

$$C = \frac{2s(b-a)}{(s-a)(b-s)}(\|T\|_a + \|T\|_b)$$

*provided* $1 < a < s < b$ *and* $s/b - 1 < \varepsilon < s/a - 1$.

This result is a reformulation of Proposition 1 from [IS93] in terms of differential forms. The idea of the proof is based on complex interpolation and can be traced back to [RW83]. We set $\varepsilon = (\lambda - 1)p$, $s = \lambda p$, $a = (p+3)/4$ and $b = sp$, where

$$1 - \frac{1}{4pq} < \lambda < 1 + \frac{1}{4pq}$$

A trivial verification shows that $s - a > (p-1)/2$, $b - s > 1$ and $s/b - 1 < \varepsilon < s/a - 1$. This yields inequality (8.53) and completes the proof of Theorem 8.8. ■

8.4. *Caccioppoli estimate.* – In nonlinear PDE theory, the interior estimates of Caccioppoli type are critical for proving regularity properties of the solutions. We will also find them useful in proving the removability of singularities for Hodge conjugate fields; see Subsect. 9.3.

The Sobolev exponents in our estimates will be independent of the open set on which we choose to consider a local solution. To keep this clear, we introduce the following

DEFINITION 8.10. – *Given* $1 < p < \infty$ *and* $K \geq 1$, *we denote by*

$$a = a_p(\mathcal{R}, K), \qquad b = b_p(\mathcal{R}, K)$$

*where* $a_p(\mathcal{R}, K)$ *andfl* $b_p(\mathcal{R}, K)$ *are those determined by Theorem* 8.8 *with* $M$ *set to* $\mathcal{R}$.

Let $\Omega$ be an arbitrary open subset of $\mathcal{R}$. We shall examine the nonhomogeneous Hodge system

$$\mathfrak{H}_p(\phi + \phi_0) = \psi + \psi_0, \qquad d\phi = d^*\psi = 0 \tag{8.55}$$

where both pairs $(\phi_0, \psi_0)$ and $(\phi, \psi)$ belong to $\mathcal{L}^{\lambda p}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{\lambda q}_{\mathrm{loc}}(\wedge^l \Omega)$ with

$$a \leqslant \lambda \leqslant b \tag{8.56}$$

The bundle map $\mathfrak{H}_p \colon \wedge^l \Omega \to \wedge^l \Omega$ verifies conditions (8.15), (8.16) and (8.17).

In order to formulate Caccioppoli's inequality, it is necessary to express $\phi$ and $\psi$ in terms of their potential forms $\xi$ and $\zeta$, respectively. This means

$$\phi = d\xi, \qquad \xi \in \mathcal{W}^{d,\lambda p}_{\mathrm{loc}}(\wedge^{l-1}\Omega) \tag{8.57}$$

$$\phi = d^*\zeta 5, \qquad \zeta \in \mathcal{W}^{d^*,\lambda p}_{\mathrm{loc}}(\wedge^{l+1}\Omega) \tag{8.58}$$

This introduces no loss of generality as long as $\Omega$ is chosen to be cohomologically trivial (e.g. a coordinate neighborhood of a point of $\mathcal{R}$).

THEOREM 8.11 (Caccioppoli Type Estimate). – *Under the definitions above, we have*

$$(8.59)\qquad \int_\Omega (|\chi^q d\xi|^p + |\chi^p d^*\zeta|^q)^\lambda \le C_p(K) \int_\Omega (|\chi^q \phi_0|^p + |\chi^p \psi_0|^q + |\xi \wedge d\chi^q|^p + |{}^*\zeta \wedge d\chi^p|^q)^\lambda$$

*for all nonnegative test functions* $\chi \in C_0^\infty(\Omega)$

It is worth pointing out that the constant $C_p(K)$ is independent of $\Omega$ and $\lambda$.

PROOF. – We begin by multiplying the Hodge system by $\chi^p$. In view of the homogeneity property of $\mathfrak{H}_p$, see (8.17), we obtain

$$\mathfrak{H}_p(\chi^q d\zeta + \chi^q \phi_0) = \chi^p d^* \zeta + \chi^p \psi_0$$

This in turn can be viewed as a system on the reference manifold $\mathcal{R}$.

$$(8.60)\qquad \widetilde{\mathfrak{H}}_p(\widetilde{\phi} + \widetilde{\phi}_0) = \widetilde{\psi} + \widetilde{\psi}_0$$

where

$$(8.61)\qquad \begin{cases} \widetilde{\phi} = d(\chi^q \xi) \in d\,\mathcal{W}^{d,\lambda p}(\wedge^{l-1}\mathcal{R}) \\ \widetilde{\psi} = d^*(\chi^q \zeta) \in d^*\,\mathcal{W}^{d^*,\lambda q}(\wedge^{l+1}\mathcal{R}) \\ \widetilde{\phi}_0 = \chi^q \phi_0 - d\chi^q \wedge \xi \in d\mathcal{L}^{\lambda p}(\wedge^l \mathcal{R}) \\ \widetilde{\psi}_0 = \chi^p \psi_0 - (-1)^{n(n-l)} * [d\chi^p \wedge * \zeta] \in d\mathcal{L}^{\lambda q}(\wedge^l \mathcal{R}) \end{cases}$$

It is immaterial which extension $\widetilde{\mathfrak{H}}_p\colon \wedge^l \mathcal{R} \to \wedge^l \mathcal{R}$ of the map $\mathfrak{H}_p\colon \wedge^l \Omega \to \wedge^l \Omega$ we choose as long as $\widetilde{\mathfrak{H}}_p$ satisfies the same conditions (8.15), (8.16) and (8.17). For instance, see formula (9.3). Applying Theorem 8.8, we obtain

$$(8.61)\qquad \int_{\mathcal{R}} (|\widetilde{\phi}|^p + |\widetilde{\psi}|^q)^\lambda \le C(p, K) \int_{\mathcal{R}} (|\widetilde{\phi}_0|^p + |\widetilde{\psi}_0|^q)^\lambda$$

The only issue which remains is that of replacing the tilde forms by $\xi$, $\zeta$, $\phi_0$ and $\psi_0$ as shown in (8.61). The rest of the calculation is straightforward. ■

## 9. – Regularity theorems for Hodge systems.

Roughly speaking, our goal in this section is to show that the solution $(\phi, \psi)$ of the Hodge system

$$(9.1)\qquad \mathfrak{H}_p(\phi + \phi_0) = \psi + \psi_0, \qquad d\phi = d^*\psi = 0$$

enjoys the same degree of integrability as does the data $(\phi_0, \psi_0)$. The system will be studied on various open regions $\Omega \subset \mathcal{R}$ of the reference manifold $\mathcal{R}$. However, the range of the integrability exponents will be independent of $\Omega$. To keep things clear, we shall use, for the rest of this chapter, only the characteristic numbers for the reference manifold $\mathcal{R}$. That is

$$a = a_p(\mathcal{R}, K) < 1 < b_p(\mathcal{R}, K) = b \tag{9.2}$$

as given in Definition 8.10. In particular, these numbers depend only on the structural constants of the Hodge system (9.1) but not on the region $\Omega$. Neither the hypothesis nor the conclusions of our theorems will be affected by extending $\mathfrak{H}_p : \Gamma(\wedge^l \Omega) \to \to \Gamma(\wedge^l \Omega)$ to sections over $\mathcal{R}$ as long as the conditions (8.15), (8.16) and (8.17) remain valid for the extension.

One such extension is given by

$$\chi_\Omega \mathfrak{H}_p(\xi) + (1 - \chi_\Omega)|\xi|^{p-2}\xi, \qquad \xi \in \Gamma(\wedge^l \mathcal{R}) \tag{9.3}$$

where $\chi_\Omega$ stands for the characteristic function of the region $\Omega$. In Subsect. 9.2, we shall discuss other possible extensions; see formula (9.39). For notational convenience, we simply assume that $\mathfrak{H}_p$ is already defined on $\mathcal{R}$ and satisfies the conditions (8.15), (8.16) and (8.17) therein.

9.1. *Interior regularity.* – Let $\Omega$ be an open subset of $\mathcal{R}$. We begin with the following preliminary result which illustrates the integrability improvement property for local solutions of a Hodge system. Recall $a$ and $b$ from Definition 8.10.

THEOREM 9.1. – *Given* $(\phi, \psi) \in \mathcal{L}^{ap}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{aq}_{\mathrm{loc}}(\wedge^l \Omega)$ *which solves the nonhomogeneous Hodge system* (9.39) *with data* $(\phi_0, \psi_0) \in \mathcal{L}^{\lambda p}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{\lambda q}_{\mathrm{loc}}(\wedge^l \Omega)$ *for some* $\lambda \in [a, b]$, *we have*

$$(\phi, \psi) \in \mathcal{L}^{\lambda p}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{\lambda q}_{\mathrm{loc}}(\wedge^l \Omega)$$

Before trying to solve this problem, it may be helpful to reduce it to special cases. First notice that it introduces no loss of generality to assume that

$$\phi = d\xi \quad \text{for some } \xi \in \mathcal{W}^{1,ap}(\wedge^{l-1}\Omega) \tag{9.4}$$

and

$$\psi = d^*\zeta \quad \text{for some } \zeta \in \mathcal{W}^{1,aq}(\wedge^{l+1}\Omega) \tag{9.5}$$

Indeed, since the result is local, we may take $\Omega$ to be a regular region of trivial cohomology (i.e. the relative cohomology groups $\mathcal{H}_T(\wedge^l \Omega)$ and $\mathcal{H}_N(\wedge^l \Omega)$ are zero. We then use the Hodge decomposition (5.17)

$$\phi = da + d^*\beta + h, \qquad h_N = \beta_N = 0 \tag{9.6}$$

where $a \in \mathcal{W}^{1,ap}(\wedge^{l-1}\Omega)$ and we recall that $ap > 1$. Since $\phi$ was closed, we find at once that $d^*\beta + h$ is a harmonic field with normal part vanishing on $\partial\Omega$. Accordingly, $\phi = = da$, as desired. We argue similarly for (9.6). In particular, by the Sobolev Imbedding Theorem, we obtain

$$\xi \in \mathcal{L}^{a'p}_{\mathrm{loc}}(\wedge^{l-1}\Omega) \quad \text{and } \zeta \in \mathcal{L}^{a'q}_{\mathrm{loc}}(\wedge^{l+1}\Omega) \tag{9.7}$$

for all $a' > a$ sufficiently close to $a$. One further assumption we can make is that

$$\xi,\ \zeta,\ \phi_0 \text{ and } \psi_0 \text{ are compactly supported in } \Omega \tag{9.8}$$

For, if not, we take an arbitrary test function $0 \leqslant \eta \in C_0^\infty(\Omega)$ and consider new equations

$$\mathcal{H}_p(d\tilde{\xi} + \tilde{\phi}_0) = d^* \tilde{\zeta} + \tilde{\psi}_0$$

for $\tilde{\xi} = \chi^q \xi$ and $\tilde{\zeta} = \chi^p \zeta$, where $(\tilde{\phi}_0, \tilde{\psi})$ can be considered as new data which is given by formulas similar to (8.61)

$$\tilde{\phi}_0 = \eta^q \phi_0 - d\eta^q \wedge \xi \in \mathcal{L}^{a'p}(\wedge^l \mathcal{R})$$

$$\tilde{\psi}_0 = \eta^p \psi_0 - (-1)^{n-nl} * [d\eta^p \wedge * \xi] \in \mathcal{L}^{a'q}(\wedge^l \mathcal{R})$$

The new Hodge system can be viewed as a system on the entire reference manifold $\mathcal{R}$. Thus, the Caccioppoli inequality applies for $\lambda = a$. Let $\chi$ be an arbitrary nonnegative function of class $C_0^\infty(\Omega)$. Inequality (8.59) yields

$$\int_\Omega (\chi^{p+q} |\phi|^p + \chi^{p+q} |\psi|^q)^a \leqslant C_p(K, \mathcal{R}) \int_\Omega (\chi^{p+q} |\phi_0|^p + \chi^{p+q} |\psi_0|^q)^a$$
$$+ C_p(K, \mathcal{R}) \int_\Omega (\chi^q |\nabla\chi|^p |x - \xi_0|^p + \chi^p |\nabla \chi|^q |z - \zeta_0|^q)^a \tag{9.9}$$

This inequality holds for arbitrary $\xi_0 \in \mathcal{L}^{ap}(\wedge^l \Omega') \cap \ker d$ and $\zeta_0 \in \mathcal{L}^{aq}(\wedge^l \Omega') \cap \cap \ker d^*$, where $\Omega'$ is any open set containing the support of the test function $\chi$. From now on, we no longer need to appeal to the Hodge system. Inequality (9.9) alone will imply the desired higher integrability. Namely,

$$(\phi, \psi) \in \mathcal{L}^{a'p}(\wedge^l \Omega) \times \mathcal{L}^{a'q}(\wedge^l \Omega) \tag{9.10}$$

for some $a' > a$. For this, we need to reduce the problem to the Euclidean space $\mathbb{R}^n$. Without loss of generality, we may think of $\Omega$ as a coordinate neighborhood in $\mathcal{R}$. Let $f\colon \mathbb{R}^n \to \Omega$ be a diffeomorphism from $\mathbb{R}^n$ onto $\Omega$. We then pullback the forms in (9.9) to $\mathbb{R}^n$ by the rules

$$\overline{\phi} = f^\#(\phi), \quad \overline{\phi_0} = f^\#(\phi_0), \quad \overline{\xi} = f^\#(\xi), \quad \overline{\xi_0} = f^\#(\xi_0)$$

$$\overline{\psi} = f_\#(\psi), \quad \overline{\psi_0} = f_\#(\psi_0), \quad \overline{\zeta} = f_\#(\zeta), \quad \overline{\zeta_0} = f_\#(\zeta_0)$$

See Sect. 2 for the definition of the pullback operations $f^\#, f_\#\colon \Gamma(\wedge \Omega) \to \Gamma(\wedge \mathbb{R}^n)$. In view of the commutation formulas $d \circ f^\# = f^\# \circ d$ and $d^* \circ f_\# = f_\# \circ d^*$, inequality (9.9) reduces to a similar one on $\mathbb{R}^n$ for the forms defined above. Of course, the constant $C_p(K, \mathcal{R})$ will change. It will depend on $f$ and the support of the forms involved there, but will not depend on the test function $\chi$. Instead of introducing new symbols, we simply assume that $\Omega = \mathbb{R}^n$.

PROOF OF THEOREM 9.1. – Our proof is divided into four steps.

*Step* 1 (Reverse Hölder Inequality).

Fix an arbitrary cube $Q \subset \mathbb{R}^n$. We denote by $2Q$ the cube with the same center as $Q$

but dilated to twice $Q$'s size. Then there is a nonnegative function $\chi \in C_0^\infty(2Q)$ such that

$$\begin{cases} \chi \leqslant 1 & \text{(with equality on } Q) \\ |\nabla\chi| \leqslant C(n)\,|Q|^{-1/n} & \text{(everywhere)} \end{cases} \tag{9.11}$$

We refer to $\chi$ as a cutoff function for the cubes $Q \subset 2Q$. Now estimate (9.9) yields

$$\int_Q (|\phi|^p + |\psi|^q)^a \leqslant C(n, p, K) \int_{2Q} (|\phi_0|^p + |\psi_0|^q)^a \tag{9.12}$$
$$+ C(n, p, K) \int_{2Q} (|Q|^{-p/n}\,|\xi - \xi_0|^p + |Q|^{-q/n}\,|\zeta - \zeta_0|^q)^a$$

Here, we are still free to choose the forms $\xi_0$ and $\zeta_0$ on the cube $2Q$, as long as

$$\xi_0 \in \mathcal{L}^{ap}(\wedge^{l-1} 2Q) \cap \ker d \quad \text{ and } \zeta_0 \in \mathcal{L}^{aq}(\wedge^{l+1} 2Q) \cap \ker d^*$$

Many of the estimates for differential forms on Euclidean space are classical. However, some of them are not well known. For the convenience of the reader, we rephrase Corollary 4.2 from [IL93].

LEMMA 9.2 (Poincare-Sobolev Lemma). – *Let $\mathcal{Q}$ be a cube in $\mathbb{R}^n$. Suppose $\xi \in \mathcal{W}^{1,r}(\wedge^{l-1}\mathcal{Q})$ and $\zeta \in \mathcal{W}^{1,s}(\wedge^{l+1}\mathcal{Q})$, where $1 < r$, $s < n$. Then there exist $\xi_0 \in \mathcal{L}^r(\wedge^{l-1}\mathcal{Q}) \cap \ker d$ and $\zeta_0 \in \mathcal{L}^s(\wedge^{l+1}\mathcal{Q}) \cap \ker d^*$ such that*

$$\|\xi - \xi_0\|_{nr/(n-r)} \leqslant C(r, n)\,\|d\xi\|_r \tag{9.13}$$

$$\|\zeta - \zeta_0\|_{ns/(n-s)} \leqslant C(s, n)\,\|d^*\zeta\|_s \tag{9.14}$$

It is important to notice that the constants $C(r, n)$ and $C(s, n)$ are independent of the cube $\mathcal{Q}$. Let us set

$$m = 1 - \frac{1}{n} + \frac{a}{n}\min\{p, q\} \tag{9.15}$$

Hence,

$$1 < m < \min\{ap, aq\}$$

We shall apply the lemma to the cube $\mathcal{Q} = 2Q$ and with exponents $r = ap/m$ and $s = aq/m$. By Hölder's inequality, we obtain

$$|\mathcal{Q}|^{-ap/n} \int_{\mathcal{Q}} |\xi - \xi_0|^{ap} \leqslant |\mathcal{Q}|^{1-m} \left( \int_{\mathcal{Q}} |\xi - \xi_0|^{nr/(n-r)} \right)^{((n-r)\,ap)/nr}$$
$$\leqslant C(n, p)\,|\mathcal{Q}|^{1-m} \left( \int_{\mathcal{Q}} |d\xi|^{ap/m} \right)^m$$

Similarly,

$$|\mathcal{D}|^{-aq/n}\int_{\mathcal{D}}|\zeta-\zeta_0|^{aq}\leqslant C(n,p)\,|\mathcal{D}|^{1-m}\left(\int_{\mathcal{D}}|d\zeta|^{aq/m}\right)^m$$

Combining this with (9.12) leads to the estimate

$$\fint_Q(|\phi|^p+|\psi|^q)^a\leqslant C(n,p,K)\fint_{2Q}(|\phi_0|^p+|\psi_0|^q)^a \tag{9.16}$$

$$+C(n,p,K)\left[\fint_{2Q}(|\phi|^p+|\psi|^q)^{a/m}\right]^m$$

Here and in the sequel, we use the symbol $\fint_{\mathcal{D}}=1/|\mathcal{D}|\int_{\mathcal{D}}$ for the integral mean over a cube $\mathcal{D}\subset\mathbb{R}^n$. Finally, it will be convenient to set

$$F=(|\phi|^p+|\psi|^q)^{a/m},\qquad F_0=(|\phi_0|^p+|\psi_0|^q)^{a/m} \tag{9.17}$$

Then, (9.16) yields the so-called *reverse Hölder inequalities*

$$\left(\fint_Q F^m\right)^{1/m}\leqslant A\fint_{2Q}F+B\left(\fint_{2Q}F_0^m\right)^{1/m} \tag{9.18}$$

for all cubes $Q\subset\mathbb{R}^n$ with the constants $A$ and $B$ independent of the cube $Q$. Note that $F\in\mathcal{L}^m(\mathbb{R}^n)$ and $F_0\in\mathcal{L}^k(\mathbb{R}^n)$ for all $k>m$ sufficiently close to $m$ $(k=a'\,m/a)$ and both $F$ and $F_0$ are compactly supported. Usually, in estimates like (9.18), the constant $A$ is greater than $1$. However, the case $A<1$ is worth discussing as well. By the Lebesgue Differentiation Theorem, one may pass to the limit as $Q$ shrinks to a point. This leads to a pointwise estimate $F\leqslant AF+BF_0$. Hence, $F\leqslant(B/(1-A))F_0$. In particular, $F$ is integrable with the same power as $F_0$. This argument obviously fails when $A\geqslant 1$. Nevertheless, it turns out that up to a certain degree, the power of integrability of $F$ remains as large as that of $F_0$. Our next step considers this case. We shall present a new approach to Gehring's Lemma [Geh73] by carefully examining the constants involved in the familiar maximal inequalities.

*Step* 2 (Maximal Inequalities).

For $F\in\mathcal{L}^m_{\mathrm{loc}}(\mathbb{R}^n)$, $m\geqslant 1$, its Hardy-Littlewood maximal function is defined by

$$\mathfrak{M}_m F(x)=\sup\left\{\left(\fint_Q|F|^m\right)^{1/m}:x\in Q\subset\mathbb{R}^n\right\}$$

where the supremum is taken over all cubes $Q$ with edges parallel to the coordinate axes of $\mathbb{R}^n$ and containing the point $x$. We abbreviate $\mathfrak{M}_1=\mathfrak{M}$.

PROPOSITION 9.3. – *For each $k > m \geq 1$ and $f \in \mathcal{L}^k(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} |\mathfrak{M}_m F|^k \leq \frac{3^n 2^k k}{k-m} \int_{\mathbb{R}^n} |F|^k \tag{9.19}$$

$$\int_{\mathbb{R}^n} |F|^k \leq \frac{2^n (k-m)}{k} \int_{\mathbb{R}^n} |\mathfrak{M}_m F|^k \tag{9.20}$$

These refinements of the classical maximal inequalities are proven in [BI83]. Note in particular the more detailed information concerning the bounding constants for these operators. As a corollary, we obtain

LEMMA 9.4. – *Suppose two functions $f$ and $f_0$ are coupled by the inequality*

$$\mathfrak{M}_m f \leq A \mathfrak{M} f + B \mathfrak{M}_m f_0, \qquad m > 1 \tag{9.21}$$

*Let $k > m$ be close enough to $m$ to satisfy*

$$6^n 4^k \frac{k-m}{k-1} A^k \leq 1 \tag{9.22}$$

*Then,*

$$\int_{\mathbb{R}^n} |f|^k \leq 6^n 2^{k+1} B^k \int_{\mathbb{R}^n} |f_0|^k \tag{9.23}$$

*provided that both $f$ and $f_0$ belong to $\mathcal{L}^k(\mathbb{R}^n)$.*

PROOF. – Applying (9.19), (9.20) and (9.21) yields

$$\|f\|_k \leq \left(2^n \frac{k-m}{k}\right)^{1/k} \|\mathfrak{M}_m f\|_k \leq 2^n \left(\frac{k-m}{k}\right)^{1/k} [A \|\mathfrak{M} f\|_k + B \|\mathfrak{M}_m f_0\|_k]$$

$$\leq \left(2^n \frac{k-m}{k}\right)^{1/k} \left(3^n 2^k \frac{k}{k-1}\right)^{1/k} A \|f\|_k + \left(2^n \frac{k-m}{k}\right)^{1/k} \left(3^n 2^k \frac{k}{k-1}\right)^{1/k} B \|f_0\|_k$$

$$\leq \frac{\|f\|_k}{2} + (6^n 2)^{1/k} B \|f_0\|_k$$

which implies (9.2). ∎

We would now like to combine Lemma 9.4 with estimate (9.18). Unfortunately, a technical difficulty arises since we do not know a priori that $F$ belongs to $\mathcal{L}^k(\mathbb{R}^n)$ for any $k > m$. Our next step resolves this concern.

*Step* 3 (An Approximation Argument).

One can approximate $F$ and $F_0$ by functions of class $C_0(\mathbb{R}^n)$ in such a way that inequality (9.18) is preserved. Indeed, let $\eta$ be an arbitrary nonnegative functions of class $C_0^\infty(\mathbb{R}^n)$ such that $\int \eta(y)\,dy = 1$. We refer to such $\eta$ as mollifiers. The approximations are defined by

$$F^\eta = \eta * F \quad \text{and} \quad F_0^\eta = (\eta * F_0^m)^{1/m}$$

where $*$ stands for convolution. Thus, $F^\eta$ and $F_0^\eta$ are continuous functions with compact support. Using Minkowski's inequality for integrals, we obtain

$$\left( \fint_Q |F^\eta|^m \right)^{1/m} = \left( \fint_Q \left| \int_{\mathbb{R}^n} \eta(y)\, F(x-y)\,dy \right|^m dx \right)^{1/m} \leqslant \int_{\mathbb{R}^n} \eta(y) \left( \fint_Q |F(x-y)|^m\,dx \right)^{1/m} dy$$

$$\leqslant A \int_{\mathbb{R}^n} \eta(y) \left( \fint_{2Q} F(x-y)\,dx \right) dy + B \int_{\mathbb{R}^n} \eta(y) \left( \fint_{2Q} |F_0(x-y)|^m\,dx \right)^{1/m} dy$$

$$\leqslant A \fint_{2Q} F^\eta + B \left[ \int_{\mathbb{R}^n} \eta(y) \fint_{2Q} |F_0(x-y)|^m\,dx\,dy \right]^{1/m} = A \fint_{2Q} F^\eta + B \left( \fint_{2Q} |F_0^\eta|^m \right)^{1/m}$$

as desired.

We now pass to the supremum over all cubes $Q \subset \mathbb{R}^n$ containing a given point $x \in \mathbb{R}^n$. This leads to a pointwise inequality for maximal functions

$$\mathfrak{M}_m(F^\eta) \leqslant A\mathfrak{M}(F^\eta) + B\mathfrak{M}_m(F_0^\eta)$$

Next, let $k > m$ be chosen close enough to $m$ to satisfy condition (9.22). This, combined with (9.23) yields

$$\int_{\mathbb{R}^n} |F^\eta|^k \leqslant 6^n 2^{k+1} B^k \int_{\mathbb{R}^n} |F_0^\eta|^k \tag{9.24}$$

for all mollifiers $\eta$. Recall that the function $F_0$ belongs to $\mathcal{L}^k(\mathbb{R}^n)$ for some $k > m$. Letting $\eta$ approach the Dirac measure, we conclude that $F$ also belongs to $\mathcal{L}^k(\mathbb{R}^n)$ and as a limit case of (9.24), we find

$$\int_{\mathbb{R}^n} F^k \leqslant 6^n 2^{k+1} B^k \int_{\mathbb{R}^n} F_0^k$$

Finally, we return to the definition of the functions $F$ and $F_0$, see (9.17). Setting $a' = ka/m > a$, we obtain

$$\int_{\mathbb{R}^n} (|\phi|^p + |\psi|^q)^{a'} \leqslant 6^n 2^{k+1} B^k \int_{\mathbb{R}^n} (|\phi_0|^p + |\psi_0|^q)^{a'}$$

Unfortunately, this proof gives no information about how large $a'$ can be. We overcome this weakness in our final step.

*Step* 4 (A continuous induction).

Let $T$ denote the set of all numbers $t$ from $[a, b]$ so that the solution pair $(\phi, \psi)$ belongs to $\mathcal{L}^{tp}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{tq}_{\mathrm{loc}}(\wedge^l \Omega)$ whenever the data is also chosen from this space. We seek to show that $T = [a, b]$.

Our hypothesis guarantees that there is some nonempty largest subinterval contained in $T \cap [a, b]$ and containing $a$. Thanks to the arguments above, we see that this subinterval is relatively open. Thus, to conclude that $T = [a, b]$, we need only argue that this subinterval is relatively closed as well. To this end, suppose

$$(\phi, \psi) \in \mathcal{L}^{tp}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{tq}_{\mathrm{loc}}(\wedge^l \Omega) \tag{9.25}$$

for all $t \in [a, \gamma)$ with $a < \gamma \leq b$.

Suppose that the data pair $(\phi_0, \psi_0)$ belongs to

$$\mathcal{L}^{\gamma p}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{\gamma q}_{\mathrm{loc}}(\wedge^l \Omega)$$

We need to show that the solution pair $(\phi, \psi)$ also belongs to this space. As in (9.4) and (9.5), the problem reduces to the case when

$$\phi = d\xi \quad \text{and} \quad \psi = d^* \zeta$$

where $\xi \in \mathcal{W}^{1, tp}(\wedge^{l-1} \Omega)$ and $\zeta \in \mathcal{W}^{1, tq}(\wedge^{l+1} \Omega)$ for all $t < \gamma$. By the Sobolev Imbedding Theorem, $(\xi, \zeta) \in \mathcal{L}^{\gamma p}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{\gamma q}_{\mathrm{loc}}(\wedge^l \Omega)$. We may assume that all of the forms $\xi$, $\zeta$, $\phi_0$ and $\psi_0$ are compactly supported in $\Omega$ and are coupled by the Hodge system

$$\mathfrak{H}_p(d\xi + \phi_0) = \psi_0 + d^* \zeta$$

on the whole manifold $\mathcal{R}$. For, if not, we replace them by tilde forms as in Subsect. 8.4; see formulas (8.61). These observations make it legitimate to use Theorem 8.8. Accordingly,

$$\int_{\mathcal{R}} (|\phi|^p + |\psi|^q)^t \leq C_p(K, \mathcal{R}) \int_{\mathcal{R}} (|\phi_0|^p + |\psi_0|^q)^t \quad \text{for all } t < \gamma \tag{9.26}$$

Of course, the fact that the constant $C_p(K, \mathcal{R})$ does not depend on the parameter $t$ is of critical importance here. We then see, that as a limit case, (9.26) holds with $t = \gamma$. Therefore, $\gamma \in T$ and the proof of Theorem 9.1 is complete. ∎

9.2. *Extending the Hodge system across the boundary.* – The estimates we derived in the previous section provide no information about higher integrability properties of solutions near $\partial M$. One practical way to answer such questions is by extending the system and its solutions beyond the region $M$. This procedure is both effective for our purposes and has a broad range of applications. Therefore, we shall discuss it in detail.

We begin by recalling the collar neighborhood $\mathcal{N}_\varepsilon \subset \mathcal{R}$ of $\partial M$ and the reflection $r: \mathcal{N}_\varepsilon \to \mathcal{N}_\varepsilon$, see (2.28). This mapping is an orientation reversing diffeomorphism of $\mathcal{N}_\varepsilon$ with itself which keeps the points of $\partial M$ fixed. Furthermore, through each point $s \in \partial M$ there passes a unique geodesic arc $\gamma_s \subset \mathcal{N}_\varepsilon$ which is orthogonal to $\partial M$ and the map $r$ restricted to $\gamma_s$ is an isometry of $\gamma_s$ onto itself. This property of $r$ will be crucial in constructing so called regular extensions of forms of class $\mathcal{W}_T^{d,p}(\Lambda^{l-1}M)$ and $\mathcal{L}^q(\Lambda^l M) \cap \ker d_N^*$ out of the boundary. This will be done with the aid of the pullbacks $r^\#$: $\Gamma(\Lambda^l \mathcal{N}_\varepsilon) \to \Gamma(\Lambda^l \mathcal{N}_\varepsilon)$ and $r_\# = (-1)^{l(n-l)} * r^\# * : \Gamma(\Lambda^l \mathcal{N}_\varepsilon) \to \Gamma(\Lambda^l \mathcal{N}_\varepsilon)$. Thus, $* r_\# = r^\# *$. Recall also that $r^\#$ commutes with exterior differentiation while $r_\#$ commutes with $d^*$. We shall often use the formula $r^\#(\alpha \wedge \beta) = r^\# \alpha \wedge r^\# \beta$. Obviously, $r^\# \circ r^\# = r_\# \circ r_\# = id$. The following uniform bounds follow by a compactness argument

(9.27) $$\frac{1}{2} |\beta| \leqslant |r^\# \beta| \leqslant 2 |\beta|$$

(9.28) $$\frac{1}{2} |\beta| \leqslant |r_\# \beta| \leqslant 2 |\beta|$$

for all $\beta \in \Gamma(\Lambda^l \mathcal{N}_\varepsilon)$ and $l = 0, 1, \ldots, n$, provided $\varepsilon$ is sufficiently small. Indeed, if $r$ was an isometry, we would have $|\beta| = |r^\# \beta| = |r_\# \beta|$. When restricted to a sufficiently narrow collar neighborhood $\mathcal{N}_\varepsilon$ of the boundary, the reflection $r$: $\mathcal{N}_\varepsilon \to \mathcal{N}_\varepsilon$ becomes arbitrarily close to an isometry. Note too that the Jacobian determinant, $J(x, r) = r_\#(1)$, is negative. Hence (9.28), for $l = 0$, reduces to

(9.29) $$\frac{1}{2} \leqslant -J(x, r) \leqslant 2 \quad \text{on } \mathcal{N}_\varepsilon$$

At each point $a \in \partial M$ the differential $Dr(a)$: $T_a \mathcal{R} \to T_a \mathcal{R}$ acts as identity on the tangent subspace $T_a(\partial M)$ and as minus the identity on the normal subspace $N_a(\partial M)$. It then follows from the definition of the pullback that

(9.30) $$(r^\# \omega)_T = \omega_T \quad \text{and} \quad (r^\# \omega)_N = -\omega_N \quad \text{on } \partial M$$

for all $\omega \in C^\infty(\Lambda \mathcal{N}_\varepsilon)$.

Let $\Omega \subset \mathcal{N}_\varepsilon$ be an arbitrary open connected set which is symmetric about $\partial M$ (i.e. $r(\Omega) = \Omega$). Denote $\Omega^+ = \Omega \cap M$ and $\Omega^- = \Omega - \overline{M}$. Thus, $r(\Omega^-) = \Omega^+$. We shall consider a nonhomogeneous Hodge system on $\Omega^+$

(9.31) $$\mathfrak{H}_p^+(\phi + \phi_0) = \psi + \psi_0, \quad \phi = d\xi \quad \text{and} \quad d^*\psi = 0$$

The aim is to reflect these equations across $\partial M$ to $\Omega^-$. As always, we will work under the assumptions (8.15), (8.16) and (8.17) for the bundle map $\mathfrak{H}_p^+$: $\Lambda^l \Omega^+ \to \Lambda^l \Omega^+$. The data pair $(\phi_0, \psi_0)$ belongs to $\mathcal{L}^{\lambda_0 p}(\Lambda^l \Omega^+) \times \mathcal{L}^{\lambda_0 q}(\Lambda^l \Omega^+)$, $\min\{\lambda_0 p, \lambda_0 q\} > 1$, while the solution pair $(\phi, \psi)$ belongs to $\mathcal{L}^{\lambda p}(\Lambda^l \Omega^+) \times \mathcal{L}^{\lambda q}(\Lambda^l \Omega^+)$, $\min\{\lambda p, \lambda q\} > 1$. The reflection will work only if we impose one of two standing constraints on $\partial M$.

*Dirichlet constraint*

(9.32) $$\xi_T = 0 \quad \text{on } \partial M$$

*Neumann constraint*

(9.33) $$\psi_N = 0 \quad \text{on } \partial M$$

These constraints are understood in the sense of distributions. That is, in the case of Dirichlet's constraint, we have

$$\xi \in \mathcal{W}^{d,\lambda p}(\wedge^{l-1}\Omega^+) \quad \text{and} \quad \int_{\Omega^+} d\xi \wedge \alpha = (-1)^{l+1}\int_{\Omega^+} \xi \wedge d\alpha \tag{9.34}$$

for all test forms $\alpha \in C_0^\infty(\wedge^{n-l}\Omega)$, and in the case of Neumann's constraint, we have

$$\int_{\Omega^+} d\beta \wedge * \psi = 0 \tag{9.35}$$

for all $\beta \in C_0^\infty(\wedge^{l-1}\Omega)$. In other words, an integration by parts produces no integrals along $\partial M$..

In order to extend system (9.31) to $\Omega^-$, we first reflect the bundle map $\mathfrak{H}_p^+ : \wedge^l\Omega^+ \to \wedge^l\Omega^+$ via the pullbacks

$$r^\# : \wedge^l\Omega^- \to \wedge^l\Omega^+ \quad \text{and } r_\# : \wedge^l\Omega^+ \to \wedge^l\Omega^-$$

by the rule

$$\mathfrak{H}_p^- = -r_\# \mathfrak{H}_p r^\# : \wedge^l\Omega^- \to \wedge^l\Omega^- \tag{9.36}$$

This mapping satisfies conditions (8.15), (8.16) and (8.17) with possibly a new constant (independant of $M$) in place of $K$. Indeed, let's take the time to verify at least the monotonicity condition (8.16). The other two are easily checked using estimates (9.27) and (9.28). To this end, we will exploit the following identity for $X \in \Gamma(\wedge^l\Omega^+)$ and $Y \in \Gamma(\wedge^l\Omega^-)$

$$\begin{aligned}\langle r_\# X, Y\rangle &= *(Y \wedge * r_\# X) = *(Y \wedge r^\# * X) = \\ &= * r^\#(r^\# Y \wedge * X) = * r^\# * \langle X, r^\# Y\rangle = J(x, r)\langle X, r^\# Y\rangle\end{aligned}$$

Now take arbitrary $\xi, \zeta \in \Gamma(\wedge^l\Omega^-)$. By virtue of the monotonicity condition (8.16) for the map $\mathfrak{H}_p^+$ and estimates (9.27), (9.28) and (9.29), we compute

$$\begin{aligned}\langle \mathfrak{H}_p^-(\xi) - \mathfrak{H}_p^-(\zeta), \xi - \zeta\rangle &= -\langle r_\#[\mathfrak{H}_p^+(r^\#\xi) - \mathfrak{H}_p^+(r^\#\zeta)], \xi - \zeta\rangle \\ &= -J(x, r)\langle \mathfrak{H}_p^+(r^\#\xi) - \mathfrak{H}_p^+(r^\#\zeta), r^\#\xi - r^\#\zeta\rangle \\ &\geqslant -J(x, r)\, K^{-1}(|r^\#\xi| + |r^\#\zeta|)^{p-2}|r^\#(\xi - \zeta)|^2 \geqslant \frac{1}{K2^{p+1}}(|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2\end{aligned}$$

as desired. From the way in which $\mathfrak{H}_p^-$ was defined, we know that

$$\mathfrak{H}_p^-(r^\#\phi + r^\#\phi_0) = -r_\#\psi - r_\#\psi_0 \tag{9.37}$$

or equivalently,

$$\mathfrak{H}_p^-(-r^\#\phi - r^\#\phi_0) = r_\#\psi + r_\#\psi_0 \tag{9.38}$$

It may be more or less clear at this point that our intent in extending the Hodge system is to define

$$\mathfrak{H}_p(\zeta) = \begin{cases} \mathfrak{H}_p^+(\zeta) & \text{if } \zeta \in \wedge^l \Omega^+ \\ \mathfrak{H}_p^-(\zeta) & \text{if } \zeta \in \wedge^l \Omega^- \end{cases} \tag{9.39}$$

In case of the Dirichlet constraint, we extend $\phi$ and $\psi$ by the rules

$$\tilde{\phi} = \begin{cases} \phi = d\xi & \text{on } \Omega^+ \\ r^{\#}\phi = dr^{\#}\xi & \text{on } \Omega^- \end{cases} \qquad \tilde{\psi} = \begin{cases} \psi & \text{on } \Omega^+ \\ -r_{\#}\psi & \text{on } \Omega^- \end{cases} \tag{9.40}$$

Accordingly defined are the extensions $\tilde{\phi}_0$ and $\tilde{\psi}_0$. In case of the Neumann constraint, we put

$$\tilde{\phi} = \begin{cases} \phi = d\xi & \text{on } \Omega^+ \\ -r^{\#}\phi = -dr^{\#}\xi & \text{on } \Omega^- \end{cases} \qquad \tilde{\psi} = \begin{cases} \psi & \text{on } \Omega^+ \\ r_{\#}\psi & \text{on } \Omega^- \end{cases} \tag{9.41}$$

and $\tilde{\phi}_0$, $\tilde{\psi}_0$ are defined analogously. In either case, we have

$$\mathfrak{H}_p(\tilde{\phi} + \tilde{\phi}_0) = \tilde{\psi} + \tilde{\psi}_0 \tag{9.42}$$

Note that the degree of integrability of the pairs $(\tilde{\phi}, \tilde{\psi})$ and $(\tilde{\phi}_0, \tilde{\psi}_0)$ remains unchanged. In fact, by virtue of (9.27), (9.28) and (9.29), we easily arrive at the uniform bounds

$$\int_{\Omega^-} (|\tilde{\phi}|^{\lambda p} + |\tilde{\psi}|^{\lambda q}) \leqslant 2^{1+\lambda p} \int_{\Omega^+} |\phi|^{\lambda p} + 2^{1+\lambda q} \int_{\Omega^+} |\psi|^{\lambda q} \tag{9.43}$$

$$\int_{\Omega^-} (|\tilde{\phi}_0|^{\lambda_0 p} + |\tilde{\psi}_0|^{\lambda_0 q}) \leqslant 2^{1+\lambda_0 p} \int_{\Omega^+} |\phi_0|^{\lambda_0 p} + 2^{1+\lambda_0 q} \int_{\Omega^+} |\psi_0|^{\lambda_0 q} \tag{9.44}$$

The only issue remaining is the behavior of $\tilde{\phi}$ and $\tilde{\psi}$ near $\partial M$. We shall have established the extension of the Hodge system if we prove that $\tilde{\phi} \in d\mathcal{W}^{d,\lambda p}(\Omega)$ and $d^*\tilde{\psi} = 0$ on $\Omega$. As might be expected, we are going to show that

$$\tilde{\phi} = d\tilde{\xi} \tag{9.45}$$

where $\tilde{\xi} \in \mathcal{W}^{d,\lambda p}(\Omega)$ is defined by the rule

$$\tilde{\xi} = \begin{cases} \xi & \text{on } \Omega^+ \\ r^{\#}\xi & \text{on } \Omega^- \end{cases} \quad \text{and } \tilde{\xi} = \begin{cases} \xi & \text{on } \Omega^+ \\ -r^{\#}\xi & \text{on } \Omega^- \end{cases} \tag{9.46}$$

in case of the Dirichlet and Neumann constraints, respectively.

In order to prove formula (9.45), we fix an arbitrary test form $\omega \in C_0^\infty(\wedge^{n-l}\Omega)$ and compute the integral

$$\int_{\Omega} \tilde{\varphi} \wedge \omega = \int_{\Omega^+} d\xi \wedge \omega \pm \int_{\Omega^-} r^{\#}(d\xi \wedge r^{\#} \omega)$$

Hereafter, the signs + and − correspond to the Dirichlet and Neumann constraints, respectively. Changing variables in the last integral via the mapping $r\colon \Omega^+ \to \Omega^-$ yields

$$\int_{\Omega} \tilde{\varphi} \wedge \omega = \int_{\Omega^+} d\xi \wedge (\omega \pm r^{\#} \omega) = \int_{\Omega^+} d\xi \wedge \alpha$$

where $\alpha = \omega \pm r^{\#} \omega \in C_0^{\infty}(\wedge^{n-l} \Omega)$. Hence, in case of the Dirichlet constraint, we may use (9.34) to write

$$\int_{\Omega} \tilde{\varphi} \wedge \omega = (-1)^{l+1} \int_{\Omega^+} \xi \wedge d\alpha$$

This formula also holds under the Neumann constraint because in this case we have $\alpha = \omega - r^{\#} \omega \in C_T^{\infty}(\Omega^+)$, see (9.30).

Continuing in this fashion, we obtain

$$\int_{\Omega} \tilde{\varphi} \wedge \omega = (-1)^{l+1} \int_{\Omega^+} \xi \wedge (d\omega \pm dr^{\#} \omega) = (-1)^{l+1} \int_{\Omega^+} \xi \wedge d\omega \pm (-1)^{l+1} \int_{\Omega^+} r^{\#}(r^{\#} \xi \wedge d\omega)$$

$$= (-1)^{l+1} \int_{\Omega} (\xi \pm r^{\#} \xi) \wedge d\omega = (-1)^{l+1} \int_{\Omega} \tilde{\xi} \wedge d\omega$$

Which means that $\tilde{\varphi} = d\tilde{\xi}$ in the distributional sense.

In order to prove that $d^* \tilde{\psi} = 0$, we proceed analogously. Take any test form $\eta \in C_0^{\infty}(\wedge^{l-1} \Omega)$ and compute the integral

$$\int_{\Omega} d\eta \wedge * \tilde{\psi} = \int_{\Omega^+} d\eta \wedge * \psi \mp \int_{\Omega^-} d\eta \wedge * r_{\#} \psi$$

In contrast with the previous proof, this time the sings − and + correspond to the Dirichlet and Neumann constraints, respectively. Using the commutation rule, $* r_{\#} = r^{\#} *$, we find that

$$\int_{\Omega} d\eta \wedge * \tilde{\psi} = \int_{\Omega^+} d\eta \wedge * \psi \mp \int_{\Omega^-} d\eta \wedge r^{\#} * \psi$$

$$= \int_{\Omega^+} d\eta \wedge * \psi \mp \int_{\Omega^-} r^{\#}[dr^{\#} \eta \wedge * \psi] = \int_{\Omega^+} d\beta \wedge * \psi$$

where $\beta = \eta \mp r^{\#} \eta \in C_0^{\infty}(\wedge^{l-1} \Omega)$. Now, in case of the Neumann constraint, thanks to identity (9.35), we have

$$\int_{\Omega^+} d\beta \wedge * \psi = 0$$

In case of the Dirichlet constraint, we have $\beta = \eta - r^{\#}\eta \in C_T^\infty(\wedge^{l-1}\Omega^+)$ while $d^*\psi = 0$ on $\Omega^+$ and so this integral still vanishes. In either case then, we arrive at the equation

$$\int_{\Omega} d\eta \wedge * \widetilde{\psi} = 0$$

which shows that $d^*\widetilde{\psi} = 0$ on $\Omega$.

9.3. *Regularity up to the boundary.* – Recall $a$ and $b$ from Definition 8.10 and fix an arbitrary $\lambda \in [a, b]$.

THEOREM 9.5. – *Given a nonhomogeneous system*

$$\mathfrak{H}_p(\phi + \phi_0) = \psi + \psi_0 \tag{9.47}$$

*with* $(\phi_0, \psi_0) \in \mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L}^{\lambda q}(\wedge^l M)$, *suppose that the solution pair* $(\phi, \psi)$ *satisfies either the Dirichlet conditions*

$$\phi \in d\mathcal{W}_T^{d,\, ap}(\wedge^{l-1} M), \qquad \psi \in \mathcal{L}^{aq}(\wedge^l M) \cap \ker d^*$$

*or the Neumann conditions*

$$\phi \in d\mathcal{W}^{d,\, ap}(\wedge^{l-1} M), \qquad \psi \in \mathcal{L}^{aq}(\wedge^l M) \cap \ker d_N^*$$

*Then,* $(\phi, \psi) \in \mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L} q^{\lambda q}(\wedge^l M)$.

PROOF. – First, extend the equations beyond $\partial M$ as in Subsect. 9.2, say to an open set $\overline{M} \subset \Omega$. Then the assertion is a straight forward consequence of Theorem 9.1. ■

REMARK 9.6. – *When* $\mathfrak{H}_p(\xi) = |\xi|^{p-2}\xi$, *Theorem* 9.5 *deals with the projection operator* $\Pi_p$ *defined in Subsect.* 7.1. *Combining this result with Theorem* 8.8 *implies Proposition* 7.1.

## 10. – Hodge conjugate fields.

We return now to the study of homogeneous systems and their solutions. That is, pairs $(\phi, \psi)$ of $l$-forms on $M$ which are coupled by the equations

$$\psi = \mathfrak{h}(\phi), \qquad d\phi = d^*\psi = 0 \tag{10.1}$$

Recall the numbers $a = a_p(K) < 1 < b_p(K) = b$ from Definition 8 and the inequalities

$ap > 1$, $aq > 1$. Notice then that for each open $\Omega \subset M$, any $\mathfrak{H}_p$-couple $(\phi, \psi) \in \mathcal{L}^{ap}(\wedge^l \Omega) \times \mathcal{L}^{aq}(\wedge^l \Omega)$ actually belongs to $\mathcal{L}^{bp}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{bq}_{\mathrm{loc}}(\wedge^l \Omega)$.

10.1. *A compactness principle.* – Our first result is an extension of the familiar normal family property of holomorphic functions.

THEOREM 10.1. – *A family of $\mathfrak{H}_p$-couples which is bounded in $\mathcal{L}^{ap}(\wedge^l \Omega) \times \mathcal{L}^{aq}(\wedge^l \Omega)$ is compact in $\mathcal{L}^{bp}(\wedge^l F) \times \mathcal{L}^{bq}(\wedge^l F)$ for each compact subset $F \subset \Omega$.*

PROOF. – Consider a sequence $(\phi_i, \psi_i)$ of $\mathfrak{H}_p$-couples satisfying

$$\|\phi_i\|_{ap,\,\Omega} + \|\psi_i\|_{aq,\,\Omega} \leqslant C_\Omega \tag{10.2}$$

for $i = 1, 2, \ldots$, with $C_\Omega$ independent of $j$. Using Theorem 9.1, we see that $(\phi_i, \psi_i) \in \mathcal{L}^{bp}_{\mathrm{loc}}(\wedge^l \Omega) \times \mathcal{L}^{bq}_{\mathrm{loc}}(\wedge^l \Omega)$. Unfortunately, this theorem does not say anything about the bounds for the $bp$-norms of $\phi_i$ or the $bq$-norms of $\psi_i$ on compact subsets of $\Omega$. To obtain such bounds, we fix a point $x \in F$. Now, there is a (small) neighborhood $U$ about $x$ such that $\phi_i = d\xi_i$ and $\psi = d^* \zeta_i$. Furthermore, with the aid of Poincare inequality (6.24), we can always take $\xi_i \in \mathcal{W}^{1,\,bp}(\wedge^{l-1} U)$ and $\zeta_j \in \mathcal{W}^{1,\,bq}(\wedge^{l+1} U)$ such that

$$\begin{cases} \|\xi_i\|_{1,\,ap} \leqslant C_p(U)\, \|\phi_i\|_{ap} \leqslant C_U \\ \|\zeta_i\|_{1,\,aq} \leqslant C_p(U)\, \|\psi_i\|_{aq} \leqslant C_U \end{cases} \tag{10.3}$$

Then, by the Sobolev Imbedding theorem, there exists $a' > a$ such that

$$\|\xi_i\|_{a'p} + \|\zeta_i\|_{a'q} \leqslant C_U \tag{10.4}$$

with possibly new constant $C_U$ independent of $i = 1, 2, \ldots$. Applying Caccioppoli inequality (8.59), with $\lambda = a'$ yields a uniform bound for $\phi_i$ and $\psi_i$

$$\|\phi_i\|_{a'p,\,U'} + \|\psi_i\|_{a'q,\,U'} \leqslant C_{U'}$$

where $U'$ is a new, possibly smaller, neighborhood of the point $x$. It is now clear that by repeating these arguments several times, we will arrive at the estimate

$$\|\phi_i\|_{bp,\,V} + \|\psi_i\|_{bq,\,V} \leqslant C_V \tag{10.5}$$

for some neighborhood $V$ about $x$. Because of the compactness of the imbeddings:

$$\mathcal{W}^{1,\,p}(\wedge^{l-1} V) \subset \mathcal{L}^p(\wedge^{l-1} V) \quad \text{and} \quad \mathcal{W}^{1,\,q}(\wedge^{l+1} V) \subset \mathcal{L}^q(\wedge^{l+1} V)$$

we may assume that $\{\xi_i\}$ and $\{\zeta_j\}$ are Cauchy sequences in $\mathcal{L}^p(\wedge^{l-1} V)$ and $\mathcal{L}^q(\wedge^{l+1} V)$, respectively. If not, we could restrict ourselves to a subsequence. Next, we multiply the equations $\psi_i = \mathfrak{H}_p(\phi_i)$ by $\chi^p$, where $\chi \in C_0^\infty(V)$ is a fixed, nonnegative function equal to 1 in a neighborhood $V'$ about $x$.

$$\mathfrak{H}_p(\chi^q\, d\xi_i) = \chi^p\, d^* \zeta_i$$

This can be viewed as a nonhomogeneous Hodge system of the form

$$\mathfrak{H}_p[d(\chi^q \xi_i) - d\chi^q \wedge \xi_i] = d^*(\chi^p \zeta_i) - (-1)^{n-nl} * (d\chi^p \wedge {}^* \zeta_i)$$

Applying (8.26) yields

$$\int_{V'} |\phi_i - \phi_j|^p + |\psi_i - \psi_j|^q$$

$$\leq C\left(\int_V |\xi_i|^p + |\xi_j|^p + |\zeta_i|^q + |\zeta_j|^q\right)^{1-t}\left(\int_V |\xi_i - \xi_j|^p + |\zeta_i - \zeta_j|^q\right)^t$$

Hence $\{\phi_i\}$ and $\{\psi_i\}$ are Cauchy sequences in $\mathcal{L}^p(\wedge^l V')$ and $\mathcal{L}^q(\wedge^l V')$, respectively. This, combined with the uniform bounds (10.5), implies that $\{\phi_i\}$ and $\{\psi_i\}$ are actually Cauchy sequences in $\mathcal{L}^{bp}(\wedge^l V')$ and $\mathcal{L}^{bq}(\wedge^l V')$. Finally, the set $F$ can be covered by a finite number of neighborhoods such as $V'$. Therefore, $\{\phi_i\}$ and $\{\psi_i\}$ are Cauchy sequences in $\mathcal{L}^{bp}(\wedge^l F)$ and $\mathcal{L}^{bq}(\wedge^l F)$, respectively. ■

10.2. *Boundary value problems for Hodge conjugate fields.* – As promised in Subsect. 8.2, we now study the Neumann type boundary value problems for an $\mathfrak{H}_p$-couple $(\phi, \psi) \in \mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L}^{\lambda q}(\wedge^l M)$, $\lambda \in [a, b]$. The Hodge system now becomes either

$$\text{(10.6)} \qquad \begin{cases} \psi = \mathfrak{H}_p(d\alpha), \quad d^*\psi = 0 \\ \alpha \in \mathcal{W}^{d, \lambda p}(\wedge^{l-1} M) \end{cases}$$

or

$$\text{(10.7)} \qquad \begin{cases} d^*\beta = \mathfrak{H}_p(\phi), \qquad d\phi = 0 \\ \beta \in \mathcal{W}^{d^*, \lambda q}(\wedge^{l+1} M) \end{cases}$$

Of course, the duality between these systems allows us to consider only (10.6). Given an arbitrary $\alpha_0 \in \mathcal{W}^{d, \lambda p}(\wedge^{l-1} M)$, the Dirichlet Problem is now

$$\text{(10.8)} \qquad \begin{cases} \psi = \mathfrak{H}_p(d\alpha), \quad d^*\psi = 0 \\ \alpha - \alpha_0 \in \mathcal{W}_T^{d, \lambda p}(\wedge^{l-1} M) \end{cases}$$

and given any $\psi_0 \in \mathcal{W}^{d^*, \lambda q}(\wedge^l M)$, the Neumann Problem is now

$$\text{(10.9)} \qquad \begin{cases} \psi = \mathfrak{H}_p(d\alpha), \quad d^*\psi = 0 \\ \psi - \psi_0 \in \mathcal{W}_N^{d^*, \lambda q}(\wedge^{l-1} M) \end{cases}$$

With the aid of obvious substitutions, both problems reduce to solving homogeneous boundary value problems for a nonhomogeneous Hodge system of the type (8.21). Thus, for $\lambda = 1$, Theorem 8.4 provides existence and uniqueness results. However, for $\lambda \neq 1$, the Browder-Minty theory of monotone operators fails. The case $1 < \lambda \leq b$ poses no difficulty since, for the data

$$(\phi_0, \psi_0) \in \mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L}^{\lambda q}(\wedge^l M) \subset \mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$$

we may use Theorem 8.4 to solve (8.21) for $(\phi, \psi) \in \mathcal{L}^p(\wedge^l M) \times \mathcal{L}^q(\wedge^l M)$. We then conclude, by using the Regularity Theorem 9.5, that this solution actually belongs to

$\mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L}^{\lambda q}(\wedge^l M)$. The solution must satisfy inequality (8.39) of Theorem 8.8. In terms of $\alpha$, $\alpha_0$ and $\psi_0$, these read

$$\int_M |d\alpha|^{\lambda p} \leq C_p(M, K) \int_M |d\alpha_0|^{\lambda p} \tag{10.10}$$

in case of the Dirichlet conditions, and

$$\int_M |d\alpha|^{\lambda p} \leq C_p(M, K) \int_M |\psi_0|^{\lambda q} \tag{10.11}$$

in case of the Neumann conditions.

The case $a \leq \lambda < 1$ requires an approximation of the data $\alpha_0$ and $\psi_0$ by smooth forms. It also relies on the compactness principle for $\mathfrak{H}_p$-couples; see Theorem 10.1. We give the arguments only for the Dirichlet problem. The Neumann problem is similar.

Given the Dirichlet data $\alpha_0 \in \mathcal{W}^{d, \lambda p}(\wedge^{l-1} M)$, problem (10.8) is not affected by adding a closed form with vanishing tangential part to $\alpha_0$. Thus $\alpha_0$ can be suited to extra regularity. Namely, by the Poincaré inequality (6.24), we may assume that $\alpha_0 \in \mathcal{W}^{1, \lambda p}(\wedge^{l-1} M)$. We approximate $\alpha_0$ by forms $\alpha_0^j \in \mathcal{W}^{1, p}(\wedge^{l-1} M)$, converging to $\alpha_0$ in the norm of the space $\mathcal{W}^{1, \lambda p}(\wedge^{l-1} M)$. For each $j = 1, 2, \ldots$, we solve (uniquely for $d\alpha^j$) the Dirichlet problem

$$\begin{cases} \psi^j = \mathfrak{H}_p(d\alpha^j), \quad & d^* \psi^j = 0 \\ \alpha^j - \alpha_0^j \in \mathcal{W}_T^{d, p}(\wedge^{l-1} M) \end{cases}$$

As before, with the aid of Theorem 6.4, we may suit the solutions $\alpha^j$ to extra regularity. Namely $\alpha^j \in \mathcal{W}^{1, p}(\wedge^{l-1} M)$ and

$$\|\alpha^j\|_{1, \lambda p} \leq C_p(M, K)(\|\alpha_0^j\|_{1, \lambda p} + \|d\alpha^j\|_{\lambda p})$$

On the other hand, by Theorem 8.8, we have

$$\|\alpha^j\|_{\lambda p} \leq C_p(M, K) \|d\alpha_0^j\|_{\lambda p}$$

Hence, we obtain a uniform bound for $\alpha^j$ in the norm of $\mathcal{W}^{1, \lambda p}(\wedge^{l-1} M)$

$$\|\alpha^j\|_{1, \lambda p} \leq C_p'(M, K) \|\alpha_0^j\|_{1, \lambda p} \leq C_p''(M, K) \|\alpha_0\|_{1, \lambda p}$$

for all $j$. In other words, the sequence $(\phi^j, \psi^j) = (d\alpha^j, \mathfrak{H}_p(d\alpha^j))$ of $\mathfrak{H}_p$-couples is bounded in $\mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L}^{\lambda q}(\wedge^l M)$, where $\lambda \geq a$. By the Compactness Theorem 10.1, we may assume that $(\phi^j, \psi^j)$ converges to an $\mathfrak{H}_p$-couple $(\phi, \psi)$ in $\mathcal{L}^{bp}_{\mathrm{loc}}(\wedge^l M) \times \mathcal{L}^{bq}_{\mathrm{loc}}(\wedge^l M)$. For, if not, we replace $(\phi^j, \psi^j)$ by an appropriately chosen subsequence. In view of the uniform bounds established above, we have $(\phi, \psi) \in \mathcal{L}^{\lambda p}(\wedge^l M) \times \mathcal{L}^{\lambda q}(\wedge^l M)$. Note that $\phi^j = d\alpha^j$, where $\{\alpha^j\}$ is a bounded sequence in $\mathcal{W}^{1, \lambda p}(\wedge^{l-1} M)$ and $\alpha^j - \alpha_0^j \in \mathcal{W}_T^{1, \lambda p}(\wedge^{l-1} M)$. From what has been already suited to $\alpha^j$, it follows that $\{\alpha^j\}$ also converges in $\mathcal{W}^{1, \lambda p}(\wedge^{l-1} M)$ to an $\alpha$ (for the purpose of this proof, weak convergence would also suffice). Hence, $\phi = d\alpha$ with $\alpha - \alpha_0 \in \mathcal{W}_T^{1, \lambda p}(\wedge^{l-1} M)$. We then arrive at the following

THEOREM 10.2. – *For each* $a \leqslant \lambda \leqslant b$, *the Dirichlet problem* (10.8) *has a solution satisfying*

$$\|\alpha\|_{1,\lambda p} \leqslant C_p(M, K)\,\|d\alpha_0\|_{\lambda p} \tag{10.12}$$

Arguments similar to the above show

THEOREM 10.3. – *For each* $a \leqslant \lambda \leqslant b$, *the Neumann problem* (10.9) *has a solution satisfying*

$$\|\alpha\|_{1,\lambda p} \leqslant C_p(M, K)\,\|\psi_0\|_{\lambda q}^{q-1} \tag{10.13}$$

10.3. *Removability of singularities.* – We recall the classical removability theorem of P. Painlevé [Zor05]. Let $\Omega$ be an open subset in the complex plane. A closed set $E \subset \mathbb{C}$ is said to be removable if each bounded holomorphic function $f\colon \Omega - E \to \mathbb{C}$ extends to a holomorphic function on $\Omega$. Painlevé's theorem states that sets of Hausdorff dimension less that 1 are removable. Until recently, there was very little known about the possible extensions of this result to nonlinear PDEs.

We have already mentioned that harmonic fields on $n$-manifolds should be viewed as counterparts of holomorphic functions. The point to make here is that the geometric behavior of Hodge conjugate fields does not differ substantially from that of holomorphic functions and harmonic fields. The key tool is the Caccioppoli estimate for $\mathfrak{H}_p$-couples $(\phi, \psi)$ below the natural exponents $p$ and $q$. That is, with $a \leqslant \lambda < 1$. The size of the removable sets will be measured in terms of $s$-capacity.

A closed set $E \subset \mathcal{R}$ is said to have zero $s$-capacity, $s > 1$, in case there is a sequence $\{\eta_j\}$ of functions $\eta_j \in C^\infty(\mathcal{R})$ such that

i) $0 \leqslant \eta_j \leqslant 1$ everywhere on $\mathcal{R}$

ii) Each $\eta_j$ equals 1 on its own neighborhood of $E$

iii) $\lim \eta_j(x) = \begin{cases} 1 & \text{for } x \in E \\ 0 & \text{otherwise} \end{cases}$

iv) $\lim \|d\eta_j\|_s = 0$

This definition is best adapted to our proofs and coincides with the customary one. Recall that sets of Hausdorff dimension less than $n - s$ have zero $s$-capacity and conversely, sets of zero $s$-capacity have Hausdorff dimension at most $n - s$. In particular, the sets of $s$-capacity zero have measure zero.

We shall examine Hodge conjugate fields

$$\phi = d\xi \quad \text{and} \quad \psi = d^*\zeta \tag{10.14}$$

on the set $U = \Omega - E$, where $\Omega$ is an open set of $\mathcal{R}$ and $E$ is a closed subset of $\mathcal{R}$. As usual, these fields are coupled by the equation

$$\psi = \mathfrak{H}_p(\phi) \tag{10.15}$$

and we assume that $(\phi, \psi) \in \mathcal{L}^p_{\text{loc}}(\wedge^l U) \times \mathcal{L}^q_{\text{loc}}(\wedge^l U)$. In order to extend $\phi$ and $\psi$ as an $\mathfrak{H}_p$-couple to all of $\Omega$, it is necessary to assume some bounds near the singular set $E$. These bounds will be made in terms of the potential forms $\xi \in \mathcal{W}^{d,p}_{\text{loc}}(\wedge^{l-1} U)$ and $\zeta \in$

$\mathcal{W}^{d^*,q}_{\mathrm{loc}}(\wedge^{l+1}U)$. Recall the numbers $a<1<b$ from Definition 8.10, fix $s\geqslant\max\{p,q\}$ and fix $\lambda\in[a,1]$.

THEOREM 10.4 (Removability Theorem). – *Let $E\subset\mathcal{R}$ be a closed set of $\lambda s$-capacity zero and let $\Omega\subset\mathcal{R}$ be an open set. Consider the Hodge system*

$$d^*\zeta=\mathfrak{H}_p(d\xi)\quad\text{on}\quad U=\Omega-E \tag{10.16}$$

*for $\xi\in\mathcal{W}^{d,p}_{\mathrm{loc}}(\wedge^{l-1}U)$ and $\zeta\in\mathcal{W}^{d^*,q}_{\mathrm{loc}}(\wedge^{l+1}U)$ such that*

$$\|\xi\|_{\lambda sp/(s-p),\,U}+\|\zeta\|_{\lambda sq/(s-q),\,U}<\infty \tag{10.17}$$

*Then $\xi$ and $\zeta$ have extensions to $\Omega$ as forms of class $\mathcal{W}^{d,p}_{\mathrm{loc}}(\wedge^{l-1}\Omega)$ and $\mathcal{W}^{d^*,q}_{\mathrm{loc}}(\wedge^{l+1}\Omega)$ respectively. We then say that $E$ is a removable singular set.*

This result is relatively trivial for $\lambda=1$. Indeed, the smaller $\lambda$ is, the stronger the result is. We note that our removability theorem is new even in the case of the Cauchy Riemann system

$$d^*\zeta=d\xi\quad\text{on } U=\Omega-E \tag{10.18}$$

Thus $d\xi$ and $d^*\zeta$ are locally square integrable on $U$, so we take $s=p=q=2$. Then the number $r=\lambda s>1$ can be made as close to 1 as one wishes because for system (10.18) we have $a=1$ and $b=\infty$. Assuming that $\dim E<n-1$, we then conclude that $E$ is removable for bounded harmonic conjugate fields. In dimension 2, we recover the theorem of Painlevé. More generally, consider the linear Hodge system

$$d^*\zeta=\mathfrak{H}(d\xi) \tag{10.19}$$

where $\mathfrak{H}\colon\wedge^l M\to\wedge^l M$ is a measurable linear bundle automorphism satisfying conditions (8.10-8.12). In many respects, this is an excellent extension of the familiar complex Beltrami equation to all dimensions. As before, we may take $s=p=q=2$ and the number $r=\lambda s<2$. Thus, sets $E\subset\mathcal{R}$ of dimension less than $n-r$ are removable for bounded solutions of (10.19) where we notice that $n-r>n-2$. The removability of sets of dimension less than $n-2$ follows easily by applying Caccioppoli's inequality with $\lambda=1$ and is largely uninteresting. There are other far reaching consequences of Theorem 10.4 (e.g. see the removability results for mappings of bounded distortion in [IM93] and [Iwa92]). Removability results for second order PDEs are studied in [BIS99]; in contrast to the first order PDEs it is necessary to impose some bounds near the singular set not only for the solution $u$, but also for its gradient $\nabla u$.

Precisely how big the removable singular sets are is not known. However, the recent work of K. Astala [Ast94] answers this question in the planar case. See also [Str95] and [GLS96]. The removability results for nonlinear Hodge systems depend strongly on the exponent $a=a_p(K)$ in Definition 8.10. So, it is desirable to identify, or at least give a good bound for, this exponent. Unfortunately, this question is beyond the scope of this paper.

PROOF OF THEOREM 10.4. – We think of $\xi$ and $\zeta$ as measurable sections of $\wedge^{l-1}\Omega$ and $\wedge^{l+1}\Omega$ respectively, which are equal to, say zero, on $E$. Such extension does not affect our arguments since $E$ has measure zero. Fix an arbitrary nonnegative test function $\chi \in C_0^\infty(\Omega)$ and consider the sequence $\chi_j = (1-\eta_j)\chi$. We have pointwise estimates $|\chi_j| \leqslant |\chi|$ and $|d\chi_j| \leqslant |d\chi| + |\chi|\,|d\eta_j|$. Applying Caccioppoli's estimate (8.59) yields

$$\begin{aligned}\int_U (|\chi_j^q d\xi|^p + |\chi_j^p d^*\zeta|^q)^\lambda &\leqslant C(p,K)\int_U (|\xi \wedge d\chi_j^q|^p + |*\zeta \wedge d\chi_j^p|^q)^\lambda \\ &\leqslant C(p,K)\int_U (|\xi|^p |d\chi_j|^p |\chi_j|^q + |\zeta|^q |d\chi_j|^q |\chi_j|^p)^\lambda \\ &\leqslant C(p,K)\int_U (|\xi|^p |d\chi|^p |\chi|^q + |\zeta|^q |d\chi|^q |\chi|^p)^\lambda \\ &\quad + C(p,K)\int_U (|\xi|^p |d\eta_j|^p + |\zeta|^q |d\eta_j|^q)^\lambda |\chi|^{\lambda pq}.\end{aligned}$$

Here $C(p,K)$ varies from line to line. As $j \to \infty$, the last integral tends to zero. Indeed, by Hölder's inequality

$$\int_U |\xi|^{\lambda p} |d\eta_j|^{\lambda p} \leqslant \|\xi\|^{\lambda p}_{\lambda sp/(s-p),\,U} \|d\eta_j\|^{\lambda p}_{\lambda s} \to 0$$

$$\int_U |\zeta|^{\lambda p} |d\eta_j|^{\lambda q} \leqslant \|\zeta\|^{\lambda q}_{\lambda sq/(s-q),\,U} \|d\eta_j\|^{\lambda q}_{\lambda s} \to 0$$

Thus, letting $j$ go to infinity, we obtain

$$\int_\Omega \chi^{\lambda pq}(|d\xi|^{\lambda p} + |d^*\zeta|^{\lambda q}) \leqslant C(p,K)\int_\Omega |\xi|^{\lambda p} |\chi|^{\lambda q} |d\chi|^{\lambda p} + C(p,K)\int_\Omega |\zeta|^{\lambda q} |\chi|^{\lambda p} |d\chi|^{\lambda q}$$

for every nonnegative function $\chi \in C_0^\infty(\Omega)$. This shows that $d\xi \in \mathscr{L}^{\lambda p}_{\mathrm{loc}}(\bigwedge^l \Omega) \subset \mathscr{L}^{ap}_{\mathrm{loc}}(\bigwedge^l \Omega)$ and $d^*\zeta \in \mathscr{L}^{\lambda q}_{\mathrm{loc}}(\bigwedge^l \Omega) \subset \mathscr{L}^{aq}_{\mathrm{loc}}(\bigwedge^l \Omega)$. By the Regularity Theorem 9.1, we then conclude that $d\xi \in \mathscr{L}^{p}_{\mathrm{loc}}(\bigwedge^l \Omega)$ and $d^*\zeta \in \mathscr{L}^{q}_{\mathrm{loc}}(\bigwedge^l \Omega)$, as desired. ■

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