ON THE ASYMPTOTIC EFFICIENCY OF ESTIMATORS IN AN AUTOREGRESSIVE PROCESS

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Summary

Let $\{X_t\}$ be defined recursively by $X_t=\theta X_{t-1}+U_t$ $(t=1, 2, \cdots)$, where $X_0=0$ and $\{U_t\}$ is a sequence of independent identically distributed real random variables having a density function f with mean 0 and variance σ^2. We assume that $|\theta|<1$. In the present paper we obtain the bound of the asymptotic distributions of asymptotically median unbiased (AMU) estimators of θ and the sufficient condition that an AMU estimator be asymptotically efficient in the sense that its distribution attains the above bound. It is also shown that the least squares estimator of θ is asymptotically efficient if and only if f is a normal density function.

1. Introduction

Let X_t $(t=1, 2, \cdots)$ be defined recursively by

$$X_t=\theta X_{t-1}+U_t \qquad t=1, 2, \cdots ,$$

where $X_0=0$ and $\{U_t: t=1, 2, \cdots\}$ is a sequence of independent identically distributed real random variables having a density function f with mean 0 and variance σ^2.

We shall define an estimator of θ to be asymptotically efficient if the asymptotic distribution of it attains the bound of the asymptotic distributions of asymptotically median unbiased (AMU) estimators of θ. We assume that $|\theta|<1$. The purpose of this paper is to obtain the bound of the asymptotic distributions of AMU estimators of θ using the asymptotic normality of the best test statistics and the sufficient condition that an AMU estimator be asymptotically efficient. We shall also give a necessary and sufficient condition that the least squares estimator of θ be asymptotically efficient. In fact, Theorem 2 of Section 4 shows that under some regularity conditions the bound of the asymptotic distributions of AMU estimators of θ is a normal distribution with mean 0 and variance $(1-\theta^2)/\sigma^2 I$, where I is the Fisher information of f. Then it is easily seen that an AMU estimator is asymptotically efficient if it has the asymptotic normal distribution. The least squares estimator $\hat{\theta}_{LS}$ of θ is given by $\left(\sum_{t=1}^{n} X_{t-1}X_t\right)\Big/\sum_{t=1}^{n} X_{t-1}^2$. It is shown by Anderson [5] that $n^{1/2}(\hat{\theta}_{LS}-\theta)$ has a limiting normal distribution with mean 0 and variance $1-\theta^2$. Then Theorem 4 of Section 4 shows that under some conditions the limiting distribution of $n^{1/2}(\hat{\theta}_{LS}-\theta)$ attains the bound of the asymptotic distributions given by Theorem 2 if and only if f is a normal density function with mean 0 and variance σ^2.

The outline of the proofs of Theorems 2 and 4 is stated in Takeuchi [9], but in this paper we shall strictly discuss them under more general conditions. The approach in this paper is similar to Bahadur [6] dealing with the bound for asymptotic variances.

The second order asymptotic efficiencies are discussed in Akahira [2] and Akahira and Takeuchi [4], [10]. Further the third order asymptotic efficiency is studied in [10].

2. Notations and definitions

Let $\mathcal{X}$ be an abstract sample space whose generic point is denoted by x, $\mathcal{B}$ a σ-field of subsets of $\mathcal{X}$, and let Θ be a parameter space, which is assumed to be an open set in a Euclidean 1-space R^1. We shall denote by $(\mathcal{X}^{(n)}, \mathcal{B}^{(n)})$ the n-fold direct products of $(\mathcal{X}, \mathcal{B})$. For each $n=1, 2, \cdots$, the points of $\mathcal{X}^{(n)}$ will be denoted by $\tilde{x}_n=(x_1, \cdots, x_n)$. We consider a sequence of classes of probability measures $\{P_{n,\theta}: \theta \in \Theta\}$ $(n=1, 2, \cdots)$ each defined on $(\mathcal{X}^{(n)}, \mathcal{B}^{(n)})$ such that for each $n=1, 2, \cdots$ and each $\theta \in \Theta$ the following holds:

$$P_{n,\theta}(B^{(n)})=P_{n+1,\theta}(B^{(n)}\times\mathcal{X})$$

for all $B^{(n)} \in \mathcal{B}^{(n)}$.

An estimator of θ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathcal{B}^{(n)}$-measurable functions $\hat{\theta}_n$ on $\mathcal{X}^{(n)}$ into Θ $(n=1, 2, \cdots)$.

For an increasing sequence of positive numbers $\{c_n\}$ (c_n tending to infinity) an estimator $\{\hat{\theta}_n\}$ is called consistent with order $\{c_n\}$ (or $\{c_n\}$-consistent for short) if for every $\varepsilon>0$ and every ϑ of Θ, there exist a sufficiently small positive number δ and a sufficiently large positive number L satisfying the following:

$$\varlimsup_{n\to\infty} \sup_{\theta: |\theta-\vartheta|<\delta} P_{n,\theta}(\{c_n|\hat{\theta}_n-\theta|\geqq L\})<\varepsilon$$

(Akahira [1], [3]). In the subsequent discussions we shall deal only with the case when $c_n=n^{1/2}$.

DEFINITION 1. A distribution function $F_{\{\hat{\theta}_n\},\theta}(y)$ is called to be the asymptotic (or limiting) distribution function of $n^{1/2}(\hat{\theta}_n-\theta)$ (or $\{\hat{\theta}_n\}$ for short) iff for each y, $F_{\{\hat{\theta}_n\},\theta}(y)$ is continuous in θ and for each y

$$\lim_{n\to\infty} |P_{n,\theta}(\{n^{1/2}(\hat{\theta}_n-\theta)\leqq y\})-F_{\{\hat{\theta}_n\},\theta}(y)|=0 .$$

Let $\{\hat{\theta}_n\}$ be a $\{n^{1/2}\}$-consistent estimator.

DEFINITION 2. $\{\hat{\theta}_n\}$ is asymptotically median unbiased (or AMU for short) iff for any $\vartheta\in\Theta$ there exists a positive number δ such that

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} |P_{n,\theta}(\{n^{1/2}(\hat{\theta}_n-\theta)\leqq 0\})-1/2|=0 ;$$

$$\lim_{n\to\infty} \sup_{\theta:|\theta-\vartheta|<\delta} |P_{n,\theta}(\{n^{1/2}(\hat{\theta}_n-\theta)\geqq 0\})-1/2|=0 .$$

If $\{\hat{\theta}_n\}$ is AMU, then $G^{+}_{\{\hat{\theta}_n\},\theta}$ and $G^{-}_{\{\hat{\theta}_n\},\theta}$ are defined as follows:

$$G^{+}_{\{\hat{\theta}_n\},\theta}(y)=\overline{\lim_{n\to\infty}}\, P_{n,\theta}(\{n^{1/2}(\hat{\theta}_n-\theta)\leqq y\}) \qquad \text{for all } y\geqq 0 , \tag{1}$$

$$G^{-}_{\{\hat{\theta}_n\},\theta}(y)=\underline{\lim_{n\to\infty}}\, P_{n,\theta}(\{n^{1/2}(\hat{\theta}_n-\theta)\leqq y\}) \qquad \text{for all } y<0 . \tag{2}$$

Let θ_0 ($\in\Theta$) be arbitrary but fixed. Consider the problem of testing hypothesis H^{+}: $\theta=\theta_0+\lambda n^{-1/2}$ ($\lambda>0$) against alternative K: $\theta=\theta_0$. We define $\beta^{+}_{\theta_0}(\lambda)$ as follows:

$$\beta^{+}_{\theta_0}(\lambda)=\sup_{\{\phi_n\}\in\Phi_{1/2}} \overline{\lim_{n\to\infty}}\, \mathrm{E}_{n,\theta_0}(\phi_n) , \tag{3}$$

where $\Phi_{1/2}=\{\{\phi_n\}: \lim_{n\to\infty} \mathrm{E}_{n,\theta_0+\lambda n^{-1/2}}(\phi_n)=1/2,\ 0\leqq\phi_n(\tilde{x}_n)\leqq 1$ for all $\tilde{x}_n$ $(n=1, 2, \cdots)\}$. Putting $A_{\{\hat{\theta}_n\},\theta_0}=\{n^{1/2}(\hat{\theta}_n-\theta_0)\leqq\lambda\}$ we have for $\lambda>0$ $P_{n,\theta_0+\lambda n^{-1/2}}(A_{\{\hat{\theta}_n\},\theta_0})$ $=P_{n,\theta_0+\lambda n^{-1/2}}(\{n^{1/2}(\hat{\theta}_n-\theta_0-\lambda n^{-1/2})\leqq 0\})\to 1/2$ $(n\to\infty)$. Since a sequence $\{I_{A\{\hat{\theta}_n\},\theta_0}\}$ of the indicators (or characteristic functions) of $A_{\{\hat{\theta}_n\},\theta_0}$ $(n=1, 2,\cdots)$ belongs to $\Phi_{1/2}$, it follows from (1) and (3) that

$$G^{+}_{\{\hat{\theta}_n\},\theta_0}(\lambda)\leqq\beta^{+}_{\theta_0}(\lambda) \tag{4}$$

for all $\lambda>0$.

Consider next the problem of the testing hypothesis H^{-}: $\theta=\theta_0+\lambda n^{-1/2}$ ($\lambda<0$) against alternative K: $\theta=\theta_0$. Then we define $\beta^{-}_{\theta_0}(\lambda)$ as follows:

$$\beta^{-}_{\theta_0}(\lambda)=\inf_{\{\phi_n\}\in\Phi_{1/2}} \underline{\lim_{n\to\infty}}\, \mathrm{E}_{n,\theta_0}(\phi_n) . \tag{5}$$

Note that

$$\beta^{-}_{\theta_0}(\lambda)=1-\sup_{\{\phi_n\}\in\Phi_{1/2}} \overline{\lim_{n\to\infty}}\, \mathrm{E}_{n,\theta_0}(\phi_n) . \tag{6}$$

In a similar way as the case $\lambda>0$, we have from (2) and (5)

$$G^{-}_{\{\hat{\theta}_n\},\theta_0}(\lambda)\geqq\beta^{-}_{\theta_0}(\lambda) \tag{7}$$

for all $\lambda<0$. Since θ_0 is arbitrary, the bounds of the asymptotic distributions of AMU estimators are obtained as follows:

$$G^{+}_{\{\hat{\theta}_n\},\theta}(\lambda)\leqq\beta^{+}_{\theta}(\lambda) \qquad \text{for all } \lambda>0\,;$$

$$G^{-}_{\{\hat{\theta}_n\},\theta}(\lambda)\geqq\beta^{-}_{\theta}(\lambda) \qquad \text{for all } \lambda<0\,.$$

For any $\theta\in\Theta$ letting $\beta^{+}_{\theta}(0)=1/2$ we make the following definition.

DEFINITION 3. For $\{\hat{\theta}_n\}$ asymptotically median unbiased it is called asymptotically efficient iff for each $\theta\in\Theta$

$$F_{\{\hat{\theta}_n\},\theta}(\lambda)=\begin{cases}\beta^{+}_{\theta}(\lambda) & \text{for all } \lambda\geqq 0\,,\\ \beta^{-}_{\theta}(\lambda) & \text{for all } \lambda<0\,.\end{cases} \tag{8}$$

It is shown by Takeuchi and Akahira [11] that the definition of the asymptotic efficiency works in the most common situation.

Throughout the subsequent sections we assume that $\mathcal{X}=R^1$ and Θ is an open interval $(-1, 1)$ and consider the autoregressive process $\{X_t\}$ given in Introduction.

3. Preliminary lemmas

The following lemma is given in Diananda [7].

LEMMA 3.1. *Let $\{Z_n\colon n=1, 2, \cdots\}$ be a sequence of random variables satisfying the following*:

(i) $Z_n=Z_{n,N}+R_{n,N}$ $(n>N)$;

(ii) *For each fixed N, the asymptotic distribution of $Z_{n,N}$ is normal with mean 0 and variance σ_N^2*;

(iii) $\lim_{N\to\infty}\sigma_N^2=\sigma^2$;

(iv) *$R_{n,N}$ converges in probability to 0 uniformly in n.*

Then Z_n has a limiting normal distribution with mean 0 and variance σ^2.

Let

$$Y_1, Y_2, \cdots \tag{9}$$

be a sequence of random variables.

If for some function $g(n)$ the inequality $s-r>g(n)$ implies that the two sets

$$(Y_1, Y_2, \cdots, Y_r)\,, \qquad (Y_s, Y_{s+1}, \cdots, Y_n)$$

are independent, then the sequence (9) is said to be $g(n)$-dependent ([8]).

Let (9) be m-dependent and such that $\mathrm{E}(Y_i)=0$, $\mathrm{E}(Y_i^2)<\infty$ $(i=1, 2,\cdots)$. Then we define

$$A_i=\mathrm{E}(Y_{i+m}^2)+2\sum_{j=1}^{m}\mathrm{E}(Y_{i+m-j}Y_{i+m}) \qquad (i=1, 2,\cdots).$$

The following lemma is given by Hoeffding and Robbins [8].

LEMMA 3.2. *Let $Y_1, Y_2,\cdots$ be an m-dependent sequence of random variables such that*

(a) $\mathrm{E}(Y_i)=0$, $\mathrm{E}(|Y_i|^3)\leqq R^3<\infty$ $(i=1, 2,\cdots)$,

(b) $\lim_{p\to\infty} p^{-1}\sum_{h=1}^{p} A_{i+h}=A$ *exists, uniformly for all* $i=0, 1,\cdots$.

Then $\sum_{i=1}^{n} Y_i$ *is asymptotically normal with mean* 0 *and variance* nA.

Throughout the remainder of this paper we assume the following:

ASSUMPTION (3.1). f is once differentiable and $f(u)>0$ for all u and $\lim_{u\to\pm\infty} f(u)=0$.

Then we get the following lemma.

LEMMA 3.3. *Under Assumption* (3.1), *if* $\mathrm{E}[|U_t|^3]<\infty$ *and* $\mathrm{E}\left[\left|\frac{f'(U_t)}{f(U_t)}\right|^3\right]<\infty$, *then the limiting distribution of* $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{f'(U_t)}{f(U_t)}X_{t-1}$ *is normal with mean* 0 *and variance* $\frac{\sigma^2 I}{1-\theta^2}$, *where* $I=\int\frac{\{f'(u)\}^2}{f(u)}du$.

PROOF. Putting $V_t=\frac{f'(U_t)}{f(U_t)}$, $(t=1, 2,\cdots)$, we have

$$\sum_{t=1}^{n} V_tX_{t-1}=\sum_{t=2}^{n} V_t\sum_{i=1}^{t-1}\theta^{t-1-i}U_i=\sum_{j=1}^{n-1}\theta^{j-1}\sum_{t=1}^{n-j}V_{t+j}U_t .$$

Put $W_{j,t}=V_{t+j}U_t$. Then for any fixed j, $\{W_{j,t}: t=1, 2,\cdots\}$ is a j-dependent sequence and $\mathrm{E}(W_{j,t})=0$ $(t=1, 2,\cdots)$. Since $\mathrm{E}(U_t^2)=\sigma^2$ and $\mathrm{E}(V_{t+j}^2)=I$ $(t=1, 2,\cdots)$, we obtain $\mathrm{E}(W_{j,t}^2)=\sigma^2 I$ $(t=1, 2,\cdots)$. Put $Z_n=(1/\sqrt{n})\sum_{j=1}^{n-1}\theta^{j-1}\sum_{t=1}^{n-j}W_{j,t}$ $(n=1, 2,\cdots)$. Then for $n\geqq N+2$

$$Z_n=Z_{n,N}+R_{n,N} ,$$

where $Z_{n,N}=(1/\sqrt{n})\sum_{j=1}^{N}\theta^{j-1}\sum_{t=1}^{n-j}W_{j,t}$ and $R_{n,N}=(1/\sqrt{n})\sum_{j=N+1}^{n-1}\theta^{j-1}\sum_{t=1}^{n-j}W_{j,t}$. We now show that $Z_{n,N}$ has an asymptotic normal distribution with mean 0 and variance $\left(\frac{1-\theta^{2N}}{1-\theta^2}\right)\sigma^2 I$. Since $\mathrm{E}(W_{j,t+j-i+h}W_{j,t+j+h})=0$ $(i=1,$

$\cdots, j;\ t, h=1, 2, \cdots)$ and $\frac{1}{p}\sum_{h=1}^{p} \mathrm{E}(W_{j,t+j+h}^2)=\sigma^2 I$, it follows from Lemma 3.2 that $\sum_{t=1}^{n-j} W_{j,t}$ is asymptotically normal with mean 0 and variance $(n-j)\sigma^2 I$. Hence $\theta^{j-1}\sum_{t=1}^{n-j} W_{j,t}$ is asymptotically normal with mean 0 and variance $(n-j)\theta^{2(j-1)}\sigma^2 I$. Put $W_j^{(n)}=(1/\sqrt{n})\theta^{j-1}\sum_{t=1}^{n-j} W_{j,t}$ $(j=1, 2, \cdots, N)$. Then it can be shown that for each fixed N, $W_1^{(n)}, W_2^{(n)}, \cdots, W_N^{(n)}$ are asymptotically jointly normal with mean 0 and variance $\left(\frac{1-\theta^{2N}}{1-\theta^2}\right)\sigma^2 I$. Hence for each fixed N, $Z_{n,N}$ has a limiting normal distribution with mean 0 and variance $\left(\frac{1-\theta^{2N}}{1-\theta^2}\right)\sigma^2 I$.

Next we show that $R_{n,N}$ converges in probability to 0 uniformly in n. For the purpose we have

$$
\begin{aligned}
\mathrm{E}_\theta(R_{n,N}^2) &= \frac{1}{n}\mathrm{E}_\theta\left[\left(\sum_{j=N+1}^{n-1} \theta^{j-1}\sum_{t=1}^{n-j} W_{j,t}\right)^2\right] \\
&= \frac{1}{n}\mathrm{E}_\theta\left[\sum_{j=N+1}^{n-1} \theta^{2(j-1)}\left(\sum_{t=1}^{n-j} W_{j,t}\right)^2\right. \\
&\qquad \left. +2\sum_{j<j'} \theta^{j+j'-2}\left(\sum_{t=1}^{n-j} W_{j,t}\right)\left(\sum_{t=1}^{n-j'} W_{j',t}\right)\right] \\
&= \frac{1}{n}\sum_{j=N+1}^{n-1} \theta^{2(j-1)}\sum_{t=1}^{n-j}\mathrm{E}(W_{j,t}^2) \\
&= \frac{1}{n}\sum_{j=N+1}^{n-1} \theta^{2(j-1)}(n-j)\sigma^2 I \\
&\leqq \frac{1}{n}\sum_{k=1}^{n-N-1} \theta^{2(n-k-1)}k\sigma^2 I \\
&= \frac{1}{n}\left\{\sigma^2 I\theta^{2(n-2)}\sum_{k=1}^{n-N-1} k\theta^{-2(k-1)}\right\} \\
&\leqq \sigma^2 I\theta^{2(N-2)}\sum_{k=1}^{n-N-1} k\theta^{-2(k-1)} \\
&\leqq \sigma^2 I\theta^{2(N-2)}\sum_{k=1}^{\infty} k\theta^{-2(k-1)} \\
&= \frac{\sigma^2 I\theta^{2(N-2)}}{(1-\theta^{-2})^2}\,.
\end{aligned}
$$

Hence we see that $\lim_{N\to\infty}\mathrm{E}_\theta(R_{n,N}^2)=0$ uniformly in n. Using Chebyshev's inequality we obtain $R_{n,N}$ converges in probability to 0 uniformly in n.

Since $\lim_{N\to\infty}\frac{1-\theta^{2N}}{1-\theta^2}\sigma^2 I=\frac{\sigma^2 I}{1-\theta^2}$ it follows from Lemma 3.1 that Z_n has a limiting normal distribution with mean 0 and variance $\frac{\sigma^2 I}{1-\theta^2}$. Thus we complete the proof.

4. Asymptotic normality and asymptotic efficiency

In this section it will be shown that the bound of the asymptotic distributions of AMU estimators of θ is obtained using the best test statistics and that the least squares estimator of θ is asymptotically efficient if and only if f is a normal density with mean 0 and variance σ^2.

Let θ_0 be arbitrary but fixed in Θ. Putting $\theta_1=\theta_0+\lambda n^{-1/2}$ $(\lambda\neq 0)$ we define Z_{tn} as follows:

$$Z_{tn}=\log \frac{f(X_t-\theta_0 X_{t-1})}{f(X_t-\theta_1 X_{t-1})} .$$

We assume that f is twice continuously differentiable. If $\theta=\theta_0$, then we have

$$\begin{aligned}(10)\qquad \sum_{t=1}^{n} Z_{tn} &=\sum_{t=1}^{n} \log \frac{f(U_t)}{f(U_t-\lambda n^{-1/2}X_{t-1})}\\ &=\sum_{t=1}^{n} \{\log f(U_t)-\log f(U_t-\lambda n^{-1/2}X_{t-1})\}\\ &=\lambda n^{-1/2}\sum_{t=1}^{n}\frac{f'(U_t)}{f(U_t)}X_{t-1}-\frac{\lambda^2}{2}n^{-1}\sum_{t=1}^{n}\frac{d^2\log f(U_t^*)}{dU_t^2}X_{t-1}^2 ,\end{aligned}$$

where for each t, U_t^* lies between U_t and $U_t-\lambda n^{-1/2}X_{t-1}$. If $\theta=\theta_1$, then we have

$$\begin{aligned}(11)\qquad \sum_{t=1}^{n} Z_{tn} &=\sum_{t=1}^{n} \log \frac{f(U_t+\lambda n^{-1/2}X_{t-1})}{f(U_t)}\\ &=\sum_{t=1}^{n} \{\log f(U_t+\lambda n^{-1/2}X_{t-1})-\log f(U_t)\}\\ &=\lambda n^{-1/2}\sum_{t=1}^{n}\frac{f'(U_t)}{f(U_t)}X_{t-1}+\frac{\lambda^2}{2}n^{-1}\sum_{t=1}^{n}\frac{d^2\log f(U_t^{**})}{dU_t^2}X_{t-1}^2 ,\end{aligned}$$

where for each t, U_t^{**} lies between U_t and $U_t+\lambda n^{-1/2}X_{t-1}$.

Throughout the subsequent discussions we make the following assumptions:

ASSUMPTION (4.1). f is three times differentiable in a real line and $\lim_{u\to\pm\infty} f'(u)=0$.

ASSUMPTION (4.2). $d^2\log f(u)/du^2$ is a bounded function and $\mathrm{E}(|U_t|^4)<\infty$.

ASSUMPTION (4.3). For each $\theta_0\in\Theta$ the following hold:

(a) $\lim_{n\to\infty} n^{-3/2}\sum_{t=1}^{n}\mathrm{E}_{\theta_j}[|X_{t-1}|^3 \sup_{0<|\eta|<\lambda n^{-1/2}|X_{t-1}|} |g'(U_t+\eta)|]=0\ (j=0, 1)$;

(b) $\lim_{n\to\infty} n^{-3} \mathrm{E}_{\theta_j}\left[\left\{\sum_{t=1}^{n} |X_{t-1}|^3 \sup_{0<|\eta|<\lambda n^{-1/2}|X_{t-1}|} |g'(U_t+\eta)|\right\}^2\right]=0$ $(j=0, 1)$,

where $g(u)=-\dfrac{d^2 \log f(u)}{du^2}$.

Putting $T_n=\sum_{t=1}^{n} X_{t-1}^2 g(U_t)$, $T_n^*=\sum_{t=1}^{n} X_{t-1}^2 g(U_t^*)$ and $T_n^{**}=\sum_{t=1}^{n} X_{t-1}^2 g(U_t^{**})$, we get the following lemmas and theorem.

LEMMA 4.1. *Under Assumptions* (3.1) *and* (4.1)–(4.3) *the following hold*:

$$\mathrm{E}_{\theta_j}(T_n)=\sum_{t=1}^{n} \mathrm{E}_{\theta_j}\left[X_{t-1}^2\left\{\frac{f'(U_t)}{f(U_t)}\right\}^2\right]=\sigma^2 I\left\{\frac{n-1}{1-\theta_j^2}-\frac{\theta_j^4(1-\theta_j^{2(n-1)})}{(1-\theta_j^2)^2}\right\} \quad (j=0, 1); \tag{12}$$

$$\lim_{n\to\infty} |\mathrm{E}_{\theta_0}(T_n^*/n)-\mathrm{E}_{\theta_0}(T_n/n)|=0\ ; \tag{13}$$

$$\lim_{n\to\infty} |\mathrm{E}_{\theta_1}(T_n^{**}/n)-\mathrm{E}_{\theta_1}(T_n/n)|=0\ ; \tag{14}$$

$$\lim_{n\to\infty} |\mathrm{E}_{\theta_0}(T_n^{*2}/n^2)-\mathrm{E}_{\theta_0}(T_n^2/n^2)|=0\ ; \tag{15}$$

$$\lim_{n\to\infty} |\mathrm{E}_{\theta_1}(T_n^{**2}/n^2)-\mathrm{E}_{\theta_1}(T_n^2/n^2)|=0\ . \tag{16}$$

PROOF. Since

$$X_{t-1}=\sum_{i=1}^{t-1} \theta_j^{t-1-i} U_i \qquad (j=0, 1)\ ,$$

we have

$$\begin{aligned}
\mathrm{E}_{\theta_j}(T_n)&=\mathrm{E}_{\theta_j}\left[\sum_{t=1}^{n} X_{t-1}^2 g(U_t)\right]\\
&=\sum_{t=1}^{n} \mathrm{E}_{\theta_j}\left[X_{t-1}^2\left\{-\frac{d^2 \log f(U_t)}{dU_t^2}\right\}\right]\\
&=-\sum_{t=1}^{n} \mathrm{E}_{\theta_j}\left[X_{t-1}^2\frac{f''(U_t)}{f(U_t)}-X_{t-1}^2\left\{\frac{f'(U_t)}{f(U_t)}\right\}^2\right]\\
&=\sum_{t=1}^{n} \mathrm{E}_{\theta_j}\left[X_{t-1}^2\left\{\frac{f'(U_t)}{f(U_t)}\right\}^2\right]\\
&=\sum_{t=1}^{n} \mathrm{E}_{\theta_j}(X_{t-1}^2)\,\mathrm{E}\left[\left\{\frac{f'(U_t)}{f(U_t)}\right\}^2\right]\\
&=\sum_{t=1}^{n} \mathrm{E}_{\theta_j}(X_{t-1}^2) I\\
&=I\sum_{t=2}^{n} \mathrm{E}_{\theta_j}\left[\left\{\sum_{i=1}^{t-1} \theta_j^{t-1-i} U_i\right\}^2\right]\\
&=I\sum_{t=2}^{n}\sum_{i=1}^{t-1} \theta_j^{2(t-1-i)}\,\mathrm{E}(U_i^2)
\end{aligned}$$

$$=I\sum_{t=2}^{n}\sigma^2\frac{1-\theta_j^{2t}}{1-\theta_j^2}$$
$$=\sigma^2 I\left\{\frac{n-1}{1-\theta_j^2}-\frac{\theta_j^4(1-\theta_j^{2(n-1)})}{(1-\theta_j^2)^2}\right\}\qquad (j=0,1).$$

Hence (12) holds.

Using the mean value theorem we have

$$\begin{aligned}&|\mathrm{E}_{\theta_0}(T_n^*/n)-\mathrm{E}_{\theta_0}(T_n/n)|\\ &\leqq(1/n)\sum_{t=1}^{n}\mathrm{E}_{\theta_0}[X_{t-1}^2|g(U_t^*)-g(U_t)|]\\ &\leqq|\lambda|n^{-3/2}\sum_{t=1}^{n}\mathrm{E}_{\theta_0}[|X_{t-1}|^3\sup_{0<|\eta|<|\lambda|n^{-1/2}|X_{t-1}|}|g'(U_t+\eta)|].\end{aligned}$$

It follows from (a) of Assumption (4.3) that (13) holds. Similarly (14) holds. By Schwarz's inequality we have

$$\begin{aligned}(17)\qquad &|\mathrm{E}_{\theta_0}(T_n^{*2}/n^2)-\mathrm{E}_{\theta_0}(T_n^2/n^2)|\\ &\leqq\{\mathrm{E}_{\theta_0}(T_n^*/n-T_n/n)^2\}^{1/2}\{\mathrm{E}_{\theta_0}(T_n^*/n+T_n/n)^2\}^{1/2}.\end{aligned}$$

It follows from Assumption (4.2) and that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\,\mathrm{E}_{\theta_0}(T_n^{*2}/n^2)&=\overline{\lim_{n\to\infty}}\,n^{-2}\mathrm{E}_{\theta_0}\left[\left\{\sum_{t=1}^{n}X_{t-1}^2 g(U_t^*)\right\}^2\right]\\ &\leqq\overline{\lim_{n\to\infty}}\,n^{-2}\mathrm{E}_{\theta_0}\left[\left\{\sum_{t=1}^{n}X_{t-1}^2\sup_{|U_t-U_t^*|<|\lambda|n^{-1/2}|X_{t-1}|}g(U_t^*)\right\}^2\right]<\infty.\end{aligned}$$

Similarly it follows that

$$\overline{\lim_{n\to\infty}}\,\mathrm{E}_{\theta_0}(T_n^2/n^2)<\infty.$$

Hence we have

$$(18)\qquad \overline{\lim_{n\to\infty}}\,\{\mathrm{E}_{\theta_0}(T_n^*/n+T_n/n)^2\}^{1/2}<\infty.$$

On the other hand it follows from (b) of Assumption (4.3) that

$$\begin{aligned}(19)\qquad &\overline{\lim_{n\to\infty}}\,\mathrm{E}_{\theta_0}(T_n^*/n-T_n/n)^2\\ &=\overline{\lim_{n\to\infty}}\,n^{-2}\mathrm{E}_{\theta_0}\left[\left\{\sum_{t=1}^{n}X_{t-1}^2(g(U_t^*)-g(U_t))\right\}^2\right]\\ &\leqq\overline{\lim_{n\to\infty}}\,n^{-2}\mathrm{E}_{\theta_0}\left[\lambda^2 n^{-1}\left\{\sum_{t=1}^{n}|X_{t-1}|^3\sup_{0<|\eta|<|\lambda|n^{-1/2}|X_{t-1}|}|g'(U_t+\eta)|\right\}^2\right]\\ &=\overline{\lim_{n\to\infty}}\,\lambda^2 n^{-3}\mathrm{E}_{\theta_0}\left[\left\{\sum_{t=1}^{n}|X_{t-1}|^3\sup_{0<|\eta|<|\lambda|n^{-1/2}|X_{t-1}|}|g'(U_t+\eta)|\right\}^2\right]=0.\end{aligned}$$

From (17), (18) and (19) we have

$$\lim_{n\to\infty}|\mathrm{E}_{\theta_0}(T_n^{*2}/n^2)-\mathrm{E}_{\theta_0}(T_n^2/n^2)|=0.$$

In a similar way it follows from Assumption (4.2) and (b) of Assumption (4.3) and (12) that

$$\lim_{n\to\infty} |\mathrm{E}_{\theta_1}(T_n^{**2}/n^2)-\mathrm{E}_{\theta_1}(T_n^2/n^2)|=0 .$$

Thus we complete the proof.

LEMMA 4.2. *Under Assumptions* (3.1) *and* (4.1)-(4.3), *if* $\mathrm{E}\left[\left|\frac{f'(U_t)}{f(U_t)}\right|^4\right]<\infty$, *then both of the sequences* $T_n^*/\mathrm{E}_{\theta_0}(T_n^*)$ *and* $T_n^{**}/\mathrm{E}_{\theta_1}(T_n^{**})$ *converge in probability to* 1.

PROOF. From (13)-(16) of Lemma 4.1 we have

$$\lim_{n\to\infty} |\mathrm{Var}_{\theta_0}(T_n^*/n)-\mathrm{Var}_{\theta_0}(T_n/n)|=0 ;$$

$$\lim_{n\to\infty} |\mathrm{Var}_{\theta_1}(T_n^{**}/n)-\mathrm{Var}_{\theta_1}(T_n/n)|=0 ;$$

$$\lim_{n\to\infty} |\{\mathrm{E}_{\theta_0}(T_n^*/n)\}^2-\{\mathrm{E}_{\theta_0}(T_n/n)\}^2|=0 ;$$

$$\lim_{n\to\infty} |\{\mathrm{E}_{\theta_1}(T_n^{**}/n)\}^2-\{\mathrm{E}_{\theta_1}(T_n/n)\}^2|=0 ,$$

where Var designates variance. Hence in order to prove that both of the sequences $T_n^*/\mathrm{E}_{\theta_0}(T_n^*)$ and $T_n^{**}/\mathrm{E}_{\theta_1}(T_n^{**})$ converge in probability to 1, it is enough to show that

$$\lim_{n\to\infty}\frac{\mathrm{Var}_{\theta_j}(T_n)}{\{\mathrm{E}_{\theta_j}(T_n)\}^2}=0 , \qquad (j=0, 1) . \tag{20}$$

Indeed, it follows from (12) of Lemma 4.1 that

$$\{\mathrm{E}_{\theta_j}(T_n)\}^2=O(n^2) \qquad (j=0, 1) .$$

Also it follows from Assumption (4.2) that

$$\begin{aligned}\mathrm{Var}_{\theta_j}(T_n)&=\sum_{t=1}^{n}\mathrm{Var}_{\theta_j}(X_{t-1}^2 g(U_t))\\ &\quad+\sum_{t\neq t'}\sum\mathrm{Cov}_{\theta_j}(X_{t-1}^2 g(U_t), X_{t'-1}^2 g(U_{t'}))=O(n) \qquad (j=0, 1) ,\end{aligned}$$

where Cov designates covariance. Hence (20) holds. Thus we complete the proof.

For Theorems 1, 2 and 3 we assume further the following:

ASSUMPTION (4.4). $\mathrm{E}\left[\left|\frac{f'(U_t)}{f(U_t)}\right|^4\right]<\infty.$

In the following theorem we shall show that the best test statistics have limiting normal distributions.

THEOREM 1. *Suppose that Assumptions* (3.1) *and* (4.1)-(4.4) *hold. If* $\theta=\theta_0$, *then* $\sum_{t=1}^{n} Z_{tn}$ *has a limiting normal distribution with mean* $\frac{\lambda^2\sigma^2 I}{2(1-\theta_0^2)}$ *and variance* $\frac{\lambda^2\sigma^2 I}{1-\theta_0^2}$. *If* $\theta=\theta_1$, *then* $\sum_{t=1}^{n} Z_{tn}$ *has a limiting normal distribution with mean* $-\frac{\lambda^2\sigma^2 I}{2(1-\theta_0^2)}$ *and variance* $\frac{\lambda^2\sigma^2 I}{1-\theta_0^2}$.

PROOF. If $\theta=\theta_0$, then it follows from (10) that

$$\sum_{t=1}^{n} Z_{tn}=\lambda n^{-1/2}\sum_{t=1}^{n}\frac{f'(U_t)}{f(U_t)}X_{t-1}+\frac{\lambda^2}{2}n^{-1}T_n^* \tag{21}$$

$$=n^{-1}\mathrm{E}_{\theta_0}(T_n^*)\left\{\lambda\frac{n^{-1/2}\sum_{t=1}^{n}(f'(U_t)/f(U_t))X_{t-1}}{n^{-1}\mathrm{E}_{\theta_0}(T_n^*)}+\frac{\lambda^2}{2}\frac{T_n^*}{\mathrm{E}_{\theta_0}(T_n^*)}\right\}.$$

If $\theta=\theta_1$, then it follows from (11) that

$$\sum_{t=1}^{n} Z_{tn}=\lambda n^{-1/2}\sum_{t=1}^{n}\frac{f'(U_t)}{f(U_t)}X_{t-1}-\frac{\lambda^2}{2}n^{-1}T_n^{**} \tag{22}$$

$$=n^{-1}\mathrm{E}_{\theta_1}(T_n^{**})\left\{\lambda\frac{n^{-1/2}\sum_{t=1}^{n}(f'(U_t)/f(U_t))X_{t-1}}{n^{-1}\mathrm{E}_{\theta_0}(T_n^{**})}-\frac{\lambda^2}{2}\frac{T_n^{**}}{\mathrm{E}_{\theta_1}(T_n^{**})}\right\}.$$

It follows from (12) and (13) of Lemma 4.1 that

$$\lim_{n\to\infty} n^{-1}\mathrm{E}_{\theta_j}(T_n^*)=\lim_{n\to\infty} n^{-1}\mathrm{E}_{\theta_j}(T_n^{**})=\frac{\sigma^2 I}{1-\theta_0^2}\qquad (j=0,1).$$

Hence it is seen from Lemma 3.3 that both of the sequences of $\frac{n^{-1/2}\sum_{t=1}^{n}(f'(U_t)/f(U_t))X_{t-1}}{n^{-1}\mathrm{E}_{\theta_0}(T_n^*)}$ and $\frac{n^{-1/2}\sum_{t=1}^{n}(f'(U_t)/f(U_t))X_{t-1}}{n^{-1}\mathrm{E}_{\theta_1}(T_n^{**})}$ have a limiting normal distribution with mean 0 and variance $\frac{1-\theta_0^2}{\sigma^2 I}$. Therefore it follows from (21), (22) and Lemma 4.2 that $\sum_{t=1}^{n} Z_{tn}$ has limiting normal distributions with means $\frac{\lambda^2\sigma^2 I}{2(1-\theta_0^2)}$ and $-\frac{\lambda^2\sigma^2 I}{2(1-\theta_0^2)}$ and common variances $\frac{\lambda^2\sigma^2 I}{1-\theta_0^2}$ for $\theta=\theta_0$ and for $\theta=\theta_1$, respectively. This completes the proof.

THEOREM 2. *Under Assumptions* (3.1) *and* (4.1)-(4.4), *the bound of the asymptotic distributions of AMU estimators* $\{\hat{\theta}_n\}$ *is given as follows: for each* $\theta\in\Theta$

$$\overline{\lim_{n\to\infty}}\, P_{n,\theta}(\{n^{1/2}(\hat{\theta}_n-\theta)\leq\lambda\})\leq\Phi\left(\frac{\lambda\sigma\sqrt{I}}{\sqrt{1-\theta^2}}\right)\qquad \textit{for all } \lambda\geq 0\,; \tag{23}$$

(24) $$\lim_{n\to\infty} P_{n,\theta}(\{n^{1/2}(\hat{\theta}_n-\theta)\leqq\lambda\})\geqq\Phi\left(\frac{\lambda\sigma\sqrt{I}}{\sqrt{1-\theta^2}}\right) \quad \textit{for all } \lambda<0\,,$$

where Φ is a normal distribution with mean 0 *and variance* 1.

PROOF. Let θ_0 be arbitrary but fixed in Θ. Let λ be an arbitrary positive number. Then we consider the problem of testing hypothesis H^+: $\theta=\theta_0+\lambda n^{-1/2}$ against alternative K: $\theta=\theta_0$. If we choose a sequence $\{k_n\}$ such that $\lim_{n\to\infty} P_{n,\theta_0+\lambda n^{-1/2}}\left(\left\{\sum_{t=1}^n Z_{tn}>k_n\right\}\right)=1/2$, then it follows by Theorem 1 that $\lim_{n\to\infty} k_n=-\dfrac{\lambda^2\sigma^2 I}{2(1-\theta_0^2)}$.

Furthermore we have from Theorem 1

$$\lim_{n\to\infty} P_{n,\theta_0}\left(\left\{\sum_{t=1}^n Z_{tn}>k_n\right\}\right)=\lim_{n\to\infty} P_{n,\theta_0}\left(\left\{\frac{\sum_{t=1}^n Z_{tn}-J/2}{\sqrt{J}}>\frac{k_n-J/2}{\sqrt{J}}\right\}\right)$$
$$=1-\Phi(-\sqrt{J})=\Phi(\sqrt{J})\,,$$

where $J=\dfrac{\lambda^2\sigma^2 I}{1-\theta_0^2}$. Hence it follows by (3) and the fundamental lemma of Neyman and Pearson that for each $\lambda>0$

$$\beta^+_{\theta_0}(\lambda)=\Phi\left(\frac{\lambda\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)\,.$$

From (1) and (4) we obtain for every $\lambda>0$

$$\overline{\lim_{n\to\infty}}\, P_{n,\theta_0}(\{n^{1/2}(\hat{\theta}_n-\theta_0)\leqq\lambda\})\leqq\Phi\left(\frac{\lambda\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)\,.$$

Since $\{\hat{\theta}_n\}$ is AMU, $\beta^+_{\theta_0}(0)=\Phi(0)=1/2$. Hence since θ_0 is arbitrary, it follows that (23) holds.

Let λ be an arbitrary negative number. Then we consider the problem of testing hypothesis H^-: $\theta=\theta_0+\lambda n^{-1/2}$ against alternative K: $\theta=\theta_0$. Henceforth by a similar way as the case $\lambda>0$, we have from (6)

$$\beta^-_{\theta_0}(\lambda)=1-\Phi\left(-\frac{\lambda\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)=\Phi\left(\frac{\lambda\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)$$

for all $\lambda<0$. Hence it follows from (2) and (7) that for each $\lambda<0$

$$\underline{\lim_{n\to\infty}}\, P_{n,\theta_0}(\{n^{1/2}(\hat{\theta}_n-\theta_0)\leqq\lambda\})\geqq\Phi\left(\frac{\lambda\sigma\sqrt{I}}{\sqrt{1-\theta_0^2}}\right)\,.$$

Since θ_0 is arbitrary, (24) holds. Thus we complete the proof.

From Theorem 2 and Definitions 1 and 3 we get the following theorem.

THEOREM 3. *Under Assumptions* (3.1) *and* (4.1)-(4.4), *an AMU estimator* $\{\hat{\theta}_n\}$ *is asymptotically efficient if and only if the limiting distribution of* $n^{1/2}(\hat{\theta}_n-\theta)$ *is normal with mean* 0 *and variance* $(1-\theta^2)/\sigma^2 I$.

The least squares estimator $\hat{\theta}_{LS}$ of θ is given by $\left(\sum_{t=1}^{n} X_{t-1}X_t\right)\Big/\sum_{t=1}^{n} X_{t-1}^2$. It is shown by Anderson [5] that if $E(U_t^2)<\infty$ then for $|\theta|<1$, $n^{1/2}(\hat{\theta}_n-\theta)$ has a limiting normal distribution with mean 0 and variance $1-\theta^2$. It is seen that under Assumptions (3.1), (4.1) and (4.2) $\hat{\theta}_{LS}$ is a $\{n^{1/2}\}$-consistent estimator. Then it is easily shown that $\hat{\theta}_{LS}$ is asymptotically median unbiased.

Throughout the remainder of this paper we assume the following:

ASSUMPTION (4.5). $\lim_{u\to\pm\infty} uf(u)=0$.

Then it will be proved that the least squares estimator of θ is asymptotically efficient if and only if $f'(u)/f(u)=cu$, where c is some constant. Indeed, since

$$\sigma^2 I=\left\{\int u^2 f(u)du\right\}\left\{\int \left(\frac{f'(u)}{f(u)}\right)^2 f(u)du\right\}\geqq\left\{\int uf'(u)du\right\}^2=1\ ,$$

"=" is obtained if and only if $f'(u)/f(u)=cu$. It follows by Theorem 2 that the limiting distribution of $n^{1/2}(\hat{\theta}_{LS}-\theta)$ attains the bound of the asymptotic distributions if and only if f is a normal density function with mean 0 and variance σ^2. Hence it is seen by Theorem 3 that the least squares estimator is asymptotically efficient if and only if f is a normal density function with mean 0 and variance σ^2. Therefore we have now established

THEOREM 4. *Under Assumptions* (3.1) *and* (4.1)-(4.5), *a necessary and sufficient condition that the least squares estimator of* θ *be asymptotically efficient is that* f *be a normal density function with mean* 0 *and variance* σ^2.

Remark. As is immediately seen from above, Assumptions (3.1) and (4.1)-(4.5) are not necessary for the proof of sufficiency.

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