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Regularity in Parabolic Phase Transition Problems

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To a great unforgettable Master.

ABSTRACT. We describe the main results obtained in a joint work with Athanasopoulos and Caffarelli on the regularity of viscosity solutions and of their free boundaries for a rather general class of parabolic phase transition problems.

1. The Setting

A rather satisfactory improvement has been made recently in the understanding of evolution free boundary problems with two phases. Three main factors are at the heart of this progress: general results on the boundary behavior in Lipschitz domains of positive solutions of the heat equation (such as backward Harnack inequality), a *monotonicity formula* for a couple of disjointly supported harmonic functions (see Section 3), and a geometrical characterization of the smoothness of the level sets of a solution (Sections 2 and 5).

The use of boundary Harnack principles to prove regularity in free boundary problems originated in the paper [2], concerning the obstacle problem. In a subsequent series of papers [9, 10, 11], Caffarelli developed a complete regularity theory for a general class of two phases of stationary problems. The main strategy in [9, 10], based on some geometrical ideas at the heart of the regularity theory for minimal surfaces, turns out to be adaptable to the parabolic case: the major source of difficulties is given by the competing double homogeneity of the heat equation and of the (hyperbolic) interface condition. Due to this reason, the regularity results in the parabolic case are, in general, weaker than the corresponding elliptic ones.

The techniques and ideas explained in these pages have been applied in other relevant situations as well. We mention, in particular, the papers [6], on the regularity for the free boundary in the porous media equation, [19, 34], where problems arising in combustion theory are considered and [21], on the plasma confinement problem.

Let us now introduce the class of free boundary problems with which we deal and establish a notion of viscosity solutions. Denote by B_R the unit ball in $\mathbf{R}^n$, $n > 1$, centered at the origin and by C_R the cylinder $B_R \times (-R^2, R^2)$. We start by defining classical sub- and supersolutions.

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By a *classical subsolution* (respectively, *supersolution*) in C_R, we mean a continuous function v subcaloric (respectively, supercaloric) in $\Omega^+(v) = \{v > 0$ and in $\Omega^-(v) = \{v < 0\}$ such that:

i) $v \in C^1(\bar{\Omega}^{\pm}(v))$

ii) its zero level set $F(v)$ (*the free boundary*) is a smooth surface

iii) on $F(v)$, $|\nabla v^+| \neq 0$ and

$$\frac{v_t^+}{v_\nu^+} \le G\left(v_\nu^+, v_\nu^-, \nu\right) \qquad \text{(respectively } \ge\text{)} \tag{1.1}$$

Here $\nu = \frac{\nabla v^+}{|\nabla v^+|}$

We require G to be Lipschitz continuous in all its arguments, strictly increasing and strictly decreasing with respect to its first and second argument, respectively.

Definition. A continuous function u in C_R is a *viscosity subsolution* (respectively, *supersolution*) if for every subcylinder $Q \subset C_R$ and every classical supersolution (respectively, *subsolution*) v in Q, $u \le v$ (respectively $u \ge v$) on the parabolic boundary

$\partial_p Q$ of Q, implies $u \le v$ (respectively $u \le v$) in Q. The function u is a *viscosity solution* if it is both a *viscosity supersolution* and a *viscosity subsolution.* □

We allow one of the two phases u^+ or u^- to vanish identically; then u is a solution of a *one phase Stefan problem.* It is easy to check that classical (sub, super) solutions are viscosity (sub, super) solutions.

Perhaps the best known example of evolution free boundary problems, the one that actually motivated our work, is the Stefan problem, a simplified model describing the melting or the solidification of a material with a solid-liquid interphase (see [29, 32, 33]). In this case the general interface condition (1.1) takes the form

$$\frac{v_t^+}{v_\nu^+} = v_\nu^+ - v_\nu^- \tag{1.2}$$

balancing the rate of absorbed or released latent heat (per unit surface) with the heat exchanged across the interface itself

In [28, 30] it is shown that classical solutions exist locally in time, under suitable compatibility conditions on the data.

On the other hand, the solid-liquid interface could develop discontinuities (see [19]) and therefore classical solutions are not expected to exist globally in time.

What one is able to construct for all times are weak solutions; that is, solutions in the sense of distributions to the equation (enthalpy formulation)

$$\Delta u \in \beta(u)_t \tag{1.3}$$

with $\beta(u) = a_1 u^+ - a_2 u^- + \frac{1}{2}\text{sign} u$.

Clearly, a classical solution is a weak solution and as usual one would like to show that a weak solution is as smooth and classical as possible.

From another perspective, the question consists in a comparison between the two notions of solution (see [33] for a more physically oriented discussion).

In principle, the enthalpy formulation (1.3) allows for the formation of the so-called *mushy regions,* corresponding to very fine solid-liquid mixtures; in this case, the two phases are not separated by a sharp interface and $F(u)$ could have positive Lebesgue measure.

However, under suitable conditions (like the absence of a distributed heat source, as in the present case, see [27]), mushy regions do not expand and in particular do not appear if initially $F(u)$ is sharp.

A first important regularity result is the continuity of a weak solution (see [13, 17]); then $F(u) = \partial\{u > 0\} \cap C_T$ becomes the weak free boundary. It is now possible to show that weak solutions of the Stefan problem are viscosity solutions so that further regularity theory enters into the general framework.

A regularity theory for the solution and its free boundary could be developed along the following lines.

a) Suppose that the free boundary is (locally) a Lipschitz graph. Can we deduce further regularity for u and $F(u)$?

For instance, is u Lipschitz continuous across $F(u)$? Is $F(u)$ a $C^{1,\alpha}$ or at least a C^1 graph?

An answer to these questions has its own interest, since many natural geometries and initial data provide solutions satisfying a) (see for instance [31]).

b) Suppose $F(u)$ is not necessarily a graph (locally) but is "flat" in a suitable Lebesgue differentiability sense. Is $F(u)$ a smooth graph?

The question is connected with asymptotic behaviors and finite time regularization: "flat" could mean "close" in L^∞-sense to a smooth asymptotic configuration.

While our answer to the above questions could be considered rather satisfactory, the problem of studying the geometric measure properties of $F(u)$ and their relation with the "flatness" hypothesis in b) is still open.

2. The Obstacle and the One Phase Stefan Problems

We start the examination of question a) by first describing the results in [2] for the obstacle problem and in [8] for the one-phase Stefan problem. In these two simplified situations, the above-mentioned geometrical ideas and the role of comparison theorems of Harnack type can be clearly seen.

In the paper [2], the authors take a local minimizer U of the energy integral ($\lambda > 0$)

$$J(v) = \int_{B_2} \left(\frac{1}{2}|\nabla v|^2 + \lambda v\right) dx \qquad B_r = \left\{x \in \mathbf{R}^n,\ |x| \le r\right\}$$

on $K = \{v \in H^1_{loc},\ v \ge 0 \text{ in } B_2,\ v - u \in H^1_0(B_2)\}$.

The set $\{U = 0\}$ is called the coincidence set and $F = \partial\{U = 0\}$ is the free boundary. It is known that $U \in C^{1,1}(B_{2-\epsilon})$ and that $\Delta U = \lambda$ in $\Omega^+ = \{U > 0\}$ (see [20]).

Assume that, due for instance to global considerations, U is increasing in some direction, say e_1. Therefore, the free boundary is the graph of a function $x_1 = f(x_2, \ldots, x_n)$. Then, we have the following:

Theorem 1.

In B_1, f is $C^{1,\alpha}$ for some $0 < \alpha < 1$.

What is remarkable from our point of view is that the result follows almost at once from a comparison theorem between harmonic functions vanishing on some part of the boundary of a Lipschitz domain. We briefly recall this result and afterward we sketch the proof of the theorem.

Theorem 2. [15, 17]

Suppose $\Omega \in \mathbf{R}^n$ is a bounded Lipschitz domain and u, v are positive harmonic functions in Ω, vanishing on the surface disc $\Delta_{2R}(Q) = B_{2R}(Q) \cap \partial\Omega$, $Q \in \partial\Omega$. Then there exists a positive constant $c = c(n, Lip(\Omega))$ such that

$$c^{-1}\frac{u(A_r)}{v(A_r)} \le \frac{u(x)}{v(x)} \le c\frac{u(A_r)}{v(A_r)} \tag{2.1}$$

for every $x \in \Psi_r(Q) = B_r(Q) \cap \Omega$, $r \le R$, where A_r is a point in Ω with $dist(A_r, \partial\Omega) \approx dist(A_r, Q) \approx r$.

Moreover, $\frac{u(x)}{v(x)}$ is Hölder continuous in $\bar{\Psi}_R(Q)$.

The proof of Theorem 1 goes as follows.

Using an argument of Alt [1] based on maximum principle, it is possible to prove the existence of a cone (*cone of monotonicity* from now on) $\Gamma(e_1, \theta)$, with axis e_1 and opening θ, such that if $\tau \in \Gamma(e_1, \theta)$, then $D_\tau U \geq 0$. This implies that f is a Lipschitz function, with Lipschitz constant $L \leq \cot\theta$. Now, any directional derivative $D_\tau U$ is harmonic in Ω^+ and vanishes on F. Applying the comparison Theorem 2 to $u = D_\tau U$ and $v = D_{e_1} U$ it follows that the ratio $\frac{D_\tau U}{D_{e_1} U}$ is Hölder continuous in $\bar{\Omega}^+ \cap B_{2-\epsilon}$. The implicit function theorem then gives the Hölder continuity of ∇f.

Let us now consider the parabolic counterpart of the obstacle problem; that is, the one-phase Stefan problem in its integrated form. Here, the new unknown is $w(x,t) = \int_0^t u(x,t)ds$ (see [17]) and, in a local setting, w satisfies the following conditions:

i) $w(x,t) \geq 0$ in $C_R = B_R(x_0) \times (t_0 - R^2, t_0 + R^2)$ with $(x_0, t_0) \in \partial\{w > 0\}$, the free boundary;

ii) $\Delta w - w_t = 1$ in $C_R \cap \{w > 0\}$.

From [14] it is known that $w \in C_x^{1,1}(C_{R-\epsilon})$, $w \in C_t^1(C_{R-\epsilon})$, for any $\epsilon > 0$.

As before, assume w is increasing along some spatial direction, say e_1, so that the free boundary is the graph of a function $x_1 = f(x_2, \dots, x_n, t) = f(x', t)$.

We ask if f is a $C^{1,\alpha}$ function for some α, in space and time.

Adapting the argument of Alt, it is again possible to show that in $C_{R-\epsilon}$ there is a space-time cone of monotonicity $\Gamma(\theta, e_1)$, and therefore, f is Lipschitz in space and time. The difficulty here is that the parabolic comparison theorem is weaker than the elliptic one, as a consequence of the time-lag in Harnack inequality. In the present situation, the theorem could be stated as follows.

Suppose $f(0,0) = 0$ (that is $(0,0) \in \partial\{w > 0\}$) and introduce the parabolic boxes and discs, where L is the lipschitz constant of f,

$$\begin{aligned} \Psi_r &= \Psi_r(0,0) = \left\{(x,t) : f\left(x',t\right) < x_1 < 4Lr,\ \left|x'\right| < r,\ |t| < r^2\right\} \\ \Delta_r &= \Delta_r(0,0) = \left\{(x,t) : x_n = f\left(x',t\right),\ \left|x'\right| < r,\ |t| < r^2\right\} . \end{aligned}$$

Moreover, put

$$\bar{A}_r = \left(0, 3Lr, r^2\right), \qquad A_r = (0, 3Lr, 0), \qquad \underline{A}_r = \left(0, 3Lr, -r^2\right) .$$

Theorem 3. *[23]*

Let u, v be a positive solution of the heat equations (i.e., caloric functions) in Ψ_{2R} continuously vanishing on Δ_{2R}. Then there exists $C = C(u, L)$ such that

$$\frac{u(x,t)}{v(x,t)} \geq C \frac{u\left(\underline{A}_{2R}\right)}{v\left(\bar{A}_{2R}\right)} \tag{2.2}$$

for every $(x,t) \in \Psi_r$, $r \leq R$..

Unfortunately, Theorem 3 does not seem to imply the Hölder continuity up to the boundary of u/v.

The following lemma, however, allows one to overcome this difficulty. Let w satisfy i) and ii) above.

Lemma 1.

There exists a constant $C > 0$, depending only on u, L, $\|w\|_{C_x^{1,1}}$, $\|w\|_{C_t^1}$ such that

$$C^{-1}\sqrt{w(x,t)} \leq |\nabla w(x,t)| \leq C\sqrt{w(x,t)}$$

for every $(x, t) \in C_{R/2}$, *where* $\nabla w = (\nabla_x w, w_t)$.

With this lemma at hand one can prove the following:

Theorem 4.

Let w satisfy i) and ii) above. Suppose that $D_{e_1} w \geq 0$. *Then, in* $C_{R/2}$, *the free boundary is a graph of a* $C^{1,\alpha}$ *function for some* $0 < \alpha < 1$.

Let us sketch the proof of Theorem 4.

The derivatives $D_\tau w$, for $\tau \in \Gamma(\theta, e_1)$ are caloric functions in, say, $\Psi_{R/2}(x_0, t_0)$, vanishing on $\Delta_{R/2}(x_0, t_0)$. Then, in $\psi_{R/2}(x_0, t_0)$, by Theorem 3 applied to $u = D_\tau w$, $v = D_{e_1} w$, we have

$$\frac{D_\tau w(x,t)}{D_{e_1} w(x,t)} \geq C \frac{D_\tau w\left(\underline{A}_{R/2}\right)}{D_{e_1} w\left(\bar{A}_{R/2}\right)} \geq C \frac{\left|\nabla w\left(\underline{A}_{R/2}\right)\right| \cos\alpha(\underline{\nabla}, \tau)}{\left|\nabla w\left(\bar{A}_{R/2}\right)\right|} \tag{2.3}$$

where $\alpha(\underline{\nabla}, \tau)$ denotes the angle between $\nabla w(\underline{A}_{R/2})$ and τ.

Now, by Lemma 1 we have

$$\left|\nabla w\left(\underline{A}_{R/2}\right)\right| \geq C^{-1}\sqrt{w\left(\underline{A}_{R/2}\right)}, \qquad \left|\nabla w\left(\bar{A}_{R/2}\right)\right| \leq C\sqrt{w\left(\bar{A}_{R/2}\right)}\,.$$

On the other hand, by the monotonicity of w along τ and Harnack inequality we have

$$w\left(\underline{A}_{R/2}\right) \geq C(n) w\left(\bar{A}_{R/2}\right)\ .$$

From the above estimates we can write

$$\frac{D_\tau w(x,y)}{D_{e_1} w(x,t)} \geq C\cos\alpha(\underline{\nabla}, \tau) \equiv \delta(\tau)$$

or

$$D_{\nu(\tau)}(u) \geq 0 \qquad \nu(\tau) = \tau - \delta(\tau) e_1\ . \tag{2.4}$$

When τ varies on $\partial\Gamma(\theta, e_1)$, $\nu(\tau)$ describes a new family of directions that contains a larger cone $\Gamma(\theta_2, e_2)$ with $\frac{\pi}{2} - \theta_2 \leq \mu(\frac{\pi}{2} - \theta)$, $\mu < 1$.

Starting now with the cone $\Gamma(\theta_2, e_2)$ in the domain $\Psi_{R/4}(x_0, t_0)$ and iterating the preceding process, we obtain a sequence of cones $\Gamma_k = \Gamma_k(\theta_k, e_k)$ such that

a) in $\Psi_{2^{-k-1}}(x_0, t_0)$ w is increasing along all directions in Γ_k

b) $|e_{k+1} - e_k| \leq C\mu^k$

c) $\frac{\pi}{2} - \theta_{k+1} \leq \mu(\frac{\pi}{2} - \theta_k)$

It follows that $e_k \to e_\infty$, f is differentiable at (x_0, t_0) and e_∞ is the normal vector to the tangent plane.

The geometric decay c) determines the C^α modulus of continuity of ∇f.

3. Lipschitz-Free Boundaries I: Regularity of Viscosity Solutions

In this section we take up question a) of the introduction and in particular the regularity of a viscosity solution with Lipschitz-free boundary. This requires a preliminary investigation on the behavior of a caloric function near its zero-level set. Therefore, the key tools are once again Harnack inequality and the comparison Theorem 3. However, due to the lack of regularity of u, it is impossible to bypass the time-lag weakness of Theorem 3 using something like Lemma 1. The right way to recover an *elliptic* version of the theorem seems to be use of a control of the oscillation of the solution away from the zero level set in order to obtain a backward Harnack inequality as in the following theorem, in which $f = f(x', t)$ is a Lipschitz function (in space and time), with Lipschitz constant L, and $f(0, 0) = 0$. We also use the notations of Section 2. We remark that the theorem also holds if f is Lipschitz with respect to the parabolic distance.

Theorem 5.

Let u be a caloric function in Ψ_2, vanishing on Δ_2, with $u(\underline{A}_{\frac{3}{2}}) = m > 0$ and $M = \sup u$. Then there exists a constant C, depending only on n, L and $\frac{m}{M}$ such that

$$u\left(x, t+\rho^2\right) \leq Cu\left(x, t-\rho^2\right)$$

for all $(x, t) \in \Psi_1$ and for all ρ, $0 < \rho \leq \frac{d_{x,t}}{4L}$ $\quad (d_{x,t} = dist((x, t), \Delta_2)$.

Backward Harnack inequalities were first introduced by Fabes et al. in the paper [22] for solutions vanishing on the lateral part of a cylinder (e.g., the Green's function), to parabolic equations in divergence form, with time independent coefficients. Extensions to non-cylindrical domains and to non-divergence form equations can be found in [23] and in [26], respectively. Finally, extensions to time dependent operators are due to Fabes et al. in [24, 25].

We list some of the consequences of Theorem 5 in the following corollary.

Corollary 1.

Let u be as in Theorem 5. Then, in Ψ_1,

i) there exists a cone of monotonicity $\Gamma(\theta, e_1)$ in space and time;

ii) there exists a constant $C = C(n, L, \frac{m}{M})$ such that

$$C^{-1}\frac{u(x,t)}{d_{x,t}} \leq \left|\nabla_{x,t}u(x,t)\right| \leq C\frac{u(x,t)}{d_{x,t}} \ ;$$

iii) there exists $b = b(n, L, \frac{m}{M})$ such that, for each t fixed in $(-\frac{3}{2}, \frac{3}{2})$, the functions

$$w^+ = u + u^{1+b} \quad w^- = u - u^{1+b}$$

are, respectively, sub- and superharmonic.

In particular, the consequence iii) says that for t fixed, u is "almost" a harmonic function. This, in turn, implies the possibility to use the following *monotonicity formula* of [3].

Theorem 6.

Let w_1, w_2 be continuous subharmonic functions in the unit ball B_1 such that $w_1(0) = w_2(0) = 0$ and $w_1 w_2 = 0$ in B_1. Then (ρ, σ are radial coordinates in $\mathbf{R}^n$)

$$g(r) = r^{-4}\int_{B_r} |\nabla w_1|^2 \rho d\rho d\sigma \int_{B_r} |\nabla w_2|^2 \rho d\rho d\sigma \tag{3.1}$$

is an increasing function of r. Moreover, if $w = w_1 + w_2$, for $r \leq \frac{1}{2}$, we have

$$g(r) \leq C(n)\|w\|^4_{L^\infty(B_1)} \ . \tag{3.2}$$

A parabolic version of this formula has been proved recently by Caffarelli [12] for the heat equation and extended by Caffarelli and Kenig [16] to general equations with Dini-continuous coefficients.

This formula is the key tool in proving the Lipschitz continuity of the solution of our free boundary problem, since it gives a control of the behavior at the origin of the gradients of w_1 and w_2.

Indeed, if u is a viscosity solution with Lipschitz-free boundary, we can apply formula (3.1) to $w_1 = u^+ + (u^+)^{1+b}$ and $w_2 = u^- + (u^-)^{1+b}$ and conclude that, at any fixed level of time, for small r

$$r^{-2n} \int_{\Omega^+(u)\cap B_r} |\nabla u^+|^2 \, dx \int_{\Omega^-(u)\cap B_r} |\nabla u^-|^2 \, dx \leq C \tag{3.3}$$

where C depends on the usual quantities.

It is then possible to prove the following result.

Theorem 7.

Let u be a viscosity solution to a free boundary problem in C_2. If $F(u)$ is Lipschitz in some direction ν, then u is Lipschitz continuous in C_1.

Clearly, the Lipschitz regularity of u is optimal, given the jump of the gradients across $F(u)$.

Here is a sketch of the proof. Let $(x_0, t_0) \in \Omega^+(u)$ at distance d (small) from the free boundary. Suppose that the $(n+1)$–dimensional ball $B_d(x_0, t_0)$ touches $F(u)$ at $(0, 0)$ and set $u(x_0) = Mh$. Notice that $cd \leq h \leq d$. We want to show that M is controlled from above.

Using a barrier function and maximum principle, it is not hard to prove that in $B_{\frac{1}{2}h}(0)$, $u^- = -\alpha x_1^- + o(|x|)$, with $\alpha \geq 0$, and $u^+(\frac{h}{8}, x', 0) \geq cMh$ for $|x'| \leq \frac{h}{16}$.

Therefore, if ℓ is any segment from $F(u)$ to $x_1 = \frac{h}{8}$ we have

$$\int_\ell \left(w^+\right)_{x_1} dx_1 \geq cMh \, .$$

Integrating in x' and using Hölder inequality we get

$$h^{-n} \int_{B_{\frac{1}{20}h}} |\nabla u^+|^2 \, dx \geq CM^2 \, .$$

Similarly we obtain

$$h^{-n} \int_{B_{\frac{1}{20}h}} |\nabla u^-|^2 \, dx \geq C\alpha^2 \, .$$

From the monotonicity formula we deduce $M^2\alpha^2 \leq c$ and therefore if M is large, then α is small.

Consider now the function

$$\psi(x,t) = \frac{M}{3}x_1 + \beta^+ t + \frac{\beta^+}{2}x_1^2 - c_1\left(t^2 + \frac{1}{2}t|x|^2\right) - c_2\left(2^{-2n}\left|x'\right|^2 - x_1^2\right)$$

where β^+ is chosen such that

$$\frac{M}{10}G\left(\frac{M}{3}, \frac{3\alpha}{2}, e_1\right) < \beta^+ < \frac{M}{3}G\left(\frac{M}{3}, \frac{3\alpha}{2}, e_1\right) \, . \tag{3.4}$$

Choosing c_1,c_2 large enough, the function

$$\phi = \psi^+ - \frac{9}{2M}\alpha\psi^-$$

is a classical subsolution in a small cylinder $B_\delta \times (0, t_0)$ such that $\phi \leq u$ on its parabolic boundary. Therefore, $\phi \leq u$ in the whole cylinder.

Now, if M is very large, from (3.4) $\frac{\phi_t^+(0,0)}{\phi_{e_1}^+(0,0)}$ becomes very large (α is small). Hence, $F(\phi)$ and $F(u)$ must cross each other which is a contradiction.

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A Counterexample

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ɪ λ is an integer.
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equals the first eigenvalue on $S_{\lambda,\mu}$. For

main becomes very narrow and α tends

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s of degree α, and vanishing on $\partial D_{\lambda,\mu}$.

Define

$$\begin{cases} w^+(x,t) = (1+Mt)h_{\lambda_0+1,t}(x) & \text{in } D_{\lambda_0+1,t} \\ w^+(x,t) = 0 & \text{otherwise}. \end{cases}$$

Observe now that $D^c_{\lambda,0}$, the complement of $D_{\lambda,0}$, can be obtained from $D_{\lambda,0}$ by a rotation R. As before, define

$$\begin{cases} w^-(x,t) = -(1+Mt)h_{\lambda_0+1,-t}(Rx) & \text{in } D_{\lambda_0+1,-t} \\ w^-(x,t) = 0 & \text{otherwise}. \end{cases}$$

For ϵ small and M large, the two functions w^+ and w^- turn out to be, respectively, supersolution and subsolution of the one-phase Stefan problem in $B_\epsilon \times [0,\epsilon]$, sharing the same initial data and the same initial free boundary. Let u now be the solution of the two-phase Stefan problem in the cylinder $B_\epsilon \times [0,\epsilon]$ with initial and lateral data satisfying

$$w^- \le u \le w^+ .$$

Then, the free boundary of u is contained in the set

$$D_{\lambda_0+1,-t} \cap D_{\lambda_0+1,+t}$$

and therefore, it has a corner singularity at the origin for $t \in [0,\epsilon]$.

The counterexample brings out the main source of difficulty in studying this kind of problem: the duality between the parabolicity of heat diffusion and the hyperbolicity of the free boundary relation. Indeed, V_ν, u^+_ν and u^-_ν are invariant under hyperbolic dilations of the graph of u in $\mathbf{R}^{n+1}$; that is, under the transformation $u \longrightarrow \frac{u(\lambda x,\lambda t)}{\lambda}$, while the heat equation is invariant under the parabolic dilation $u \longrightarrow \frac{u(\lambda x,\lambda^2 t)}{\lambda}$. A closer look reveals that two main factors seem to prevent immediate smoothing:

i) the simultaneous vanishing of the heat fluxes from both sides of the free boundary, and

ii) the size of the Lipshitz constant.

Therefore, a regularity theory can be developed only under additional conditions, able to prevent i) and/or ii) above. The next section is devoted to the examination of i).

5. Lipschitz-Free Boundary III. A Nondegenerate Case

The counterexample of Section 4 shows that one cannot expect Lipschitz-free boundary to regularize instantaneously. On the other hand, we have seen two kinds of conditions that can be imposed to recover regularity. Here we treat the first one; that is, a *nondegeneracy condition* which prevents simultaneous vanishing of the two fluxes from both sides of the free boundary. In this case, one can prove that viscosity solutions are actually classical ones. More precisely we have the following:

Theorem 9. [6]

Let u be a viscosity solution of a free boundary problem in C_2 such that $F(u)$ is given by the graph of a Lipschitz function $x_1 = f(x',t)$ with Lipschitz constant L. Assume that $(0,0) \in F(u)$, that $M = \sup_{C_2} u$, $u(e_1, -\frac{3}{2}) = 1$, and that

i) $G = G(a,b,\nu)$ is Lipschitz in all its arguments with Lipschitz constant L_G and, for some positive c^*

$$D_a G \ge c^* \qquad D_b G \le -c^* ;$$

ii) (nondegeneracy condition) there exists $m_0 > 0$ such that if (x_0,t_0) is a regular point for $F(u)$ then, for any small r,

$$\int_{B_r(x_0)} |u| dx \ge m_0 r^{n+1} .$$

Then, the following conclusions hold:

There exist positive constants c_1, c_2, depending only on n, L, M, L_G, c^*, m_0, such that,

1) In C_1, $F(u)$ is a C^1 graph in space and time, and for every $(x_1, x', t), (y_1, y', s) \in F(u)$

$$\left|\nabla_x' f\left(x', t\right) - \nabla_x' f\left(y', t\right)\right| \leq c_1 \left(-log\left|x' - y'\right|\right)^{-\frac{4}{3}} ;$$
$$\left|D_t f\left(x', t\right) - D_t f\left(x', s\right)\right| \leq c_1 (-log|t - s|)^{-\frac{1}{3}} ;$$

2) $u \in C^1(\bar{\Omega}^+) \cap C^1(\bar{\Omega}^-)$ and on $F(u) \cap C_1$

$$u_\nu^+ \geq c_2 > 0$$

Therefore, u is a classical solution.

We already said that the general strategy to attack the problem is the same as in [9], for the elliptic cae. In particular the starting point is the existence of a cone of monotonicity $\Gamma(\theta, e_1)$ in space and time, such that, in a neighborhood of $F(u)$, $D_\tau u \geq 0$ for every $\tau \in \Gamma(\theta, e_1)$. This follows from Corollary 1.

The second step consists in improving the opening of the cone away from $F(u)$.

The third one is to carry this improvement to the free boundary, in a smaller cylinder, decreasing in this way the Lipschitz constant of f.

Finally, an appropriate rescaling and iteration of the above steps gives the result.

Let us go back to Step 2. In enlarging the cone away from $F(u)$ one realizes that the derivatives of u along purely spatial directions behave differently from those involving a time component. This different behavior will of course result in a different opening speed between the spatial section of the cone $\Gamma^x(e_1, \theta_1^x)$ and the space-time section $\Gamma^t(\nu_1, \theta_1^t)$, where ν_1 belongs to the plane spanned by e_1, e_t.

The improvement of the cone $\Gamma^x(e_1, \theta_1^x)$ away from $F(u)$ can be done in parabolic homogeneity as in Theorem 4, using the backward Harnack inequality. This means that in a cylinder, say $B_{\frac{1}{8}}(\frac{2}{3}e_1) \times (-\frac{1}{64}, \frac{1}{64})$, $D_\tau u \geq 0$ for every $\tau \in \Gamma^x(\bar{e}_1, \bar{\theta}_1^x)$ with

$$\frac{\bar{\delta}}{\delta} \leq b < 1 \tag{5.1}$$

where we call $\delta = \frac{\pi}{2} - \theta^x$ the *defect angle in space.*

It turns out that parabolic homogeneity is not enough here; hence, one has to refine the calculations to achieve the gain in a hyperbolically scaling cylinder.

The cylinder that can be obtained is of the form $B_{\frac{1}{8}}(\frac{2}{3}e_1) \times (-\frac{\delta}{\mu}, \frac{\delta}{\mu})$, where $\mu = \frac{\pi}{2} - \theta^t$ is the *defect angle in time.*

We only mention that at this point one has to take care of a delicate question. The nondegeneracy condition ii) in the statement of Theorem 9 assures in a weak sense that, at regular points of $F(u)$, the heat fluxes from both sides of the free boundary are not simultaneously vanishing. However, there is no information about which of the two is not zero. On the other hand, at each step of the iteration process, one needs this information. One way to bypass the problem is to improve the cone of monotonicity also from the negative side of $F(u)$, making sure that in both sides there is a common enlarged cone, which we still denote by $\Gamma^x(\bar{e}_1, \bar{\theta}_1^x)$.

The enlargement in time requires new ideas. Observe first of all that the monotonicity of u along the directions in $\Gamma^t(\nu_1, \theta_1^t)$ amounts to the existance of real c, A, B, such that $0 \leq B - A \leq c\mu$ and

$$A \leq -\frac{D_t u^+}{D_{e_1} u^+} \left(-\frac{D_t u^-}{D_{e_1} u^-}\right) \leq B$$

almost everywhere on $F(u)$ and everywhere else in C_2. To enlarge the opening is equivalent to lowering B or increasing A.

The following lemma gives a simultaneous improvement in time of $\Gamma^t(\nu_1, \theta_1^t)$ on both sides of the free boundary. We set

$$\alpha_+ = D_{e_1}u\left(\frac{3}{4}e_1, 0\right) \quad \alpha_- = D_{e_1}u\left(-\frac{3}{4}e_1, 0\right) .$$

Lemma 2.

Suppose $G(\alpha_+, \alpha_-, e_1) \geq -\frac{A+B}{2}$ (respectively, $G(\alpha_+, \alpha_-, e_1) \leq -\frac{A+B}{2}$. Then, there exist positive C, c such that if δ is small and $\delta \leq c\mu^3$, then

$$-\frac{D_t u}{D_{e_1}u} \leq B - C\mu$$

(respectively, $-\frac{D_t u}{D_{e_1}u} \geq A + C\mu$) in $B_{\frac{1}{8}}(\pm\frac{1}{8}e_1) \times (-\frac{\delta}{\mu}, \frac{\delta}{\mu})$.

From Lemma 2, the enlargement of the monotonicity cone away from and in both sides of the free boundary is obtained.

We now come to Step 3: the improvement of the Lipschitz constant up to the free boundary. Since in Lemma 2 we require the defect angle to be much smaller in space than in time, essentially we have to prove first regularity in space and then in time. In both cases the idea is to use a perturbation argument introduced by Caffarelli in [9], based on the construction of a family of subsolutions, able to *measure* the opening of the cone of monotonicity and to carry the interior enlargement to the free boundary.

The key lemma in the present case is the following.

Lemma 3.

Let u be a caloric function in a domain $D \subset \mathbf{R}^{n+1}$ and suppose that $|u_t| \leq |\nabla_x u|$ in D. Assume ϕ is a C^2 function such that $1 \leq \phi \leq 2$ and, for some small positive constant c_1, c_2, and $c_3 > 1$, satisfies

$$\Delta\phi - c_1 u_t - c_2|\nabla\phi| - c_3\frac{|\nabla\phi|^2}{\phi} \geq 0 \qquad u_t \geq 0 \tag{5.2}$$

in a compactly contained subdomain D' of D.

Then the function

$$v_\phi(x,t) = \sup_{B_{\phi(x,t)}(x,t)} u$$

is subcaloric in $\{v_\phi > 0\} \cap D'$ and in $\{v_\phi < 0\} \cap D'$.

Define now

$$D = \left[B_1 \setminus \bar{B}_{\frac{1}{8}}\left(\pm\frac{3}{4}e_1\right)\right] \times (-T, T)$$

and construct in D the perturbation family ϕ_η, $0 \leq \eta \leq 1$, satisfying (5.2) and also the following conditions, for suitable small positive numbers h, k:

i) $1 \leq \phi_\eta \leq 1 + \eta h$

ii) $\phi_\eta \geq 1 + \eta hk$ in $B_{\frac{1}{2}} \times (-\frac{T}{2}, \frac{T}{2})$

iii) $\phi_\eta = 1$ outside $B_{\frac{8}{9}} \times (-\frac{7}{8}T, T)$

iv) $|\nabla\phi_\eta| \leq C\eta h$.

Let u be our viscosity solution; choose $T = \frac{\delta}{\mu}$ and put ($p = (x,t)$, $q = (y,s)$)

$$v_{\phi_\eta}(p) = \sup_{B_{\epsilon\phi_\eta(p)\sin\theta_1}(p)} u(q - \epsilon e_1) .$$

We ask for every $\epsilon > 0$ and every η, $0 \le \eta \le 1$,

1) is $v_{\phi_\eta} \le u$?

2) is v_{ϕ_η} a subsolution of the free boundary problem?

The answer to both question is *yes* if we add to v_{ϕ_η} a correction term that takes into account the enlargement of the cone of monotonicity from both sides of $F(u)$ away from it. This term can be chosen of the form $W^+ - W^-$, where $W^\pm$ is the caloric measure in $\Omega^\pm(u)$ of the set $\partial B_{\frac{1}{8}}(\pm\frac{3}{4}e_1) \times (-T, T)$ multiplied by $u(\pm\frac{3}{4}e_1, 0)$.

The correction term produces a shift in the speed of the free boundary of the new family of perturbations $\bar{v}_{\phi_\eta} = v_{\phi_\eta} + W^+ - W^-$ that can be controlled by means of the nondegeneracy condition ii) of Theorem 9. Indeed, here is the only point in which this condition has to be used.

Using the family $\bar{v}_{\phi_\eta}$, the answer to questions 1) and 2) above is positive and, therefore, thanks to property ii) of ϕ_η, we conclude that in the cylinder $B_{\frac{1}{2}} \times (-\frac{\delta_1}{2\mu_1}, \frac{\delta-1}{2\mu_1})$, u is increasing along the directions in a spatial cone $\Gamma^x(e_2, \theta_2^x)$ and in a space-time cone $\Gamma^t(\nu_2, \theta_2^t)$.

The *defect angles* in space and time of the new cones, that is $\delta_2 = \frac{\pi}{2} - \theta_2^x$, $\quad \mu_2 = \frac{\pi}{2} - \theta_2^t$, satisfy the relations

$$\begin{aligned}\delta_2 &= \delta_1 - c\frac{\delta_1^2}{\mu_1} \\ \mu_2 &= \mu_1 - c\delta_1 .\end{aligned}$$

Iteration of the whole procedure gives that in a sequence of contracting cylinders $B_{2^{-k}} \times (-\frac{\delta_k}{2^k\mu_k}, \frac{\delta_k}{2^k\mu_k})$, u is increasing along all directions in a sequence of cones $\Gamma^x(e_k, \theta_k^x)$ and $\Gamma^t(\nu_k, \theta_k^t)$ where the respective defect angles satisfy the recurrence relations

$$\delta_{k+1} = \delta_k - c\frac{\delta_k^2}{\mu_k} \tag{5.3}$$

$$\mu_{k+1} = \mu_k - c\delta_k \tag{5.4}$$

with $\delta_k \ll \mu_k^3$.

From (5.3) and (5.4) it easily follows the asymptotic behavior

$$\delta_k \sim \frac{c_1(\eta)}{k^{\frac{3}{2}-\eta}} \qquad \mu_k \sim \frac{c_2(\eta)}{k^{\frac{1}{2}-\eta}}$$

for any small $\eta > 0$.

These asymptotic behaviors correspond exactly to the modulus of continuity of $|\nabla_x' f|$ and $D_t f$ in Theorem 10.

To prove the other assertions of the theorem, notice that for each time level $t_0 \in (-1, 1)$, $\Omega^\pm(u) \cap \{t = t_0\}$ is a Liapunov-Dini domain. Since u_t is bounded, the results of Widman [34] apply and therefore $\nabla_x u^\pm$ are continuous up to the free boundary at each level of time. Finally, using the free boundary condition the proof is easily completed.

Notice that the size of the contracting cylinders reveal very well the underlying double homogeneity of the problem. The proper scaling at each step, $x \longrightarrow 2^{-k}x$, $t \longrightarrow 2^{-k}\frac{\delta_k}{\mu_k}t$, is intermediate between parabolic and hyperbolic scaling. This produces the logarithmic modulus of continuity of $\nabla_{x'} f$.

6. Flatness and Finite Time Regularization

In this section we examine the other factor that seems to prevent immediate smoothing, namely the (large) size of the Lipschitz constant of the free boundary.

A first result that can be proved is the following.

Theorem 10. *[7]*

Let u be a viscosity solution satisfying the hypotheses of Theorem 9 except for the non-degeneracy condition ii).

If the Lipschitz constant in space of f is small enough, then the same conclusions of Theorem 9 hold.

Therefore, if the (Lipschitz) free boundary is *flat* enough in space, an intantaneous regularization takes place.

As an immediate application, consider the counterexample of Section 4. When time increases, the corner at the origin widens until it reaches a critical angle beyond which the free boundary starts regularizing. Of course, the two functions w^+, w_- are no longer supersolution and subsolution, respectively.

The main point in the proof of Theorem 10 is to exploit the flatness of $F(u)$ to recover a sort of non-degeneracy. This can be done by controlling the behavior of superharmonic functions near regular points of their zero level set, in the situation described below, that one encounters at each step in the iteration process. Let $\Gamma_k := \Gamma_k(\theta_k, \nu_k)$ be a sequence of spatial cones and D the domain

$$\{(x', x_n) \in R^n : |x'| < 2, g(x') < x_n < 2\}$$

with g a Lipschitz function.

Suppose that $0 \in F := \{x_n = g(x')\}$ and that

a) $\delta_k := \frac{\pi}{2} - \theta_k \leq \frac{c}{(k+\bar{k})^{1+a}}$, $a > 0$, $\bar{k} \gg 1$, and $|\nu_k - \nu_{k+1}| \leq \bar{c}(\delta_k - \delta_{k+1})$

b) there exist k_0 such that, for $k < k_0$,

$$\Gamma_k \cap [B_{2^{-k}}(0) \backslash B_{2^{-k-1}}(0)$$

and a ball $B_{2^{-4k_0}}$ tangent from inside to F at 0, with inward normal ν_{k_0}, such that $\Gamma_{k_0} \cap B_{2^{-4k_0}} \subset D \cap \Gamma_{k_0}$.

Then the following lemma holds.

Lemma 4.

Let $\{\Gamma_k\}$ and D be as above and w be a positive superharmonic function, continuous in $\bar{D}$ and vanishing on F. Then there exists a constant $C = C(n, a, \bar{k})$ such that, near 0

$$w(x) \geq C w_0 < x, \nu_{k_0} >$$

where w_0 = denotes the minimum of w on the set $\bar{D} \cap \{x_n = 2\}$.

Following now the strategy in Theorem 9, one constructs the family of perturbations $\bar{v}_{\phi_\eta}$. Thanks to Lemma 4, these functions are subsolutions but only for fixed $\epsilon > 0$.

As a consequence, it is impossible to carry to the free boundary the full interior cone enlargement. What it is possible to prove near the free boundary is that along all the directions τ in (space and space-time) larger cones, u is only ϵ-monotone; that is,

$$u(p + \lambda\tau) - u(p) \geq 0 \quad \lambda \geq \epsilon .$$

On the other hand, ϵ-monotonicity implies full monotonicity $\sqrt{\epsilon}$ away from the free boundary, so that it is possible to improve ϵ-monotonicity itself, i.e., decreasing ϵ, at the price of giving up a small portion of the enlarged cone. This improvement, just as in [10], requires a new family of subsolutions.

To achieve the result, one performs a double iteration procedure which consists at each step of a cone enlargement and of an ϵ-monotonicity improvement in a sequence of contracting cylinders with the previous intermediate homogeneity.

The result in Theorem 10 does not really give complete information about the question of the finite time regularization, due to the starting hypothesis of Lipschitz continuity of the free boundary.

For instance, if as time goes to infinity we have convergence to a nice and regular stationary configuration, it could be interesting to know if, after a finite time, the solution becomes a classical one, regardless of its initial data.

A rather satisfactory answer can be obtained without asking $F(u)$ to be a graph and only assuming u to be ϵ-monotone along all directions τ belonging to a space cone $\Gamma^x(e, \theta_x)$ with large enough opening and to a space-time cone $\Gamma^t(\nu, \theta_t)$ containing the direction e (to avoid infinit speed of the free boundary).

Leaving unchanged the other conditions, we reach the same conclusion of Theorem 10.

The ϵ-monotonicity hypothesis turns out to be quite natural in studying the above question. Indeed, it is easy to see that a function u is ϵ-monotone along the directions of a suitable cone as soon as it comes close in L^∞ sense to a strictly increasing function in space with small oscillations in time. This is exactly what happens when a nice asymptotic configuration occurs.

We give a couple of applications to the two-phase Stefan problem. For the first one notice that uniform limits as $t \to \infty$ of solutions of the two-phase Stefan problem are Harmonic functions.

Theorem 11.

Suppose u is a viscosity solution of a two-phase Stefan problem in the cylinder $B_1 \times (0, \infty)$ converging in L^∞_{loc} for $t \to \infty$ to a harmonic function u_∞. Assume that at a point $x_0 \in F(u_\infty)$, $|\nabla_\infty(x_0)| \neq 0$.

Then there exists a neighborhood V of x_0 and $T^* > 0$ such that in $V \times (T^*, \infty)$, $F(u)$ is a C^1 graph and u is a classical solution.

T^* depends clearly on a bound from below of $|\nabla_\infty(x_0)|$ in V.

In the second application, we deal with solutions close to traveling waves; that is, global solutions of the form

$$u_0(x, t) = (A + 1)\left(e^{t-x_n} - 1\right)^+ - A\left(e^{t-x_n} - 1\right)^- .$$

Theorem 12.

Let u be a global solution to the Stefan problem. Suppose there exists a compactly supported and smooth function ϕ such that

$$u_0(x, 0) - \phi(x) \leq u(x, 0) \leq u_0(x, 0) + \phi(x) .$$

Then, after a finite time T^*, depending on u_0, ϕ, the free boundary of u is a smooth graph

$$x_n = g\left(x', t\right)$$

For the proofs of the results in this section see [7].

References

[1] Alt, H.W. (1977). The fluid flow through porous media. Regularity of the free surface, *Manuscripta Math.*, **21**, 255–272.

[2] Athanasopoulos, I. and Caffarelli, L.A. (1985). A theorem of real analysis and its application to free boundary problems, *Comm. Pure Appl. Math.*, **38**, 499–502.

[3] Alt, H.W., Caffarelli, L.A., and Friedman, A. (1984). Variational problems with two phases and their free boundaries, *T.A.M.S.*, **282**, 431–461.

[4] Aronson, D., Caffarelli, L.A., and Vasquez, J.L. (1985). Interfaces with a corner point in one dimensional porous medium flow, *Comm. Pure Appl. Math.*, **38**, 374–404.

[5] Athanasopoulos, I., Caffarelli, L.A., and Salsa, S. (1996). Caloric functions in Lipschitz domains and the regularity of solutions to phase transition problems, *Annals of Math.*, **143**, 413–434.

[6] Athanasopoulos, I., Caffarelli, L.A., and Salsa, S. (1996). Regularity of the free boundary in parabolic phase transition problems, *Acta Math.*, **176**, 245–282.

[7] Athanasopoulos, I., Caffarelli, L.A., and Salsa, S. (1998). Phase Transition Problems of Parabolic Type. Flat Free Boundary are Smooth, *Comm. Pure Appl. Math.*, **51**, 75–110.

[8] Athanasopoulos, I. and Salsa, S. (1991). An application of a parabolic comparison principle to free boundary problems, *Indiana Univ. Math. J.*, **40**, 29–32.

[9] Caffarelli, L.A. (1987). A Harnack inequality approach to the regularity of free boundaries. Part I: Lipschitz free boundaries are $C^{1,\alpha}$, *Revista Math. Iberoamericana*, **3**, 139–162.

[10] Caffarelli, L.A. (1989). A Harnack inequality approach to the regularity of free boundaries. Part II: Flat free boundaries are Lipschitz, *Comm. Pure Appl. Math.*, **42**, 55–78.

[11] Caffarelli, L.A. (1988). A Harnack inequality approach to the regularity of free boundaries. Part III: Existence theory,compactness, and dependence on x, *Ann. Scuola Norm. Sup. Pisa*, **15**, 583–602.

[12] Caffarelli, L.A. (1995). Uniform Lipschitz regularity of a singular perturbation problem, *Diff. and Int. Eqs.*, **8**, 1585–1590.

[13] Caffarelli, L.A. and Evans, L.C. (1983). Continuity of the temperature in the two phase Stefan problem, *Arch. Rat. Mech. Anal.*, **81**, 199–220.

[14] Caffarelli, L.A. and Friedman, A. (1979). Continuity of the temperature in the Stefan problem, *Indiana Univ. Math. J.*, **28**, 53–70.

[15] Caffarelli, L.A., Fabes, E.B., Mortola, S., and Salsa, S. (1981). Boundary behavior of non-negative solutions of elliptic operators in divergence form, *Indiana Univ. Math. J.*, **30**, 621–640.

[16] Caffarelli, L.A. and Kenig, C. Gradient estimates for variable coefficient parabolic equations and singular perturbation problems, Preprint.

[17] DiBenedetto, E. (1982). Continuity of weak solutions to certain singular parabolic equations, *Ann. Mat. Pura Appl.*, **121**, 131–176.

[18] Duvaut, G. (1973). Résolution d'un probleme de Stefan (fusion d'un bloc de glace à zéro degrés), *C.R. Acad. Sci. Paris*, Série I, **276**-A, 1461–1463.

[19] Friedman, A. (1968). The Stefan problem in several space variables, *T.A.M.S.*, **133**, 51–87.

[20] Friedman, A. (1970). *Variational Principles and Free Boundary Problems*, Wiley, New York.

[21] Friedman, A. and Liu, Y. A free boundary problem arising in magneto hidrodynamics system, *Ann. Scuola Norm. Sup. Pisa, cl. sci.*, **22**, 375–448.

[22] Fabes, E.B., Garofalo, N., and Salsa, S. A Backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations, *Ill Journal of Math.*, **30**(4), 536–565.

[23] Fabes, E.B., Garofalo, N., and Salsa, A. (1984). Comparison theorems for temperatures in non-cylindrical domains, *Atti Accad. Naz. Lincei*, Ser. 8 Rend., **78**, 1–12.

[24] Fabes, E.B. and Safonov, M.V. Behavior near the boundary of positive solutions of second order parabolic equations, to appear in *Proc. of El Escorial 96*.

[25] Fabes, E.B., Safonov, M.V., and Yuan, Y. Behavior near the boundary of positive solutions of second order parabolic equations II. to appear in *Trans. Amer. Math. Soc.*

[26] Garofalo, N. (1984). Second order parabolic equations in non-variational form: Boundary Harnack principle and comparison theorems for non-negative solutions, *Ann. Mat. Pura Appl.*, **138**(4), 267–296.

[27] Götz, I.G. and Zaltzman, B.B. (1991). Non-increase of mushy region in a nonhomogeneous Stefan problem, *Quart. Appl. Math.*, **XLIX**, 741–746.

[28] Hanzawa, E.I. (1981). Classical solution oft he Stefan problem, *Tohoku Math.*, **33**, 297–335.

[29] Ladyzenskaja, O.A., Solonnikov, V.A., and Ural'ceva, N.N. (1968). Linear and quasilinear equations of parabolic type, *Trans. Math. Monographs*, **23**, A.M.S., Providence, RI.

[30] Meirmanov, A.M. (1981). On the classical solution of the multidimensional Stefan problem for quasilinear parabolic equations, *Math. U.S.S.R-Sbornik*, **40**, 157–178.

[31] Nochetto, R.H. (1987). A class of non-degenerate two phase Stefan problems in several space variables, *Comm. P.D.E.*, **12**, 21–45.

[32] Rubenstein, L. (1979). The Stefan problem: comments on its present state, *J. Inst. Maths. Applics.*, **124**, 259–277.

[33] Visintin, A. (1996). *Models of Phase Transitions*, Birkhäuser, Boston.

[34] Widman, K.O. (1967). Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations, *Math. Scand.*, **21**, 7–27.

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