The Journal of Fourier Analysis and Applications

Volume 4, Issues 4 and 5, 1998

On the John-Strömberg-Torchinsky Characterization of BMO

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Dedicated to the memory of our dear friend, Gene Fabes

1. Introduction

Functions of bounded mean oscillation were defined by John and Nirenberg in [7] by the condition

$$\|f\|_{BMO(R^d)} = \sup_Q \inf_c \frac{1}{\lambda(Q)} \int_Q |f - c| < \infty \tag{1.1}$$

where Q are cubes and λ stands for Lebesgue measure. As is well known, given a cube Q, the best choice for c is the median value of f on Q, but the mean value

$$f_Q := \frac{1}{\lambda(Q)} \int_Q f$$

is good enough. The main theorem on these functions states that the L^1 condition, (1.1), implies an exponential decay of their distribution functions:

Theorem 1. *[7]*
If $f \in BMO\left(R^d\right)$, then for all $t > 0$

$$\lambda\left\{x \in Q \,\middle|\, \left|f(x) - f_Q\right| > t\right\} \le 2\lambda(Q)\, e^{-\frac{c(d)t}{\|f\|_{BMO(R^d)}}} .$$

This implies that for $1 < p < \infty$ the condition

$$\sup_Q \inf_c \left(\frac{1}{\lambda(Q)} \int_Q |f - c|^p\right)^{\frac{1}{p}} < \infty \tag{1.2}$$

Math Subject Classifications. Primary: 46E30, 46E35; Secondary: 46B25.

Keywords and Phrases. BMO, Dyadic BMO, Local Polynomial Approximation.

Acknowledgements and Notes. *Y. Sagher* – Supported in part by grant #95 – 00225 from the U.S.-Israel Binational Science Foundation.

P. Shvartsman – Supported by the Center for Absorption in Science, Israel Ministry of Absorption of Immigrants and by grant #95 – 00225 from the U.S.-Israel Binational Science Foundation.

ISSN 1069-5869

is equivalent to (1.1). It follows from Theorem 2 that (1.2) is equivalent to (1.1) for $0 < p < \infty$.

Moreover, the weaker condition

$$\sup_{Q} \inf_{P \in \mathcal{P}_k} \left(\frac{1}{\lambda(Q)} \int_Q |f - P|^p \right)^{\frac{1}{p}} < \infty \tag{1.3}$$

where P are polynomials of degree at most k, $P \in \mathcal{P}_k$, any k, also turns out to be equivalent to (1.1).

All these definitions assume a certain local integrability of the functions. Very soon after the definition of $BMO\left(R^d\right)$, John, in [6], gave a characterization of $BMO\left(R^d\right)$ which does not assume any local integrability.

We denote

$$\lambda_Q(E) = \frac{\lambda\left(E \bigcap Q\right)}{\lambda(Q)} .$$

Definition 1.

Let $0 < s < 1$ *and let* f *be a measurable function. We define*

$$\|f\|_{BMO_s(R^d)} = \sup_Q \left(\inf_{c \in R} \left(\inf_{t \geq 0} \lambda_Q \{|f - c| > t\} < s \right) \right) . \quad \square \tag{1.4}$$

Thus, $\gamma > \|f\|_{BMO_s(R^d)}$ iff for each cube Q there exists $c = c(Q)$ so that

$$\lambda_Q \{|f - c| > \gamma\} < s .$$

Theorem 2. *[6]*

If $0 < s < \frac{1}{2}$, *then* $BMO_s\left(R^d\right) = BMO\left(R^d\right)$.

It is easy to see that no result holds for $s > \frac{1}{2}$.

The case $s = \frac{1}{2}$ is considerably harder.

Theorem 3. *[9]*

$BMO_{\frac{1}{2}}\left(R^d\right) = BMO\left(R^d\right)$.

It was pointed out in [8] that $\|f\|_{BMO_s(R^d)} = \sup_Q \inf_c (f - c)^{*,Q}(s)$ where $f^{*,Q}$ is the left-continuous[1] non-increasing rearrangement of f with respect to λ_Q. This connects (1.4) with the Real Interpolation Theory.

Another way of considering (1.4) is its expression in the language of Orlicz spaces. We define

$$\phi_s(u) = \begin{cases} 0 & if \quad u \leq 1 \\ \frac{1}{s} & if \quad u > 1 \end{cases}$$

giving us

$$\begin{aligned} \|f\|_{L^{\phi_s}(Q)} &= \inf \left\{ t \,\middle|\, \int \phi_s\left(\frac{|f|}{t}\right) d\lambda_Q < 1 \right\} \\ &= \inf \left\{ t \,\middle|\, \lambda_Q \{|f| > t\} < s \right\} \\ &= f^{*,Q}(s) . \end{aligned}$$

Of course ϕ_s is not a Young function. In particular, we do not have even a quasi-triangle inequality: Take $f = \chi_{\left[0, \frac{s}{2}\right]}$, $g = \chi_{\left[1 - \frac{s}{2}, 1\right]}$, $Q = [0, 1]$. Then

$$\|f\|_{L^{\phi_s}(Q)} = \|g\|_{L^{\phi_s}(Q)} = 0$$

[1]In the literature f^* usually stands for the right-continuous non-increasing rearrangement.

but

$$\|f+g\|_{L^{\phi_s}(Q)} = 1 .$$

We will see, however, that for $0 < s < \frac{1}{2}$, if $f+g \in \mathcal{P}_k$, then

$$\|f+g\|_{L^{\phi_s}(Q)} \leq c(k,d,s)\left(\|f\|_{L^{\phi_s}(Q)} + \|g\|_{L^{\phi_s}(Q)}\right) \tag{1.5}$$

and for $0 < s < 1$ if $f-g \in \mathcal{P}_k$, then

$$\left| \ \|f\|_{L^{\phi_s}(Q)} - \|g\|_{L^{\phi_s}(Q)} \ \right| \leq c(k,d,s)\|f-g\|_{L^{\phi_s}(Q)} . \tag{1.6}$$

In this paper we generalize the John-Strömberg result in the spirit of (1.3). We say that $f \in BMO_s^k\left(R^d\right)$ if

$$\|f\|_{BMO_s^k(R^d)} := \sup_Q \inf_{P \in \mathcal{P}_{k-1}} (f-P)^{*,Q}(s) < \infty$$

and we prove that for $0 < s < \frac{1}{2}$

$$BMO\left(R^d\right)/\mathcal{P}_{k-1} = BMO_s^k\left(R^d\right) . \tag{1.7}$$

This implies the characterizations (1.1), (1.2), and (1.3). It turns out that this characterization fails for $s = \frac{1}{2}$, unless, of course, $k = 1$.

The characterization (1.7) for some $s > 0$ also follows from Theorem 3.2 in [10]. The approach in [10] is more general than in the present work; the authors consider a space of approximating functions which satisfies a certain set of axioms. The space $\mathcal{P}_{k-1}$ satisfies these axioms, and hence (1.7) follows. However, the axiomatic setup does not yield the critical value of s. In particular the proof of Theorem 3.2 is given for constants and so it does not make it clear that there is a difference between $k = 1$, where the result holds for $0 < s \leq \frac{1}{2}$, and $k > 1$ where the result holds for $0 < s < \frac{1}{2}$ and this range is maximal.

We also consider similar characterizations of dyadic BMO. Interestingly, in the dyadic case the critical value for s depends on the dimension of the space.

The main tool we use is the theory of Local Polynomial Approximation. We refer the reader to [1, 2, 3] for an exposition of this theory.

2. BMO and Local Polynomial Approximation

We begin by proving (1.5) and (1.6). The key to these results is the following important theorem:

Theorem 4. ***[4]***
Let $E \subset Q$ and $\lambda(E) > 0$. If $P \in \mathcal{P}_k$, then

$$\sup_{x \in Q} |P(x)| \leq b(k,d)\left(\frac{\lambda(Q)}{\lambda(E)}\right)^k \sup_{x \in E} |P(x)| . \tag{2.1}$$

Remark 1.
In the rest of the article, $b(k,d)$ will be the constant appearing in (2.1). All other constants in the article may have different values at different occurrences. □

Remark 2.

The Brudnyi-Ganzburg theorem is more general: they consider convex sets instead of cubes, and give the best possible constant in this context. □

Theorem 5.

If $P \in \mathcal{P}_k$, then

$$P^{*,Q}(0) \le b(k,d) \inf_{0<s<1} \frac{P^{*,Q}(s)}{(1-s)^k} . \tag{2.2}$$

Proof. For any $\varepsilon > 0$

$$\lambda_Q \left\{ |P| > P^{*,Q}(s) + \varepsilon \right\} < s .$$

Let

$$A = \left\{ x \in Q \,\middle|\, |P(x)| \le P^{*,Q}(s) + \varepsilon \right\}$$

so that $\lambda_Q(A) > 1 - s$ and thus

$$\begin{aligned} \sup_{x \in Q} |P(x)| &\le b(k,d) \left(\frac{\lambda(Q)}{\lambda(A)} \right)^k \sup_{x \in A} |P(x)| \\ &\le b(k,d) \frac{P^{*,Q}(s) + \varepsilon}{(1-s)^k} \end{aligned}$$

and since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Corollary 1.

If $P \in \mathcal{P}_k$, then for $0 < s < 1$

$$\|P\|_{L^\infty(Q)} \le c(k,d,s) \|P\|_{L^{\phi_s}(Q)} .$$

Remark 3.

Inequality (2.2) is sharp: take $P(x) = (1-x)^k$ on $[0,1]$. $P^{*,Q}(0) = 1$, $P^{*,Q}(s) = P(s) = (1-s)^k$. □

Remark 4.

A precise inequality connecting $P^*(\sigma)$ and $P^*(\tau)$ for polynomials on a convex set was given by Ganzburg in [5].[2] □

Theorem 6.

If $P \in \mathcal{P}_k$, then for $0 < s < 1$

$$(f + P)^{*,Q}(s) \le f^{*,Q}(s) + \frac{b(k,d)}{(1-s)^k} P^{*,Q}(s) .$$

Proof. Since

$$\lambda_Q \{ |f + P| > t \} \le \lambda_Q \left\{ |f| + \sup_{y \in Q} |P(y)| > t \right\}$$

we have

$$\inf \left\{ t \,\middle|\, \lambda_Q \{ |f+P| > t \} < s \right\} \le \inf \left\{ t \,\middle|\, \lambda_Q \left\{ |f| + \sup_{y \in Q} |P(y)| > t \right\} < s \right\}$$

[2]The notation in [5] may be misleading: * stands for the non-decreasing rearrangement there.

$$= \inf\left\{t \left| \lambda_Q \left\{|f| > t - \left(\sup_{y\in Q}|P(y)|\right)\right\} < s\right.\right\}$$
$$= \inf\left\{t \left| \lambda_Q \{|f| > t\} < s\right.\right\} + \left(\sup_{y\in Q}|P(y)|\right)$$

and we have

$$\begin{aligned}(f+P)^{*,Q}(s) &\le f^{*,Q}(s) + \left(\sup_{y\in Q}|P(y)|\right)\\ &\le f^{*,Q}(s) + \frac{b(k,d)}{(1-s)^k} P^{*,Q}(s)\,. \qquad \square\end{aligned}$$

Corollary 2.

If $0 < s < 1$, $P_1, P_2 \in \mathcal{P}_k$ *and* f *is a measurable function, then*

$$\Big|\, \|f - P_1\|_{L^{\phi_s}(Q)} - \|f - P_2\|_{L^{\phi_s}(Q)} \,\Big| \le \frac{b(k,d)}{(1-s)^k} \|P_1 - P_2\|_{L^{\phi_s}(Q)}\,.$$

Proof. From the previous theorem

$$(g + P_2 - P_1)^{*,Q}(s) \le g^{*,Q}(s) + \frac{b(k,d)}{(1-s)^k}(P_2 - P_1)^{*,Q}(s)\,.$$

Taking $g = f - P_2$ we have

$$\|f - P_1\|_{L^{\phi_s}(Q)} - \|f - P_2\|_{L^{\phi_s}(Q)} \le \frac{b(k,d)}{(1-s)^k} \|P_1 - P_2\|_{L^{\phi_s}(Q)}\,. \qquad \square$$

The previous theorem is, of course, equivalent to (1.6).

Theorem 7.

If $k > 0$, $P \in \mathcal{P}_k$ *and* f *is a measurable function on a* Q, *then for* $0 < s < \frac{1}{2}$

$$P^{*,Q}(0) \le \frac{b(k,d)}{(1-2s)^k}\left[(f+P)^{*,Q}(s) + f^{*,Q}(s)\right]\,.$$

If $k = 0$ *the inequality holds for* $0 < s \le \frac{1}{2}$.

Proof. Consider the case $k > 0$. From $P = (f + P) + (-f)$ it follows that

$$P^{*,Q}(2s) \le \left[(f+P)^{*,Q}(s) + f^{*,Q}(s)\right]$$

so that, using (2.2), we have

$$\begin{aligned}P^{*,Q}(0) &\le b(k,d)\frac{P^{*,Q}(2s)}{(1-2s)^k}\\ &\le \frac{b(k,d)}{(1-2s)^k}\left[(f+P)^{*,Q}(s) + f^{*,Q}(s)\right]\,.\end{aligned}$$

The proof for $k = 0$ is clear. $\square$

This enables us to prove (1.5):

Corollary 3.

If $k > 0$, $0 < s < \frac{1}{2}$, *and* $f + g = P \in \mathcal{P}_k$, *then*

$$\|f + g\|_{L^{\phi_s}(Q)} \leq \frac{b(k,d)}{(1-2s)^k} \left(\|f\|_{L^{\phi_s}(Q)} + \|g\|_{L^{\phi_s}(Q)}\right) .$$

If $k = 0$ *the inequality holds for* $0 < s \leq \frac{1}{2}$.

Proof. From the previous theorem,

$$\begin{aligned} \|f + g\|_{L^{\phi_s}(Q)} &= \|P\|_{L^{\phi_s}(Q)} \\ &= P^{*,Q}(s) \\ &\leq P^{*,Q}(0) \\ &\leq \frac{b(k,d)}{(1-2s)^k} \left[(P - f)^{*,Q}(s) + f^{*,Q}(s)\right] \\ &= \frac{b(k,d)}{(1-2s)^k} \left(\|g\|_{L^{\phi_s}(Q)} + \|f\|_{L^{\phi_s}(Q)}\right) . \end{aligned}$$

The proof for $k = 0$ is clear. □

Remark 5.

Let us see that the inequality

$$P^{*,Q}(s) \leq b(k,d) \left[(f + P)^{*,Q}(s) + f^{*,Q}(s)\right]$$

does not hold for $\frac{1}{2} \leq s < 1$.

Let $0 < \varepsilon < 1 - s$. *We take on* $[0, 1]$

$$f(x) = \begin{cases} -x & if \quad 0 \leq x < s + \varepsilon \\ 0 & if \quad s + \varepsilon \leq x \leq 1 \end{cases}$$

and $P(x) = x$ *so that*

$$\begin{aligned} (f + P)^{*,Q}(s) &= 0 \\ f^{*,Q}(s) &= \varepsilon \end{aligned}$$

but

$$P^{*,Q}(s) = 1 - s . \quad \square$$

Although L^{ϕ_s} is not an Orlicz space we use the standard notation of Local Approximation Theory:

Definition 2.

Given a measurable function f *we define*

$$\mathcal{E}_k(f; Q)_{L^{\phi_s}(Q)} = \inf_{P \in \mathcal{P}_{k-1}} \|f - P\|_{L^{\phi_s}(Q)} . \quad \square$$

Theorem 8.

Let f *be a measurable function.*

For $0 < s \leq \frac{1}{2}$ *and for every cube* Q *there exists a constant* c_Q *so that*

$$\mathcal{E}_1(f; Q)_{L^{\phi_s}} = \left\|f - c_Q\right\|_{L^{\phi_s}(Q)} .$$

If $k > 1$ and $0 < s < \frac{1}{2}$ then for every cube Q there exists a polynomial $P_Q \in \mathcal{P}_{k-1}$ so that

$$\mathcal{E}_k(f; Q)_{L^{\phi_s}} = \left\|f - P_Q\right\|_{L^{\phi_s}(Q)} .$$

Proof. Let us consider the case $k > 1$. Since

$$\begin{aligned}\left| \|f - P_1\|_{L^{\phi_s}(Q)} - \|f - P_2\|_{L^{\phi_s}(Q)} \right| &\leq \frac{b(k,d)}{(1-s)^{k-1}} \|P_1 - P_2\|_{L^{\phi_s}(Q)} \\ &\leq \frac{b(k,d)}{(1-s)^{k-1}} \|P_1 - P_2\|_{L^\infty(Q)}\end{aligned}$$

we have that

$$F_f(P) := \|f - P\|_{L^{\phi_s}(Q)}$$

is continuous on $\left(\mathcal{P}_{k-1}, \|.\|_{L^\infty(Q)}\right)$. The set

$$A_f := \left\{P \in \mathcal{P}_{k-1} \,\middle|\, \|f - P\|_{L^{\phi_s}(Q)} \leq \mathcal{E}_k(f; Q)_{L^{\phi_s}} + 1\right\}$$

is closed in $\left(\mathcal{P}_{k-1}, \|.\|_{L^\infty(Q)}\right)$:

If $\|f - P_n\|_{L^{\phi_s}(Q)} \leq \mathcal{E}_k(f; Q)_{L^{\phi_s}} + 1$ and $\lim_{n\to\infty} \|P_n - P\|_{L^\infty(Q)} = 0$, then

$$\begin{aligned}\|f - P\|_{L^{\phi_s}(Q)} &\leq \|f - P_n\|_{L^{\phi_s}(Q)} + \frac{b(k,d)}{(1-s)^{k-1}} \|P - P_n\|_{L^{\phi_s}(Q)} \\ &\leq \|f - P_n\|_{L^{\phi_s}(Q)} + \frac{b(k,d)}{(1-s)^{k-1}} \|P - P_n\|_{L^\infty(Q)}\end{aligned}$$

and so

$$\|f - P\|_{L^{\phi_s}(Q)} \leq \mathcal{E}_k(f; Q)_{L^{\phi_s}} + 1 .$$

Since

$$\begin{aligned}\|P\|_{L^\infty(Q)} &\leq \frac{b(k,d)}{(1-2s)^{k-1}} \left(\|f - P\|_{L^{\phi_s}(Q)} + \|f\|_{L^{\phi_s}(Q)}\right) \\ &\leq \frac{b(k,d)}{(1-2s)^{k-1}} \left(\mathcal{E}_k(f; Q)_{L^{\phi_s}} + 1 + \|f\|_{L^{\phi_s}(Q)}\right)\end{aligned}$$

we have that A_f is a compact set in $\left(\mathcal{P}_{k-1}, \|.\|_{L^\infty(Q)}\right)$ and so F_f has a minimum value on the set.

The proof for $k = 1$ is elementary. □

Definition 3.

Given a function f on a cube Q we will denote by $P_{Q,k,s}(f)$ a polynomial in $\mathcal{P}_k$ of best approximation in $L^{\phi_s}(Q)$ for f. We will write P_Q when f,k, s are clear from the context. □

Theorem 9.

If $T \in \mathcal{P}_k$ and $0 < s < 1$, then for all m

$$\mathcal{E}_m(f + T; Q)_{L^{\phi_s}} \leq \mathcal{E}_m(f; Q)_{L^{\phi_s}} + \frac{b(k,d)}{(1-s)^k} \mathcal{E}_m(T; Q)_{L^{\phi_s}} .$$

Proof. We can assume $m < k$.

By Theorem 6

$$\begin{aligned}\mathcal{E}_m(f + T; Q)_{L^{\phi_s}} &= \mathcal{E}_m\left(\left(f - P_{Q,m-1,s}(f)\right) + \left(T - P_{Q,m-1,s}(T)\right); Q\right)_{L^{\phi_s}} \\ &\leq \left\|\left(f - P_{Q,m-1,s}(f)\right) + \left(T - P_{Q,m-1,s}(T)\right)\right\|_{L^{\phi_s}(Q)} \\ &\leq \left\|\left(f - P_{Q,m-1,s}(f)\right)\right\|_{L^{\phi_s}(Q)} + \frac{b(k,d)}{(1-s)^k} \left\|T - P_{Q,m-1,s}(T)\right\|_{L^{\phi_s}(Q)} \\ &= \mathcal{E}_m(f; Q)_{L^{\phi_s}} + \frac{b(k,d)}{(1-s)^k} \mathcal{E}_m(T; Q)_{L^{\phi_s}} . \quad \square\end{aligned}$$

Lemma 1.

Let $0 < s < \frac{1}{2}$. *Given a cube* Q *and a measurable function* f, *if*

$$\lambda\left(Q\Delta Q'\right) \leq \frac{\frac{1}{2}-s}{2}\lambda(Q),$$

then

$$\left\|P_Q - P_{Q'}\right\|_{L^\infty(Q)} \leq \frac{b^2(k,d)\,4^{k-1}}{[(1-s)(1-2s)]^{k-1}}\left(\mathcal{E}_k(f;Q)_{L^{\phi_s}(Q)} + \mathcal{E}_k\left(f;Q'\right)_{L^{\phi_s}(Q)}\right).$$

If $s = \frac{1}{2}$, *then given* $\varepsilon > 0$ *there exists* $\delta = \delta(Q,\varepsilon,f)$ *so that for every* Q' *which satisfies*

$$\lambda\left(Q\Delta Q'\right) \leq \delta\lambda(Q)$$

we have

$$\left|c_Q - c_{Q'}\right| \leq \mathcal{E}_1(f;Q)_{L^{\phi_{\frac{1}{2}}}(Q)} + \mathcal{E}_1\left(f;Q'\right)_{L^{\phi_{\frac{1}{2}}}(Q)} + \varepsilon.$$

Proof. Let us denote

$$s_1 = s + \frac{\frac{1}{2}-s}{2}.$$

We have

$$\begin{aligned}
\left\|P_Q - P_{Q'}\right\|_{L^\infty(Q)} &\leq \frac{b(k,d)}{(1-s_1)^{k-1}}\left\|P_Q - P_{Q'}\right\|_{L^{\phi_{s_1}}(Q)} \\
&\leq \frac{b^2(k,d)}{[(1-s_1)(1-2s_1)]^{k-1}}\left(\left\|f-P_Q\right\|_{L^{\phi_{s_1}}(Q)} + \left\|f-P_{Q'}\right\|_{L^{\phi_{s_1}}(Q)}\right) \\
&\leq \frac{b^2(k,d)}{[(1-s_1)(1-2s_1)]^{k-1}}\left(\left\|f-P_Q\right\|_{L^{\phi_s}(Q)} + \left\|f-P_{Q'}\right\|_{L^{\phi_{s_1}}(Q)}\right) \\
&= \frac{b^2(k,d)}{[(1-s_1)(1-2s_1)]^{k-1}}\left(\mathcal{E}_k(f;Q)_{L^{\phi_s}} + \left\|f-P_{Q'}\right\|_{L^{\phi_{s_1}}(Q)}\right).
\end{aligned}$$

But for any $\varepsilon > 0$

$$\begin{aligned}
&\lambda\left\{x \in Q \,\middle|\, \left|f(x) - P_{Q'}(x)\right| > \mathcal{E}_k\left(f;Q'\right)_{L^{\phi_s}} + \varepsilon\right\} \\
\leq\ &\lambda\left\{x \in Q' \,\middle|\, \left|f(x) - P_{Q'}(x)\right| > \mathcal{E}_k\left(f;Q'\right)_{L^{\phi_s}} + \varepsilon\right\} + \lambda\left(Q\backslash Q'\right) \\
<\ &s\lambda\left(Q'\right) + \lambda\left(Q\backslash Q'\right) \\
\leq\ &s\lambda(Q) + \lambda\left(Q'\backslash Q\right) + \lambda\left(Q\backslash Q'\right) \\
\leq\ &\left(s + \frac{\frac{1}{2}-s}{2}\right)\lambda(Q) \\
=\ &s_1\lambda(Q)
\end{aligned}$$

so that

$$\left\|f - P_{Q'}\right\|_{L^{\phi_{s_1}}(Q)} \leq \mathcal{E}_k\left(f;Q'\right)_{L^{\phi_s}} + \varepsilon$$

and hence:

$$\left\|f - P_{Q'}\right\|_{L^{\phi_{s_1}}(Q)} \leq \mathcal{E}_k\left(f;Q'\right)_{L^{\phi_s}}.$$

Therefore:

$$\begin{aligned}\|P_Q - P_{Q'}\|_{L^\infty(Q)} &\leq \frac{b^2(k,d)}{[(1-s_1)(1-2s_1)]^{k-1}}\left(\mathcal{E}_k(f;Q)_{L^{\phi_s}(Q)} + \mathcal{E}_k(f;Q')_{L^{\phi_s}(Q)}\right)\\ &\leq \frac{b^2(k,d)\,4^{k-1}}{[(1-s)(1-2s)]^{k-1}}\left(\mathcal{E}_k(f;Q)_{L^{\phi_s}(Q)} + \mathcal{E}_k(f;Q')_{L^{\phi_s}(Q)}\right).\end{aligned}$$

Let us consider the case $k = 1$, $s = \frac{1}{2}$.

Given $\varepsilon > 0$ we choose $\delta > 0$ so that

$$\delta < \frac{1}{2} - \lambda_Q\left\{|f - c_Q| > \mathcal{E}_1(f;Q)_{L^{\phi_{\frac{1}{2}}}} + \varepsilon\right\}.$$

Then

$$\begin{aligned}|c_Q - c_{Q'}| &\leq \|f - c_{Q'}\|_{L^{\phi_{\frac{1}{2}}}(Q')} + \|f - c_Q\|_{L^{\phi_{\frac{1}{2}}}(Q')}\\ &= \mathcal{E}_1(f;Q')_{L^{\phi_{\frac{1}{2}}}} + \|f - c_Q\|_{L^{\phi_{\frac{1}{2}}}(Q')}.\end{aligned}$$

But if $\lambda(Q\Delta Q') \leq \delta\lambda(Q)$, we have

$$\begin{aligned}&\lambda\left\{x \in Q' \,\middle|\, |f(x) - c_Q| > \mathcal{E}_1(f;Q)_{L^{\phi_{\frac{1}{2}}}} + \varepsilon\right\}\\ \leq\ &\lambda\left\{x \in Q \,\middle|\, |f(x) - c_Q| > \mathcal{E}_1(f;Q)_{L^{\phi_{\frac{1}{2}}}} + \varepsilon\right\} + \lambda(Q'\backslash Q)\\ <\ &\frac{1}{2}\lambda(Q) - \delta\lambda(Q) + \lambda(Q'\backslash Q)\\ \leq\ &\frac{1}{2}\lambda(Q') + \frac{1}{2}\lambda(Q\backslash Q') - \delta\lambda(Q) + \lambda(Q'\backslash Q)\\ \leq\ &\frac{1}{2}\lambda(Q') - \delta\lambda(Q) + \lambda(Q'\Delta Q)\\ \leq\ &\frac{1}{2}\lambda(Q')\end{aligned}$$

which implies

$$\|f - c_Q\|_{L^{\phi_{\frac{1}{2}}}(Q')} \leq \mathcal{E}_1(f;Q)_{L^{\phi_{\frac{1}{2}}}} + \varepsilon$$

and

$$|c_Q - c_{Q'}| \leq \mathcal{E}_1(f;Q)_{L^{\phi_{\frac{1}{2}}}(Q)} + \mathcal{E}_1(f;Q')_{L^{\phi_{\frac{1}{2}}}(Q)} + \varepsilon. \qquad \square$$

Theorem 10.

Let $Q_0 \subset Q_1$ and $\lambda(Q_1) \leq \frac{3}{2}\lambda(Q_0)$.

If $k > 1$ and $0 < s < \frac{1}{2}$, then

$$\|P_{Q_0} - P_{Q_1}\|_{L^\infty(Q_1)} \leq c(k,d,s) \sup_{Q_0 \subseteq Q \subseteq Q_1} \mathcal{E}_k(f;Q)_{L^{\phi_s}}.$$

If $k = 1$ we have for $0 < s \leq \frac{1}{2}$

$$|c_{Q_0} - c_{Q_1}| \leq 6 \sup_{Q_0 \subseteq Q \subseteq Q_1} \mathcal{E}_1(f;Q)_{L^{\phi_s}}.$$

Proof. Let

$$\gamma > \sup_{Q_0 \subseteq Q \subseteq Q_1} \mathcal{E}_k(f;Q)_{L^{\phi_s}}$$

and suppose that

$$\|P_{Q_0} - P_{Q_1}\|_{L^\infty(Q_1)} > 6\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma . \tag{2.3}$$

Let us prove that if (2.3) holds, then there is a cube Q_2, $Q_0 \subset Q_2 \subset Q_1$ so that for $i = 0,1$,

$$\|P_{Q_2} - P_{Q_i}\|_{L^\infty(Q_1)} > 2\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma . \tag{2.4}$$

We define

$$\alpha = \sup_{Q_0 \subseteq Q \subseteq Q_1}\left\{\lambda(Q) \,\middle|\, \|P_Q - P_{Q_0}\|_{L^\infty(Q_1)} \le 2\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma\right\} .$$

Let $Q_0 \subset Q^m \subset Q_1$ be such that

$$\|P_{Q^m} - P_{Q_0}\|_{L^\infty(Q_1)} \le 2\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma \tag{2.5}$$

and such that

$$\lim_{m\to\infty}\lambda(Q^m) = \alpha .$$

We can assume that the sequence of centers of Q^m converges, and define $\widetilde{Q}$ to be the cube centered at the limit point with $\lambda(\widetilde{Q}) = \alpha$. Clearly,

$$\lim_{m\to\infty}\lambda(Q^m \Delta \widetilde{Q}) = 0 .$$

There are two cases:

1. If

$$\left\|P_{\widetilde{Q}} - P_{Q_0}\right\|_{L^\infty(Q_1)} \le 2\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma \tag{2.6}$$

for all Q which satisfy

$$\lambda(Q\Delta\widetilde{Q}) < \frac{\frac{1}{2}-s}{2}\lambda(\widetilde{Q})$$

we have

$$\left\|P_{\widetilde{Q}} - P_Q\right\|_{L^\infty(\widetilde{Q})} \le \frac{2b^2(k,d)\,4^{k-1}}{[(1-s)(1-2s)]^{k-1}}\gamma .$$

We take $Q^\wedge$ so that $\widetilde{Q} \subset Q^\wedge$ and

$$\lambda(Q^\wedge \setminus \widetilde{Q}) < \frac{\frac{1}{2}-s}{2}\lambda(\widetilde{Q}) .$$

Since

$$\lambda(Q^\wedge) > \lambda(\widetilde{Q}) = \alpha$$

we have that

$$\|P_{Q^\wedge} - P_{Q_0}\|_{L^\infty(Q_1)} > 2\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma .$$

We also have

$$\begin{aligned}\left\|P_{Q^\wedge}-P_{\widetilde{Q}}\right\|_{L^\infty(Q_1)} &\le b(k,d)\left(\frac{\lambda(Q_1)}{\lambda(\widetilde{Q})}\right)^{k-1}\left\|P_{Q^\wedge}-P_{\widetilde{Q}}\right\|_{L^\infty(\widetilde{Q})} \\ &\le b(k,d)\left(\frac{3}{2}\right)^{k-1}\left\|P_{Q^\wedge}-P_{\widetilde{Q}}\right\|_{L^\infty(\widetilde{Q})} \\ &\le 2\frac{b^3(k,d)\,6^{k-1}}{[(1-s)(1-2s)]^{k-1}}\gamma\end{aligned} \tag{2.7}$$

so that, using (2.3), (2.6), and (2.7) we have:

$$\begin{aligned}\left\|P_{Q^\wedge}-P_{Q_1}\right\|_{L^\infty(Q_1)} &\ge \left\|P_{Q_0}-P_{Q_1}\right\|_{L^\infty(Q_1)}-\left\|P_{Q_0}-P_{\widetilde{Q}}\right\|_{L^\infty(Q_1)} \\ &\quad -\left\|P_{Q^\wedge}-P_{\widetilde{Q}}\right\|_{L^\infty(Q_1)} \\ &> 2\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma\end{aligned}$$

and in this case we take $Q_2 = Q^\wedge$.

2. If

$$\left\|P_{\widetilde{Q}}-P_{Q_0}\right\|_{L^\infty(Q_1)} > 2\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma$$

we take $Q_2 = \widetilde{Q}$. If

$$\lambda(Q\Delta Q_2) < \frac{\frac{1}{2}-s}{2}\lambda(Q_2)$$

then

$$\begin{aligned}\left\|P_Q-P_{Q_2}\right\|_{L^\infty(Q_1)} &\le b(k,d)\left(\frac{3}{2}\right)^{k-1}\left\|P_Q-P_{Q_2}\right\|_{L^\infty(Q_2)} \\ &\le \frac{2b^3(k,d)\,6^{k-1}}{[(1-s)(1-2s)]^{k-1}}\gamma\,.\end{aligned} \tag{2.8}$$

If m is sufficiently large so that

$$\lambda\left(Q^m\Delta Q_2\right) < \frac{\frac{1}{2}-s}{2}(Q_2)$$

then, using (2.3), (2.5), and (2.8) we have:

$$\begin{aligned}\left\|P_{Q_1}-P_{Q_2}\right\|_{L^\infty(Q_1)} &\ge \left\|P_{Q_1}-P_{Q_0}\right\|_{L^\infty(Q_1)}-\left\|P_{Q_0}-P_{Q^m}\right\|_{L^\infty(Q_1)} \\ &\quad -\left\|P_{Q^m}-P_{Q_2}\right\|_{L^\infty(Q_1)} \\ &> 2\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma\,.\end{aligned}$$

Proving that (2.3) implies (2.4).

Define for $i = 0,1,2$:

$$A_i = \left\{ x \in Q_i \;\middle|\; \left| f(x) - P_{Q_i}(x) \right| \le \gamma \right\}$$

and let us estimate $\lambda\left(A_i \bigcap A_j\right)$.

If $x \in A_i \bigcap A_j$, then

$$\left| P_{Q_i}(x) - P_{Q_j}(x) \right| \le 2\gamma$$

so that

$$\left\| P_{Q_i} - P_{Q_j} \right\|_{L^\infty\left(A_i \bigcap A_j\right)} \le 2\gamma .$$

From (2.4) and the Brudnyi-Ganzburg Theorem

$$\begin{aligned} 2\frac{18^{k-1} b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma &< \left\| P_{Q_i} - P_{Q_j} \right\|_{L^\infty(Q_1)} \\ &\le b(k,d)\left(\frac{\lambda(Q_1)}{\lambda\left(A_i \bigcap A_j\right)} \right)^{k-1} \left\| P_{Q_i} - P_{Q_j} \right\|_{L^\infty\left(A_i \bigcap A_j\right)} \\ &\le b(k,d)\left(\frac{\lambda(Q_1)}{\lambda\left(A_i \bigcap A_j\right)} \right)^{k-1} 2\gamma \end{aligned}$$

and so for $k > 1$,

$$\lambda_{Q_1}\left(A_i \bigcap A_j\right) < \left[\frac{b(k,d)\,[(1-s)(1-2s)]^{k-1}}{18^{k-1} b^3(k,d)} \right]^{\frac{1}{k-1}} < \frac{1}{18} .$$

Of course

$$\left(A_0 \bigcup A_2\right) \backslash A_1 \subseteq Q_1 \backslash A_1$$

so that

$$\begin{aligned} \lambda(A_0) + \lambda(A_2) &= \lambda\left(\left(A_0 \bigcup A_2\right) \backslash A_1\right) + \lambda\left(A_1 \bigcap A_0\right) \\ &\quad + \lambda\left(A_0 \bigcap A_2\right) + \lambda\left(A_1 \bigcap A_2\right) \\ &\le \lambda(Q_1 \backslash A_1) + \frac{1}{6}\lambda(Q_1) \\ &< \left(s + \frac{1}{6}\right)\lambda(Q_1) . \end{aligned}$$

We also have

$$\lambda(A_0) > (1-s)\lambda(Q_0)$$

and similarly

$$\begin{aligned} \lambda(A_2) &> (1-s)\lambda(Q_2) \\ &> (1-s)\lambda(Q_0) . \end{aligned}$$

Therefore,

$$2(1-s)\lambda(Q_0) < \left(s + \frac{1}{6}\right)\lambda(Q_1) ,$$

i.e.,

$$\begin{aligned}\lambda(Q_0) &< \frac{\left(s+\frac{1}{6}\right)}{2(1-s)}\lambda(Q_1)\\ &\le \frac{2}{3}\lambda(Q_1)\end{aligned}$$

a contradiction. This shows that the original hypothesis, (2.3), fails, proving

$$\|P_{Q_0}-P_{Q_1}\|_{L^\infty(Q_1)} \le 6\frac{18^{k-1}b^3(k,d)}{[(1-s)(1-2s)]^{k-1}}\gamma\,.$$

Since $\gamma > \sup_{Q_0\subseteq Q\subseteq Q_1}\mathcal{E}_k(f;Q)_{L^{\phi_s}}$ is arbitrary, the theorem for $k>1$ is proved.
The proof for $k=1$ is clear. □

The following result is in [2]. For the convenience of the reader we include a proof.

Theorem 11.
Let Q be a cube in R^d with side length r_Q. If $P\in\mathcal{P}_{k-1}$ and $m<k$, then

$$\mathcal{E}_m(P;Q)_{L^\infty} \le c(k,d)\,r_Q^m\sum_{|\alpha|=m}\|\partial^\alpha P\|_{L^\infty(Q)}\,.$$

Proof. Let x_Q be the center of Q. For any $y\in R^d$,

$$\begin{aligned}P(y) &= \sum_{|\beta|\le m-1}\frac{\partial^\beta P(x_Q)}{\beta!}(y-x_Q)^\beta+\sum_{m\le|\beta|\le k-1}\frac{\partial^\beta P(x_Q)}{\beta!}(y-x_Q)^\beta\\ &= P_{m-1}+\sum_{m\le|\beta|\le k-1}\frac{\partial^\beta P(x_Q)}{\beta!}(y-x_Q)^\beta\end{aligned}$$

so that

$$\begin{aligned}\mathcal{E}_m(P;Q)_{L^\infty} &\le \|P-P_{m-1}\|_{L^\infty(Q)}\\ &\le \sum_{m\le|\beta|\le k-1}\frac{r_Q^{|\beta|}}{\beta!}\|\partial^\beta P\|_{L^\infty(Q)}\,.\end{aligned}$$

Let us write $\beta=\alpha+\delta$ where $|\alpha|=m$. By Markov's inequality

$$\begin{aligned}\|\partial^\beta P\|_{L^\infty(Q)} &= \|\partial^\delta(\partial^\alpha P)\|_{L^\infty(Q)}\\ &\le c(k,d)\,r_Q^{-|\delta|}\|\partial^\alpha P\|_{L^\infty(Q)}\,.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{E}_m(P;Q)_{L^\infty} &\le \sum_{0\le|\delta|\le k-m-1\,\&\,|\alpha|=m} r_Q^{m+|\delta|}\|\partial^\delta(\partial^\alpha P)\|_{L^\infty(Q)}\\ &\le c(k,d)\,r_Q^m\sum_{|\alpha|=m}\|\partial^\alpha P\|_{L^\infty(Q)}\,.\end{aligned}$$ □

Theorem 12.

Let f be a measurable function and let $Q_0 \subset Q_1$, $m \le k$ and $0 < s < \frac{1}{2}$. Then there exists a chain of cubes

$$Q_0 = Q^0 \subseteq Q^1 \subseteq \ldots \subseteq Q^\ell \subseteq Q_1$$

so that

$$\left(\frac{3}{2}\right)^i \lambda\left(Q_0\right) \le \lambda\left(Q^i\right) \le \left(\frac{3}{2}\right)^{i+1} \lambda\left(Q_0\right)$$

and

$$\mathcal{E}_m\left(f - P_{Q_1,k,s}(f); Q_0\right)_{L^{\phi_s}} \le c(k,d,s) \sum_{i=0}^{\ell} \left(\frac{2}{3}\right)^{\frac{im}{d}} \mathcal{E}_k\left(f; Q^i\right)_{L^{\phi_s}} .$$

Proof. Let ℓ be an integer such that

$$\left(\frac{3}{2}\right)^{\ell-1} < \frac{\lambda(Q_1)}{\lambda(Q_0)} \le \left(\frac{3}{2}\right)^{\ell}$$

and let us construct

$$Q_0 = \widetilde{Q}^0 \subset \widetilde{Q}^1 \subset \ldots \subset \widetilde{Q}^\ell = Q_1$$

so that for $i = 0, \ldots, \ell - 2$

$$\lambda\left(\widetilde{Q}^{i+1}\right) = \frac{3}{2}\lambda\left(\widetilde{Q}^i\right) .$$

Let us denote $P_i = P_{\widetilde{Q}^i,k,s}(f)$ so that $P_\ell = P_{Q_1,k,s}(f)$. By Theorem 9,

$$\begin{aligned}
\mathcal{E}_m\left(f - P_\ell; Q_0\right)_{L^{\phi_s}} &= \mathcal{E}_m\left(f - P_0 + \sum_{i=0}^{\ell-1}\left(P_i - P_{i+1}\right); Q_0\right)_{L^{\phi_s}} \\
&\le \mathcal{E}_m\left(f - P_0; Q_0\right)_{L^{\phi_s}} + \frac{b(k,d)}{(1-s)^{k-1}} \mathcal{E}_m\left(\sum_{i=0}^{\ell-1}\left(P_i - P_{i+1}\right); Q_0\right)_{L^{\phi_s}} .
\end{aligned}$$

But

$$\begin{aligned}
\mathcal{E}_m\left(f - P_0; Q_0\right)_{L^{\phi_s}} &\le \|f - P_0\|_{L^{\phi_s}(Q_0)} \\
&= \mathcal{E}_k\left(f; Q_0\right)_{L^{\phi_s}}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_m\left(\sum_{i=0}^{\ell-1}\left(P_i - P_{i+1}\right); Q_0\right)_{L^{\phi_s}} &\le \mathcal{E}_m\left(\sum_{i=0}^{\ell-1}\left(P_i - P_{i+1}\right); Q_0\right)_{L^\infty} \\
&\le \sum_{i=0}^{\ell-1} \mathcal{E}_m\left(P_i - P_{i+1}; Q_0\right)_{L^\infty} \\
&\le c(k,d)\, r_{Q_0}^m \sum_{i=0}^{\ell-1} \sum_{|\alpha|=m} \left\|\partial^\alpha\left(P_i - P_{i+1}\right)\right\|_{L^\infty(Q_0)} \\
&\le c(k,d)\, r_{Q_0}^m \sum_{i=0}^{\ell-1} \sum_{|\alpha|=m} \left\|\partial^\alpha\left(P_i - P_{i+1}\right)\right\|_{L^\infty\left(\widetilde{Q}^{i+1}\right)} .
\end{aligned}$$

From Markov's inequality,

$$\left\|\partial^{\alpha}\left(P_i - P_{i+1}\right)\right\|_{L^{\infty}\left(\widetilde{Q}^{i+1}\right)} \leq c(k,d)\, r_{\widetilde{Q}^{i+1}}^{-m} \left\|P_i - P_{i+1}\right\|_{L^{\infty}\left(\widetilde{Q}^{i+1}\right)}$$

and from Theorem 10,

$$\left\|P_i - P_{i+1}\right\|_{L^{\infty}\left(\widetilde{Q}^{i+1}\right)} \leq c(k,d,s) \sup_{\widetilde{Q}^i \subseteq Q \subseteq \widetilde{Q}^{i+1}} \mathcal{E}_k(f;Q)_{L^{\phi_s}} .$$

We choose Q^i, $0 \leq i \leq \ell - 1$ so that

$$\widetilde{Q}^i \subseteq Q^{i+1} \subseteq \widetilde{Q}^{i+1}$$

and

$$\sup_{\widetilde{Q}^i \subseteq Q \subseteq \widetilde{Q}^{i+1}} \mathcal{E}_k(f;Q)_{L^{\phi_s}} \leq 2\mathcal{E}_k\left(f;Q^i\right)_{L^{\phi_s}} .$$

Thus:

$$\begin{aligned}
\mathcal{E}_m\left(f - P_{Q_1};Q_0\right)_{L^{\phi_s}} &\leq \mathcal{E}_m\left(f - P_0;Q_0\right)_{L^{\phi_s}} + c(k,d,s)\, r_{Q_0}^m \\
&\quad \sum_{i=0}^{\ell-1} \sum_{|\alpha|=m} \left\|\partial^{\alpha}\left(P_i - P_{i+1}\right)\right\|_{L^{\infty}\left(\widetilde{Q}^{i+1}\right)} \\
&\leq \mathcal{E}_k\left(f;Q_0\right)_{L^{\phi_s}} + c(k,d,s)\, r_{Q_0}^m \\
&\quad \sum_{i=0}^{\ell-1} r_{\widetilde{Q}^{i+1}}^{-m} \left\|P_i - P_{i+1}\right\|_{L^{\infty}\left(\widetilde{Q}^{i+1}\right)} \\
&\leq \mathcal{E}_k\left(f;Q_0\right)_{L^{\phi_s}} + c(k,d,s) \sum_{i=0}^{\ell-1} \left(\frac{r_{Q_0}}{r_{\widetilde{Q}^{i+1}}}\right)^m \mathcal{E}_k\left(f;Q^i\right)_{L^{\phi_s}} \\
&\leq \mathcal{E}_k\left(f;Q_0\right)_{L^{\phi_s}} + c(k,d,s) \sum_{i=0}^{\ell-1} \left(\frac{2}{3}\right)^{\frac{im}{d}} \mathcal{E}_k\left(f;Q^i\right)_{L^{\phi_s}} . \quad \square
\end{aligned}$$

Definition 4.

We say that

$$f \in BMO_s^k\left(R^d\right)$$

if

$$\|f\|_{BMO_s^k(R^d)} = \sup_Q \mathcal{E}_k(f;Q)_{L^{\phi_s}} < \infty . \quad \square$$

Clearly, $BMO_s^k\left(R^d\right)$ is a space of equivalence classes of functions mod $(\mathcal{P}_{k-1})$. We are now able to prove an extension of Theorem 2.

Theorem 13.

For $0 < s < \frac{1}{2}$ *and* $k > 1$

$$BMO\left(R^d\right)/\mathcal{P}_{k-1} = BMO_s^k\left(R^d\right) . \tag{2.9}$$

Proof. From the previous theorem it follows that if $Q \subset Q'$, then

$$\mathcal{E}_1\left(f - P_{Q',k,s}(f);Q\right)_{L^{\phi_s}} \leq c(k,d,s)\, \|f\|_{BMO_s^k(R^d)} . \tag{2.10}$$

To show that this implies $BMO\left(R^d\right)/\mathcal{P}_{k-1} = BMO_s^k\left(R^d\right)$ we need to show that if $f \in BMO_s^k\left(R^d\right)$ then there exists a polynomial $P \in \mathcal{P}_{k-1}$ so that $f - P \in BMO\left(R^d\right)$.

We follow in outline an idea of Brudnyi.

Let $Q^i = \left[-2^i, 2^i\right]^d$ and let $P_i = P_{Q^i,k,s}(f)$. Then

$$\mathcal{E}_1\left(f - P_{Q^i}; Q^0\right)_{L^{\phi_s}} \le c(k,d,s)\,\|f\|_{BMO_s^k(R^d)}\ .$$

Let us prove that $\left\{\left\|P_{Q^i}\right\|_{L^\infty(Q^0)}\right\}$ is bounded:

We denote by $c_{Q^0}\left(f - P_{Q^i}\right)$ a constant of best approximation in $L^{\phi_s}\left(Q^0\right)$ to $f - P_{Q^i}$. Let us denote

$$\widetilde{P}_{Q^i} = P_{Q^i} + c_{Q^0}\left(f - P_{Q^i}\right)\ .$$

Then

$$\begin{aligned}
\left\|\widetilde{P}_{Q^i}\right\|_{L^\infty(Q^0)} &\le \frac{b(k,d)}{(1-s)^{k-1}}\left\|\widetilde{P}_{Q^i}\right\|_{L^{\phi_s}(Q^0)} \\
&\le \frac{b^2(k,d)}{[(1-s)(1-2s)]^{k-1}}\left(\|f\|_{L^{\phi_s}(Q^0)} + \left\|f - \widetilde{P}_{Q^i}\right\|_{L^{\phi_s}(Q^0)}\right) \\
&= \frac{b^2(k,d)}{[(1-s)(1-2s)]^{k-1}}\left(\|f\|_{L^{\phi_s}(Q^0)} + \mathcal{E}_1\left(f - \widetilde{P}_{Q^i}; Q^0\right)_{L^{\phi_s}}\right) \\
&\le c(k,d,s)\left(\|f\|_{L^{\phi_s}(Q^0)} + \|f\|_{BMO_s^k(R^d)}\right)
\end{aligned}$$

so that $\left\{\left\|\widetilde{P}_{Q^i}\right\|_{L^\infty(Q^0)}\right\}$ is bounded. This implies that the family of coefficients of the polynomials $\widetilde{P}_{Q^i}$ is bounded. Therefore there exists a subsequence, $\left\{\widetilde{P}_{Q^{i_n}}\right\}$, and a polynomial P so that for any cube Q

$$\lim_{n\to\infty}\left\|\widetilde{P}_{Q^{i_n}} - P\right\|_{L^\infty(Q)} = 0\ .$$

Let Q be an arbitrary cube, we can find Q^{i_n} so that $Q \subseteq Q^{i_n}$. By (2.10)

$$\begin{aligned}
\mathcal{E}_1\left(f - \widetilde{P}_{Q^{i_n}}; Q\right)_{L^{\phi_s}} &= \mathcal{E}_1\left(f - P_{Q^{i_n}}; Q\right)_{L^{\phi_s}} \\
&\le c(k,d,s)\,\|f\|_{BMO_s^k(R^d)}
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{E}_1(f - P; Q)_{L^{\phi_s}} &= \mathcal{E}_1\left(f - \widetilde{P}_{Q^{i_n}} + \left(\widetilde{P}_{Q^{i_n}} - P\right); Q\right)_{L^{\phi_s}} \\
&\le \mathcal{E}_1\left(f - \widetilde{P}_{Q^{i_n}}; Q\right)_{L^{\phi_s}} + \frac{b(k,d)}{(1-s)^{k-1}}\mathcal{E}_1\left(\widetilde{P}_{Q^{i_n}} - P; Q\right)_{L^{\phi_s}} \\
&\le c(k,d,s)\left(\|f\|_{BMO_s^k(R^d)} + \mathcal{E}_1\left(\widetilde{P}_{Q^{i_n}} - P; Q\right)_{L^{\phi_s}}\right) \\
&\le c(k,d,s)\left(\|f\|_{BMO_s^k(R^d)} + \left\|\widetilde{P}_{Q^{i_n}} - P\right\|_{L^\infty(Q)}\right)
\end{aligned}$$

and so

$$\|f - P\|_{BMO(R^d)} \le c(k,d,s)\,\|f\|_{BMO_s^k(R^d)}\ .$$

The converse inequality is clear. $\square$

Remark 6.

If $s \geq \frac{1}{2}$ *and* $k > 1$, *then* $BMO_s^k(R^d) \neq BMO(R^d)/\mathcal{P}_{k-1}$. *Any choice of* $P_1 \neq P_2$ *below gives a counter example. Let A be a measurable set and*

$$\begin{aligned} f\,|A &= P_1 \in \mathcal{P}_{k-1} \\ f\,|A^c &= P_2 \in \mathcal{P}_{k-1} \end{aligned}$$

then for $s > \frac{1}{2}$ *we have*

$$\|f\|_{BMO_s(R^d)} = 0$$

and if f *is continuous, then also*

$$\|f\|_{BMO^k_{\frac{1}{2}}(R^d)} = 0 . \qquad \square$$

Let us see that Theorem 13 implies that for $0 < p < \infty$ condition (1.3) characterizes $f \in BMO(R^d)/\mathcal{P}_{k-1}$.

We denote by $[f]$ the equivalence class of f (mod$\mathcal{P}_{k-1}$). Condition (1.3) can be written

$$\sup_Q \|[f]\|_{L^p(Q,d\lambda_Q)} < \infty .$$

By Chebycheff's inequality we have for $0 < s \leq 1$

$$\sup_Q s^{\frac{1}{p}} [f]^{*,Q}(s) \leq \sup_Q \|[f]\|_{L^p(Q,d\lambda_Q)} .$$

This implies

$$\begin{aligned} \|f\|_{BMO_s^k(R^d)} &= \sup_Q [f]^{*,Q}(s) \\ &\leq s^{-\frac{1}{p}} \sup_Q \|[f]\|_{L^p(Q,d\lambda_Q)} \end{aligned}$$

which, by Theorem 13, implies that for $0 < s < \frac{1}{2}$ and $k > 1$

$$\begin{aligned} \|f\|_{BMO(R^d)/\mathcal{P}_{k-1}} &\leq c(k,d,s)\,\|f\|_{BMO_s^k(R^d)} \\ &\leq c(k,d,s) \sup_Q \|[f]\|_{L^p(Q,d\lambda_Q)} . \end{aligned}$$

The converse for $1 \leq p < \infty$ is well known and implies the converse in the case $0 < p < 1$.

3. Local Polynomial Approximation and Dyadic BMO

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \{(\pm 1)^d\}$. We denote

$$R^d_\alpha = \left\{x \in R^d \mid sign\, x_j = \alpha_j\right\} .$$

Since each dyadic cube is contained in an R^d_α it makes sense to consider dyadic $BMO(R^d_\alpha)$ rather than dyadic $BMO(R^d)$. Since the theory does not depend on the choice of α, we carry it out in $R^d_{(1,1,\ldots,1)}$. In this section functions are defined on $R^d_{(1,1,\ldots,1)}$ and Q stands for a dyadic cube in $R^d_{(1,1,\ldots,1)}$, i.e.,

$$Q = \left\{x \mid k_j 2^{-n} < x_j < (k_j+1)\,2^{-n};\ 1 \leq j \leq d\right\}$$

where $k_j \in Z_+$ and $n \in Z$. We denote by $c_Q = c_{Q,s}(f)$ a constant of best approximation for f in $L^{\phi_s}(Q)$. We denote by $DBMO$ the dyadic version of BMO.

Definition 5.

Let $0 < s < 1$ and let f be a measurable function. We define

$$\|f\|_{DBMO_s^k} = \sup_Q \mathcal{E}_k(f;Q)_{L^{\phi_s}}$$

and write $DBMO_s$ for $DBMO_s^1$. □

We will show:

Theorem 14.

If $0 < s \le \frac{1}{1+2^d}$, then $DBMO_s = DBMO$.

We will prove this by showing:

Theorem 15.

Let $f \in DBMO_s$, $0 < s \le \frac{1}{1+2^d}$. Then for all Q and $t > 0$ we have

$$\lambda_Q\left\{\left|f - c_Q\right| > t\right\} \le 2^d e^{-\frac{c(d)t}{\|f\|_{DBMO_s}}} .$$

Theorem 16.

Let $Q \subset Q^\wedge$ and $\lambda(Q) = 2^{-d}\lambda(Q^\wedge)$. If

$$s \le \frac{1}{1+2^d}$$

then

$$\left|c_Q - c_{Q^\wedge}\right| \le \mathcal{E}_1(f;Q)_{L^{\phi_s}} + \mathcal{E}_1\left(f;Q^\wedge\right)_{L^{\phi_s}} . \tag{3.1}$$

Proof. Assume that (3.1) does not hold. Let $\varepsilon > 0$ be such that

$$\left|c_Q - c_{Q^\wedge}\right| > \mathcal{E}_1(f;Q)_{L^{\phi_s}} + \mathcal{E}_1\left(f;Q^\wedge\right)_{L^{\phi_s}} + 2\varepsilon .$$

We have

$$\begin{aligned} &\lambda_Q\left\{\left|f - c_Q\right| > \mathcal{E}_1(f;Q)_{L^{\phi_s}} + \varepsilon\right\} < s \\ &\lambda_{Q^\wedge}\left\{\left|f - c_{Q^\wedge}\right| > \mathcal{E}_1\left(f;Q^\wedge\right)_{L^{\phi_s}} + \varepsilon\right\} < s . \end{aligned}$$

We define:

$$\begin{aligned} A &= \left\{x \in Q \,\middle|\, \left|f(x) - c_Q\right| \le \mathcal{E}_1(f;Q)_{L^{\phi_s}} + \varepsilon\right\} \\ A^\wedge &= \left\{x \in Q^\wedge \,\middle|\, \left|f(x) - c_{Q^\wedge}\right| \le \mathcal{E}_1\left(f;Q^\wedge\right)_{L^{\phi_s}} + \varepsilon\right\} . \end{aligned}$$

Clearly, $A \subseteq Q^\wedge \backslash A^\wedge$ and therefore

$$\begin{aligned} \lambda(A) &\le \lambda\left(Q^\wedge \backslash A^\wedge\right) \\ &< s\lambda\left(Q^\wedge\right) . \end{aligned}$$

But

$$\begin{aligned} \lambda(A) &> (1-s)\lambda(Q) \\ &= (1-s)2^{-d}\lambda\left(Q^\wedge\right) \end{aligned}$$

so that

$$(1-s)\,2^{-d} < s\,,$$

i.e.,

$$s > \frac{1}{1+2^d}$$

so that (3.1) holds when $s \leq \frac{1}{1+2^d}$. □

We define a maximal function corresponding to best approximation in L^{ϕ_s}:

Definition 6.

$$M_s^{\#} f(x) = \sup_{\{Q \mid x \in Q\}} \mathcal{E}_1 (f; Q)_{L^{\phi_s}}\,. \quad \square$$

Thus, $\|f\|_{DBMO_s} = \left\| M_s^{\#} f \right\|_{L^\infty}$.

The following lemma, which has some independent interest, will be used in a version of the Calderón-Zygmund Lemma which we prove below.

Lemma 2.

Let f be a measurable function and $0 < s < 1$. Let A be a set with the following property: for all $x \in A$ there exist arbitrarily small cubes Q so that $x \in Q$ and $c_Q = c_{Q,s}(f) \in [a, b]$. Then at a.e. $x \in A$ we have $f(x) \in [a, b]$.

Proof. Let us assume that

$$\lambda \{x \in A \mid f(x) > b\} > 0\,.$$

Then for some $\delta > 0$ we have that

$$\lambda \{x \in A \mid f(x) > b + \delta\} > 0$$

from which follows that there exists $1 \leq j$ so that

$$\lambda \left\{ x \in A \,\middle|\, \left| f(x) - \left(b + \left(j + \frac{1}{2} \right) \delta \right) \right| \leq \frac{\delta}{2} \right\} > 0\,.$$

Let x_0 be a point of density of this set. Let Q be such that $c_Q \in [a, b]$, $x_0 \in Q$, and Q is sufficiently small so that

$$\lambda_Q \left\{ \left| f - \left(b + \left(j + \frac{1}{2} \right) \delta \right) \right| \leq \frac{\delta}{2} \right\} > \max\{s, 1-s\}$$

which proves $\mathcal{E}_1 (f; Q)_{L^{\phi_s}} \leq \frac{\delta}{2}$. But

$$\lambda_Q \left\{ \left| f - c_Q \right| \leq \mathcal{E}_1 (f; Q)_{L^{\phi_s}} \right\} > 1 - s$$

so that since $c_Q \leq b$ we must have $\mathcal{E}_1 (f; Q)_{L^{\phi_s}} \geq \delta$, a contradiction. □

The next lemma, with minor differences, is proved in [9].

Lemma 3.

(Calderón-Zygmund Lemma for best approximants) Let $0 < s \leq \frac{1}{1+2^d}$, $\delta > 0$, $\beta > 0$. If g is a measurable function defined on Q so that

$$M_s^{\#} g(y) \leq \beta$$

for some $y \in Q$, and

$$\left| c_Q \right| \leq \delta\,,$$

then there exists a family $\{Q_j\}$ of (dyadic) subcubes of Q so that:

1. *Q_j are non-overlapping.*
2. *There exists $y \in Q_j$ so that $M_s^{\#} g(y) \leq \beta$.*
3.

$$\delta < \left|c_{Q_j}\right| \leq \delta + 2\beta . \tag{3.2}$$

4. *$|g(x)| \leq \delta$ a.e. on*

$$Q \backslash \left[\left\{ x \in Q \left| M_s^{\#} g(x) > \beta \right. \right\} \bigcup \left(\bigcup_j Q_j \right) \right]$$

Proof. We bisect each side of Q to get 2^d dyadic cubes. Let Q' be a cube in this generation. Then:

1. If $M_s^{\#} g(x) > \beta$ for all $x \in Q'$ we leave it. The union of these cubes will be $\left\{ x \in Q \left| M_s^{\#} g(x) > \beta \right. \right\}$.
2. If for some $y \in Q'$ we have $M_s^{\#} g(y) \leq \beta$ and if $\left|c_{Q'}\right| > \delta$, then we select Q' into our sequence $\left\{Q_j\right\}$. From Theorem 16,

   $$\left|c_Q - c_{Q'}\right| \leq 2 M_s^{\#} g(y) \leq 2\beta .$$

   But from the hypothesis,

   $$\left|c_Q\right| \leq \delta$$

   so that

   $$\left|c_{Q'}\right| \leq \delta + 2\beta$$

   and we have

   $$\delta < \left|c_{Q'}\right| \leq \delta + 2\beta .$$

3. Finally, if for some $y \in Q'$ we have $M_s^{\#} g(y) \leq \beta$ and $\left|c_{Q'}\right| \leq \delta$, we continue to divide Q'. Observe that in this case f on Q' has the same properties as on Q so that the argument above repeats verbatim, yielding $\left\{Q_j\right\}$.

The cubes Q_j do not overlap and $\delta < \left|c_{Q_j}\right| \leq \delta + 2\beta$. Let

$$x \in Q \backslash \left[\left\{ x \in Q \left| M_s^{\#} g(x) > \beta \right. \right\} \bigcup \left(\bigcup_j Q_j \right) \right] .$$

Then there are arbitrarily small cubes Q so that $x \in Q$ and $\left|c_Q\right| \leq \delta$. By Lemma 2 this implies that $|g| \leq \delta$ a.e. in this set. □

Theorem 17.

Let f be a measurable function defined in R^d and let Q be a cube, $0 < s \leq \frac{1}{1+2^d}$. Then for all $t, \gamma > 0$

$$\lambda_Q \left\{ \left|f - c_Q\right| > t \right\} \leq 2^d e^{-\frac{c(d)t}{\gamma}} + \lambda_Q \left\{ M_s^{\#} f > \gamma \right\} . \tag{3.3}$$

Proof. The proof follows the ideas of Strömberg's proof [9].

If

$$Q \subseteq \left\{ M_s^{\#} f > \gamma \right\} ,$$

then the theorem holds trivially. We can assume therefore that there exists $y \in Q$ so that $M_s^{\#} f(y) \leq \gamma$. We apply the Calderón-Zygmund Lemma to $g := f - c_Q(f)$ with $\beta = \gamma$ and $\delta = 2\gamma$, getting a sequence of cubes $\{Q_i\}$. We have

$$c_{Q_i}(g) = c_{Q_i}(f) - c_Q(f)$$

so that from (3.2)

$$\begin{aligned} 2\gamma &< \left|c_{Q_i}(g)\right| \\ &= \left|c_{Q_i}(f) - c_Q(f)\right| \\ &\leq 2\gamma + 2\gamma = 4\gamma\ . \end{aligned}$$

From here on we write c_Q for $c_Q(f)$. Let

$$\begin{aligned} A &= \left\{x \in Q \mid \left|f(x) - c_Q\right| \leq \gamma\right\} \\ B_i &= \left\{x \in Q_i \mid \left|f(x) - c_{Q_i}\right| \leq \gamma\right\}\ . \end{aligned}$$

Clearly,

$$A \subseteq Q \backslash \left(\bigcup_i B_i\right)\ .$$

Also, of course, if $i \neq j$ then

$$\lambda\left(B_i \bigcap B_j\right) = 0$$

so that

$$\begin{aligned} \sum_i \lambda_Q(B_i) &\leq \lambda_Q(Q\backslash A) \\ &= \lambda_Q\left\{\left|f - c_Q\right| > \gamma\right\} \\ &\leq \lambda_Q\left\{\left|f - c_Q\right| > M_s^{\#} f(y)\right\} \\ &\leq s\ . \end{aligned}$$

For each i there exists $y \in Q_i$ so that $M_s^{\#} f(y) \leq \gamma$ and so that repeating the argument above we have

$$\lambda_{Q_i}\left\{\left|f - c_{Q_i}\right| > \gamma\right\} \leq s\ .$$

Therefore,

$$\lambda(B_i) \geq (1 - s)\,\lambda(Q_i)$$

and so

$$\begin{aligned} \sum_i \lambda(Q_i) &\leq \frac{1}{1-s}\sum_i \lambda(B_i) \\ &\leq \frac{s}{1-s}\lambda(Q) \\ &\leq 2^{-d}\lambda(Q)\ . \end{aligned}$$

Let

$$t_k = 4k\gamma$$

and recall that $\left|c_{Q_i} - c_Q\right| \leq t_1$.

Since $M_s^\# g = M_s^\# f$, from the Calderón-Zygmund decomposition, for almost every x in

$$Q\backslash\left[\left\{x \in Q \,\middle|\, M_s^\# g(x) > \gamma\right\} \bigcup \left(\bigcup_j Q_j\right)\right] = Q\backslash\left[\left\{x \in Q \,\middle|\, M_s^\# f(x) > \gamma\right\} \bigcup \left(\bigcup_j Q_j\right)\right]$$

we have

$$\left|f(x) - c_Q\right| = |g(x)| \le t_1 .$$

If we ignore sets of measure 0 below we have:

$$\left\{x \in Q \,\middle|\, \left|f(x) - c_Q\right| > t_k\right\} \subseteq \left\{x \in Q \,\middle|\, M_s^\# f(x) > \gamma\right\} \bigcup \left(\bigcup_j Q_j\right)$$

and so also

$$\begin{aligned} &\left\{x \in Q \,\middle|\, \left|f(x) - c_Q\right| > t_k\right\} \\ \subseteq\ &\left\{x \in Q \,\middle|\, M_s^\# f(x) > \gamma\right\} \bigcup \left(\bigcup_j \left\{x \in Q_j \,\middle|\, \left|f(x) - c_Q\right| > t_k\right\}\right) . \end{aligned}$$

Consider Q_j :

$$\begin{aligned} &\left\{x \in Q_j \,\middle|\, \left|f(x) - c_Q\right| > t_k\right\} \\ \subseteq\ &\left\{x \in Q_j \,\middle|\, \left|f(x) - c_{Q_j}\right| + \left|c_Q - c_{Q_j}\right| > t_k\right\} \\ \subseteq\ &\left\{x \in Q_j \,\middle|\, \left|f(x) - c_{Q_j}\right| + 4\gamma > t_k\right\} \\ =\ &\left\{x \in Q_j \,\middle|\, \left|f(x) - c_{Q_j}\right| > t_{k-1}\right\} \end{aligned}$$

and hence

$$\begin{aligned} &\left\{\left|f(x) - c_Q\right| > t_k\right\} \\ \subseteq\ &\left\{x \in Q \,\middle|\, M_s^\# f(x) > \gamma\right\} \bigcup \left(\bigcup_j \left\{x \in Q_j \,\middle|\, \left|f(x) - c_{Q_j}\right| > t_{k-1}\right\}\right) . \end{aligned}$$

We repeat the process. We consider only cubes Q_j which contain a point $y = y_Q$ so that $M_s^\# f(y) \le \gamma$. The other Q_j we can reject from the union since they are contained in $\left\{x \in Q \,\middle|\, M_s^\# f(x) > \gamma\right\}$. We get

$$\begin{aligned} &\left\{x \in Q_j \,\middle|\, \left|f(x) - c_{Q_j}\right| > t_{k-1}\right\} \\ \subseteq\ &\left\{x \in Q \,\middle|\, M_s^\# f(x) > \gamma\right\} \bigcup \left(\bigcup_{j_{k-1}} \left\{x \in Q_{j,j_{k-1}} \,\middle|\, \left|f(x) - c_{Q_{j,j_{k-1}}}\right| > t_{k-2}\right\}\right) \end{aligned}$$

so that

$$\begin{aligned} &\left\{x \in Q \,\middle|\, \left|f(x) - c_Q\right| > t_k\right\} \\ \subseteq\ &\left\{x \in Q \,\middle|\, M_s^\# f(x) > \gamma\right\} \bigcup \left(\bigcup_{j,j_{k-1}} \left\{x \in Q_{j,j_{k-1}} \,\middle|\, \left|f(x) - c_{Q_{j,j_{k-1}}}\right| > t_{k-2}\right\}\right) \end{aligned}$$

and

$$\begin{aligned}\sum_j \sum_{j_{k-1}} \lambda\left(Q_{j,j_{k-1}}\right) &\leq 2^{-d} \sum_j \lambda\left(Q_j\right) \\ &\leq 2^{-2d} \lambda(Q)\ .\end{aligned}$$

Repeat the procedure k times and get

$$\begin{aligned}&\left\{x \in Q \,\middle|\, \left|f(x) - c_Q\right| > t_k\right\} \\ \subseteq\ &\left\{x \in Q \,\middle|\, M_s^{\#} f(x) > \gamma\right\} \\ &\bigcup\left(\bigcup_{j, j_{k-1}, \ldots, j_1}\left\{x \in Q_{j, j_{k-1}, \ldots, j_1} \,\middle|\, \left|f(x) - c_{Q_{j, j_{k-1}, \ldots, j_1}}(f)\right| > 0\right\}\right) \\ \subseteq\ &\left\{x \in Q \,\middle|\, M_s^{\#} f(x) > \gamma\right\} \bigcup\left(\bigcup_{j, j_{k-1}, \ldots, j_1} Q_{j, j_{k-1}, \ldots, j_1}\right)\ .\end{aligned}$$

Therefore,

$$\lambda\left\{x \in Q \,\middle|\, \left|f(x) - c_Q\right| > t_k\right\} \leq \lambda\left\{x \in Q \,\middle|\, M_s^{\#} f(x) > \gamma\right\} + 2^{-kd} \lambda(Q)\ .$$

If

$$t_k < t \leq t_{k+1}$$

we have

$$k \geq \frac{t}{4\gamma} - 1$$

and

$$\begin{aligned}\lambda_Q\left\{\left|f - c_Q\right| > t\right\} &\leq \lambda_Q\left\{\left|f - c_Q\right| > t_k\right\} \\ &\leq \lambda_Q\left\{M_s^{\#} f > \gamma\right\} + 2^{-kd} \\ &\leq \lambda_Q\left\{M_s^{\#} f > \gamma\right\} + 2^d e^{-\frac{c(d)t}{\gamma}}\end{aligned}$$

where

$$c(d) = \frac{d \log 2}{4}\ .$$

Of course if $t < t_1$, (3.3) holds trivially. □

Let $f \in DBMO_s$, $0 < s \leq \frac{1}{1+2^d}$. Let us take $\gamma = \|f\|_{DBMO_s}$ so that

$$\lambda_Q\left\{M_s^{\#} f > \gamma\right\} = 0$$

and then for all Q and $t > 0$ we have

$$\lambda_Q\left\{\left|f(x) - c_Q\right| > t\right\} \leq 2^d e^{-\frac{c(d)t}{\|f\|_{DBMO_s}}}$$

proving Theorem 15, and hence, also Theorem 14.

Example 1. Let us see that

$$\|f\|_{DBMO} \leq c(s,d)\, \|f\|_{DBMOs}$$

fails when

$$s > \frac{1}{1+2^d}.$$

Define

$$f(x) = \begin{cases} 1 & if \quad x \in \bigcup_{-\infty<m<\infty} \left(\left[0, 2^{2m+1}\right]^d \setminus \left[0, 2^{2m}\right]^d \right) \\ 0 & if \quad x \in \bigcup_{-\infty<m<\infty} \left(\left[0, 2^{2m}\right]^d \setminus \left[0, 2^{2m-1}\right]^d \right). \end{cases}$$

We claim that if $s > \frac{1}{1+2^d}$, then $\mathcal{E}_1(f; Q)_{L^{\phi_s}} = 0$ for all dyadic cubes.

Clearly f is a constant on any dyadic cube which is not of the form $\left[0, 2^j\right]^d$. If $Q = \left[0, 2^{2m+1}\right]^d$, then

$$\begin{aligned} \lambda \{x \in Q | f(x) = 1\} &= \sum_{k=0}^{\infty} (-1)^k 2^{(2m+1-k)d} \\ &= \lambda(Q) \frac{2^d}{1+2^d} \end{aligned}$$

so that

$$\lambda_Q \{|f - 1| > 0\} = \frac{1}{1+2^d}.$$

Similarly, if $Q = \left[0, 2^{2m}\right]^d$ then

$$\lambda_Q \{|f| > 0\} = \frac{1}{1+2^d}.$$

Clearly, we have, for $s > \frac{1}{1+2^d}$,

$$\|f\|_{DBMO_s} = 0$$

and

$$\|f\|_{DBMO} = \frac{1}{1+2^d}.$$

Observe that this example also shows that Theorem 16 fails when $s > \frac{1}{1+2^d}$. For cubes of the form $\left[0, 2^j\right]^d$ we have

$$|c_Q - c_{Q^\wedge}| = 1$$

whereas

$$\mathcal{E}_1(f; Q)_{L^{\phi_s}} = \mathcal{E}_1\left(f; Q^\wedge\right)_{L^{\phi_s}} = 0.$$

We consider now the dyadic version of BMO_s^k.

As elsewhere in this section we consider functions on $R^d_{(1,\dots,1)}$ and Q stands for dyadic cubes in $R^d_{(1,\dots,1)}$.

We will prove:

For $0 < s < \frac{1}{1+2^d}$ and $k > 1$

$$DBMO / \mathcal{P}_{k-1} = DBMO_s^k.$$

Theorem 18.
Let $Q_0 \subset Q_1$ *and* $\lambda(Q_0) = 2^{-d}\lambda(Q_1)$, $0 < s < \frac{1}{1+2^d}$ *and* $k > 1$. *If* f *is a measurable function, then*

$$\left\| P_{Q_0} - P_{Q_1} \right\|_{L^\infty(Q_1)} \le \left(\frac{2^d}{1 - s\left(2^d + 1\right)} \right)^{k-1} b(k,d)\left(\mathcal{E}_k(f;Q_0)_{L^{\phi_s}} + \mathcal{E}_k(f;Q_1)_{L^{\phi_s}}\right) .$$

Proof. Let

$$\gamma_i > \mathcal{E}_k(f;Q_i)_{L^{\phi_s}}$$

$i = 0, 1$ and let

$$\gamma = \gamma_0 + \gamma_1 .$$

Suppose that

$$\left\| P_{Q_0} - P_{Q_1} \right\|_{L^\infty(Q_1)} > \left(\frac{2^d}{1 - s\left(2^d + 1\right)} \right)^{k-1} b(k,d)\,\gamma . \tag{3.4}$$

Define

$$A_i = \left\{ x \in Q_i \;\middle|\; \left| f(x) - P_{Q_i}(x) \right| \le \gamma_i \right\}$$

and let us estimate $\lambda\left(A_0 \bigcap A_1\right)$.

Since

$$\left\| P_{Q_0} - P_{Q_1} \right\|_{L^\infty\left(A_0 \bigcap A_1\right)} \le \gamma$$

from the Brudnyi-Ganzburg Theorem

$$\begin{aligned}
\left(\frac{2^d}{1 - s\left(2^d + 1\right)} \right)^{k-1} b(k,d)\,\gamma &< \left\| P_{Q_0} - P_{Q_1} \right\|_{L^\infty(Q_1)} \\
&\le b(k,d)\left(\frac{\lambda(Q_1)}{\lambda\left(A_0 \bigcap A_1\right)} \right)^{k-1} \left\| P_{Q_0} - P_{Q_1} \right\|_{L^\infty\left(A_0 \bigcap A_1\right)} \\
&\le b(k,d)\left(\frac{\lambda(Q_1)}{\lambda\left(A_0 \bigcap A_1\right)} \right)^{k-1} \gamma
\end{aligned}$$

and so for $k > 1$,

$$\lambda_{Q_1}\left(A_0 \bigcap A_1\right) < \frac{1 - s\left(2^d + 1\right)}{2^d} .$$

Therefore,

$$\begin{aligned}
\lambda(A_0) &= \lambda(A_0 \backslash A_1) + \lambda\left(A_0 \bigcap A_1\right) \\
&< \lambda(Q_1 \backslash A_1) + \frac{1 - s\left(2^d + 1\right)}{2^d}\lambda(Q_1) \\
&< \left(s + \frac{1 - s\left(2^d + 1\right)}{2^d} \right)\lambda(Q_1) \\
&= \frac{1 - s}{2^d}\lambda(Q_1) .
\end{aligned}$$

On the other hand we also have

$$\lambda(Q_0 \backslash A_0) < s\lambda(Q_0)$$

so that

$$\begin{aligned}(1-s)\lambda(Q_0) &< \lambda(A_0)\\ &< \frac{1-s}{2^d}\lambda(Q_1)\,,\end{aligned}$$

i.e.,

$$\lambda(Q_0) < 2^{-d}\lambda(Q_1)$$

a contradiction. This shows that the original hypothesis, (3.4), fails, proving

$$\|P_{Q_0} - P_{Q_1}\|_{L^\infty(Q_1)} \le \left(\frac{2^d}{1-s\left(2^d+1\right)}\right)^{k-1} b(k,d)\,\gamma\,.$$

Since $\gamma > \left(\mathcal{E}_k(f;Q_0)_{L^{\phi_s}} + \mathcal{E}_k(f;Q_1)_{L^{\phi_s}}\right)$ is arbitrary, the theorem is proved. □

Theorem 19.

Let $Q_0 \subset Q_1$, $m \le k$ *and* $0 < s < \frac{1}{1+2^d}$. *Let*

$$Q_0 = Q^0 \subset Q^1 \subset \ldots \subset Q^\ell = Q_1$$

be such that

$$\lambda(Q_0) = 2^{-id}\lambda\left(Q^i\right).$$

We then have

$$\mathcal{E}_m\left(f - P_{Q_1,k,s}(f);Q_0\right)_{L^{\phi_s}} \le c(k,d,s)\sum_{i=0}^{\ell} 2^{-im}\mathcal{E}_k\left(f;Q^i\right)_{L^{\phi_s}}\,.$$

Proof. Let $P_i = P_{Q^i,k,s}(f)$ so that $P_\ell = P_{Q_1,k,s}(f)$. By Theorem 9,

$$\begin{aligned}\mathcal{E}_m(f-P_\ell;Q_0)_{L^{\phi_s}} &= \mathcal{E}_m\left(f - P_0 + \sum_{i=0}^{\ell-1}(P_i - P_{i+1});Q_0\right)_{L^{\phi_s}}\\ &\le \mathcal{E}_m(f-P_0;Q_0)_{L^{\phi_s}} + \frac{b(k,d)}{(1-s)^{k-1}}\mathcal{E}_m\left(\sum_{i=0}^{\ell-1}(P_i - P_{i+1});Q_0\right)_{L^{\phi_s}}\end{aligned}$$

But

$$\begin{aligned}\mathcal{E}_m(f-P_0;Q_0)_{L^{\phi_s}} &\le \|f-P_0\|_{L^{\phi_s}(Q_0)}\\ &= \mathcal{E}_k(f;Q_0)_{L^{\phi_s}}\end{aligned}$$

and by Theorem 11

$$\begin{aligned}\mathcal{E}_m\left(\sum_{i=0}^{\ell-1}(P_i - P_{i+1});Q_0\right)_{L^{\phi_s}} &\le \mathcal{E}_m\left(\sum_{i=0}^{\ell-1}(P_i - P_{i+1});Q_0\right)_{L^\infty}\\ &\le \sum_{i=0}^{\ell-1}\mathcal{E}_m(P_i - P_{i+1};Q_0)_{L^\infty}\\ &\le c(k,d)\,r_{Q_0}^m\sum_{i=0}^{\ell-1}\sum_{|\alpha|=m}\left\|\partial^\alpha(P_i - P_{i+1})\right\|_{L^\infty(Q_0)}\\ &\le c(k,d)\,r_{Q_0}^m\sum_{i=0}^{\ell-1}\sum_{|\alpha|=m}\left\|\partial^\alpha(P_i - P_{i+1})\right\|_{L^\infty(Q^{i+1})}\,.\end{aligned}$$

From Markov's inequality,

$$\left\| \partial^{\alpha} \left(P_i - P_{i+1} \right) \right\|_{L^{\infty}(Q^{i+1})} \leq c(k,d)\, r_{Q^{i+1}}^{-m} \left\| P_i - P_{i+1} \right\|_{L^{\infty}(Q^{i+1})}$$

and from Theorem 18,

$$\left\| P_i - P_{i+1} \right\|_{L^{\infty}(Q^{i+1})} \leq c(k,d,s) \left(\mathcal{E}_k \left(f; Q^i \right)_{L^{\phi_s}} + \mathcal{E}_k \left(f; Q^{i+1} \right)_{L^{\phi_s}} \right) .$$

Thus:

$$\begin{aligned} \mathcal{E}_m \left(f - P_{Q_1}; Q_0 \right)_{L^{\phi_s}} &\leq \mathcal{E}_m \left(f - P_0; Q_0 \right)_{L^{\phi_s}} + c(k,d,s)\, r_{Q_0}^{m} \sum_{i=0}^{\ell-1} \sum_{|\alpha|=m} \\ &\qquad \left\| \partial^{\alpha} \left(P_i - P_{i+1} \right) \right\|_{L^{\infty}(Q^{i+1})} \\ &\leq \mathcal{E}_k \left(f; Q_0 \right)_{L^{\phi_s}} + c(k,d,s)\, r_{Q_0}^{m} \sum_{i=0}^{\ell-1} r_{Q^{i+1}}^{-m} \left\| P_i - P_{i+1} \right\|_{L^{\infty}(Q^{i+1})} \\ &\leq \mathcal{E}_k \left(f; Q_0 \right)_{L^{\phi_s}} + c(k,d,s) \sum_{i=0}^{\ell-1} \left(\frac{r_{Q_0}}{r_{Q^{i+1}}} \right)^{m} \left(\mathcal{E}_k \left(f; Q^i \right)_{L^{\phi_s}} \right. \\ &\qquad \left. + \mathcal{E}_k \left(f; Q^{i+1} \right)_{L^{\phi_s}} \right) \\ &\leq \mathcal{E}_k \left(f; Q_0 \right)_{L^{\phi_s}} + c(k,d,s) \sum_{i=0}^{\ell-1} 2^{-im} \left(\mathcal{E}_k \left(f; Q^i \right)_{L^{\phi_s}} \right. \\ &\qquad \left. + \mathcal{E}_k \left(f; Q^{i+1} \right)_{L^{\phi_s}} \right) \\ &\leq \mathcal{E}_k \left(f; Q_0 \right)_{L^{\phi_s}} + c(k,d,s) \sum_{i=0}^{\ell} 2^{-im} \mathcal{E}_k \left(f; Q^i \right)_{L^{\phi_s}} . \qquad \square \end{aligned}$$

Theorem 20.
For $0 < s < \frac{1}{1+2^d}$

$$DBMO / \mathcal{P}_{k-1} = DBMO_s^k .$$

Proof. In the proof of Theorem 13 replace the cubes $\left[-2^i; 2^i\right]^d$ by the dyadic cubes $\left[0, 2^i\right]$. The rest of the proof holds without change. $\square$

Let us see an example which shows that the last theorem is sharp in the case of $d = 1$.

Example 2. Let

$$f(x) = \min \left\{ x, \frac{1}{3} \right\} .$$

Let us show that $\|f\|_{DBMO_{\frac{1}{3}}^2} = 0$ although clearly, $\|f\|_{DBMO/\mathcal{P}_1} \neq 0$.

Let us see: If

$$\frac{1}{3} \in Q = \left(k2^{-n}, (k+1)\, 2^{-n} \right)$$

$k \geq 0$ and we denote $L = \left(k2^{-n}, \frac{1}{3} \right)$, then either $\frac{\lambda(L)}{2^{-n}} \leq \frac{1}{3}$ or $\frac{\lambda(L)}{2^{-n}} = \frac{2}{3}$.

If $n < 0$, then $\frac{1}{3} \in Q$ implies $k = 0$ in which case $\frac{\lambda(L)}{2^{-n}} < \frac{1}{3}$.

If $n \geq 0$, then

$$k2^{-n} < \frac{1}{3} < (k+1)\, 2^{-n}$$

implies

$$3k < 2^n < 3k + 3$$

so that either $2^n = 3k + 1$, or $2^n = 3k + 2$ which correspond to $\frac{\lambda(L)}{2^{-n}} = \frac{1}{3}$ and $\frac{\lambda(L)}{2^{-n}} = \frac{2}{3}$.

On each interval which contains $\frac{1}{3}$ we subtract from f the polynomial (i.e., x or $\frac{1}{3}$) which agrees with f on at least $\frac{2}{3}$ of the interval. This proves $\|f\|_{DBMO^{2}_{\frac{1}{3}}} = 0$. Since f is not a linear function $\|f\|_{DBMO/\mathcal{P}_1} \neq 0$.

Acknowledgment

We thank Yu. Brudnyi for his encouragement and helpful discussions.

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Received June 11, 1998

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