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The Absolute Continuity of Elliptic Measure Revisited

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In this note we will describe the main results in [28] and [27]. We will start out by making some historical remarks, in order to put our work in perspective.

In [19], Fatou showed that bounded harmonic functions in the upper half-plane have non-tangential limits for almost every (dx) boundary point in $\mathbb{R}$. This result easily extends to the higher dimensional situation in $\mathbb{R}^{n+1}_+$. A far-reaching extension was obtained by Calderón [4]: If a harmonic function u in $\mathbb{R}^{n+1}_+$ is non-tangentially bounded on a set $E \subset \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$, then u has non-tangential boundary values at almost every point (dx) on E. This was extended by Carleson [6], who obtained the same conclusion, under the weaker assumption that u is non-tangentially bounded from below on E. Related results were also obtained in terms of the square function

$$S^2(u)(x) = \int_{\Gamma(x)} |\nabla u(y,t)|^2 \, y^{1-n} \, dydt \ ,$$

where $\Gamma(x)$ denotes a (truncated) circular cone with vertex at x. Here we have [5], [35]: On a set $E \subset \mathbb{R}^n$, the following two conditions are equivalent:

(i) u is non-tangentially bounded for almost every x in $E(dx)$.

(ii) $S(u)(x) < +\infty$ for a.e. x in $E(dx)$.

In the mid 1960s, Hunt and Wheeden [24] extended Fatou's theorem (as well as Carleson's theorem) to bounded Lipschitz domains $\Omega \subset \mathbb{R}^{n+1}$. These domains have uniform interior (and exterior) cones, and thus, the notion of "non-tangential convergence" makes sense. To recall their results precisely, we recall the notion of "harmonic measure."

Consider the classical Dirichlet problem:

$$\text{(D)} \qquad \begin{cases} \Delta u = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = f \in C(\partial\Omega), \ u \in C(\overline{\Omega}) \ . \end{cases}$$

Then, by the exterior cone condition, (D) is uniquely solvable, and thus, by the maximum principle and the Riesz representation theorem, there exists a family of positive Borel probability measures $\{\omega^X\}$, so that, for $X \in \Omega$, we have

$$u(X) = \int_{\partial\Omega} f \, d\omega^X$$

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for the solution to (D). This family of measures is called *harmonic measure*. Harnack's principle implies that they are mutually absolutely continuous. We usually fix $X_* \in \Omega$, and set $d\omega = d\omega^{X_*}$, and by abuse of notation, call this measure harmonic measure. In [24] it was proved that

$$\omega(B(Q, 2r) \cap \partial\Omega) \leq C\omega(B(Q, r) \cap \partial\Omega) ,$$

for $Q \in \partial\Omega$, $r > 0$ and small, i.e., $d\omega$ is a "doubling measure." It was also proved that, for $f \geq 0$, $N(u)(Q) \simeq M_\omega(f)(Q)$ uniformly for $Q \in \partial\Omega$, where $N(u)(Q) = \sup_{X \in \Gamma(Q)} |u(X)|$ denotes the non-tangential maximal function (here $\Gamma(Q)$ denotes a "regular" family of interior cones), and

$$M_\omega(f)(Q) = \sup_{r>0} \frac{1}{\omega(B(Q, r) \cap \partial\Omega)} \int_{B(Q,r)\cap\partial\Omega} f \, d\omega$$

denotes the Hardy-Littlewood maximal function with respect to the measure $d\omega$.

As a consequence of these facts, they showed that the following Fatou-type theorem holds: If $\Delta u = 0 \in \Omega$, u is bounded in Ω, Ω Lipschitz, then u has non-tangential boundary values at almost every $Q(d\omega)$ in $\partial\Omega$. (In fact, the analog of Carleson's theorem is also shown in [24], with the exceptional set having zero harmonic measure.) A natural question is then, whether $d\omega$ and $d\sigma$, the surface measure, are mutually absolutely continuous. This was answered by Dahlberg [10], who showed:

Theorem 1.

If Ω is a bounded Lipschitz domain in $\mathbb{R}^{n+1}$, harmonic measure and surface measure are mutually absolutely continuous. Moreover, $k = \frac{d\omega}{d\sigma}$ belongs to $A_\infty(d\sigma)$, i.e., it verifies a scale-invariant version of absolute continuity. More precisely, $k \in L^2(\partial\Omega, d\sigma)$, and

$$\left(\frac{1}{\sigma(B(Q, r) \cap \partial\Omega)} \int_{B(Q,r)\cap\partial\Omega} k^2 \, d\sigma\right)^{\frac{1}{2}} \leq C \left(\frac{1}{\sigma(B(Q, r) \cap \partial\Omega)} \int_{B(Q,r)\cap\partial\Omega} k \, d\sigma\right) ,$$

for all $Q \in \partial\Omega$ and all small $r > 0$ ($k \in B_2(d\sigma)$).

In a related work [11] Dahlberg also proved that, for harmonic functions u on Lipschitz domains Ω for which $u(X_*) = 0$, we have

$$||N(u)||_{L^p(\partial\Omega, d\sigma)} \simeq ||S(u)||_{L^p(\partial\Omega, d\sigma)}$$

for $0 < p < \infty$, and thus extended the Calderón-Stein theorem to Lipschitz domains. The key difficulty in doing this is that, unlike in the case of the upper half-plane, the distance function is far from being harmonic, making it difficult to apply Green's theorem. On the other hand, if $G(X) = G(X, X_*)$ is the Green's function for Ω, $\Delta G = -\delta_{X_*}$, and if $\Delta u = 0$, $u(X_*) = 0$, we have:

$$\int_{\partial\Omega} u^2 \, d\omega = 2 \int_\Omega |\nabla u|^2 G(X) \, dX \simeq \int_{\partial\Omega} S^2(u) \, d\omega .$$

From this, it is easy to obtain the result mentioned above with $d\sigma$ replaced by $d\omega$. Then, real-variable techniques (good λ inequalities [1, 11]) allow one, since $d\sigma \in A_\infty(d\omega)$ (unlike absolute continuity, A_∞ is an equivalence relationship [8]), to obtain the result for $d\sigma$.

In 1979, in [25] a new proof of Dahlberg's theorem was found, using an integral identity. It is the following: Assume for simplicity that $\Omega = \{(x, y) : x \in \mathbb{R}^n, y > \varphi(x)\}$, where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and $\varphi(0) = 0$. Let $X_* = (0, 1)$. Then,

$$\int_{\partial\Omega} k^2(Q) \vec{N}_{n+1}(Q) \, d\omega(Q) = c_n \int_{\partial\Omega} \frac{k(Q) d\sigma(Q)}{|(Q - (0, 1))|^{n-1}} ,$$

where $\vec{N}_{n+1}$ denotes the $n + 1$ component of the outward unit normal to the boundary ("Rellich Identity"). Note that since ω is a probability measure, $\int k \, d\sigma = 1$, and that, for $Q \in \partial\Omega$,

$|(Q-(0,1))| \geq c_0$, $\vec{N}_{n+1}(Q) \geq c_0$, where c_0 depends only on the Lipschitz constant of φ. This immediately shows that $k \in L^2(\partial\Omega, d\sigma)$.

A possible approach to the Lipschitz domain results explained above is as follows: Consider the change of variables $\varrho : \mathbb{R}^{n+1}_+ \to \Omega$ given by $\varrho(z,t) = (z, t+\varphi(z))$. Then, if $\Delta u = 0$ in Ω, then $v = u \circ \varrho$ verifies $Lv = 0$ in $\mathbb{R}^{n+1}_+$, where $L = \operatorname{div} A\nabla$, and A is real, symmetric, and bounded (since it depends on the Jacobian matrix of ϱ), i.e.,

$$\lambda|\xi|^2 \leq \langle A(z,t)\xi, \xi\rangle \leq \lambda^{-1}|\xi|^2, \quad \xi \in \mathbb{R}^{n+1}.$$

The natural question that arises is whether the Hunt-Wheeden-Dahlberg theory extends to this general situation. The works of De Giorgi, Nash, and Moser [16, 33, 34] give local Hölder continuity of the solutions, and the work of Littman et al., [30] shows that, under the exterior cone condition, we have unique solvability of the classical Dirichlet problem for such L, and hence an "elliptic measure" $d\omega_L$ (Morrey [32] and Gruter-Widman [23] observed that the symmetry of A is not necessary for this). Caffarelli et al. [3] showed the analog of the Hunt-Wheeden estimates explained above. (Again, these estimates hold without the symmetry of A). Nevertheless, in 1981, Caffarelli et al. [2], and, independently, Modica and Mortola [31] found examples of A symmetric and even continuous, so that ω_L is purely singular with respect to surface measure. The natural question is then: What distinguishes the A coming from Lipschitz domains? Since A depends on the Jacobian of ϱ above, it is easy to see that $A(z,t) = A(z)$. In fact, in [26] it was shown that, if A is symmetric, and A is smooth in t, then $k_L = \frac{d\omega_L}{dz}$ is also in $B_2(dz)$, via another "Rellich Identity."

In a paper in 1982, Fabes et al. [20] showed that if A is uniformly continuous, and if its modulus of continuity η in the t direction verifies the "Square Dini Condition"

$$\int_0 \eta^2(s)\frac{ds}{s} < +\infty,$$

then $k_L = \frac{d\omega_L}{dz}$ is in $B_2(dz)$ (and, in fact, in every $L^q(dz)$, $q > 1$, and in $B_q(dz)$). The "Square Dini Condition" is, in a precise sense, sharp to guarantee absolute continuity [7, 2]. This work led to a series of works by Dahlberg [13], and R. Fefferman, and culminated with the work of R. Fefferman et al. [21]. The result in this paper deals with "perturbation theory:" Suppose that we have two operators L_1 and L_2 as above, with coefficient matrices A_1 and A_2, so that

$$E(z,t) = \sup_{\substack{|z-x|\leq \frac{t}{2} \\ \frac{t}{2}\leq s\leq t}} |A_1(x,s) - A_2(x,s)|$$

verifies the "Carleson measure condition"

$$\frac{1}{h^n}\int_{|z-z_0|\leq h}\int_0^h E^2(z,t)\,dz\frac{dt}{t} \leq C,$$

for all $z_0 \in \mathbb{R}^n$, $h \in \mathbb{R}^+$. Then, if $\omega_{L_1} \in A_\infty(dz)$, then $\omega_{L_2} \in A_\infty(dz)$. Moreover, in [21] it was shown that the theorem is optimal in a number of ways. It is worth noting that only in the result of Jerison and Kenig [26] mentioned above does symmetry play a role.

Another significant development that played a role in the recent results we want to describe occurred in the paper of Dahlberg [12], with a simplified proof due to Kenig and Stein (unpublished). Here a "better" change of variables than the naive ϱ described above was found and used. In fact, one sets

$$\varrho(z,t) = (z, ct + (\theta_t * \varphi)(z)),$$

where $\theta \in C_0^\infty(\mathbb{R}^n)$ is even. It can then be shown that ϱ is still bi-Lipschitz, and that, in addition, $t\,|\nabla^2\varrho(z,t)|^2\,dzdt$ is a Carleson measure. If we now pull back Δ from Ω to $\mathbb{R}^{n+1}_+$, using the new ϱ, we obtain $A(z,t)$ elliptic, symmetric, with two extra properties:

(i) $|\nabla A(z,t)| \leq C/t$;

(ii) $t\,|\nabla A(z,t)|^2\,dzdt$ is a Carleson measure.

Two natural questions arose from these considerations:

Question 1 (Dahlberg ~ 1984) If A verifies (i) and (ii) above, does $\omega_L \in A_\infty(dz)$?

Question 2 (Fabes ~ 1984) If $A(z,t) = A(z)$, but A is not necessarily symmetric, does $k_L \in B_2(dz)$? Does (at least) $\omega_L \in A_\infty(dz)$?

These questions seemed to be beyond the methods developed in the 1980s. We will now discuss the new methods developed recently in order to begin understanding these questions.

In [18], Dahlberg et al. found a way to use the improved "distance function" generated by the Kenig–Stein mapping ϱ described above to give a "direct" proof of

$$\int_{\partial\Omega} N(u)^2\,d\sigma \simeq \int_{\partial\Omega} S^2(u)\,d\sigma$$

for solutions of $\Delta u = 0$ in Ω, without the use of harmonic measure, by integration by parts. This then extended to solutions to $E\vec{u} = 0$ in Ω, where E is any homogeneous, symmetric, constant coefficient, higher-order elliptic system. The "integration by parts" yielded two estimates, valid on all Lipschitz domains Ω: (for $\Delta u = 0$)

$$(1) \qquad \int_{\partial\Omega} S^2(u)\,d\sigma \leq C\int_{\partial\Omega} N(u)^2\,d\sigma\,,$$

$$(2) \qquad \int_{\partial\Omega} u^2\,d\sigma \leq C\left(\int_{\partial\Omega} N(u)^2\,d\sigma\right)^{\frac{1}{2}}\left(\int_{\partial\Omega} S^2(u)\,d\sigma\right)^{\frac{1}{2}} + C\int_{\partial\Omega} S^2(u)\,d\sigma\,.$$

One then obtains $\int N(u)^2 d\sigma \simeq \int S^2(u)\,d\sigma$, using representation formulas for solutions in terms of layer potentials, Rellich identities, and the theorem of Coifman et al. [9], to show that $\int_{\partial\Omega} N(u)^2\,d\sigma \leq C\int_{\partial\Omega} u^2\,d\sigma$. This was the starting point for our new methods that provided some progress in understanding the two questions posed above. We have:

Theorem 2. [28]

Suppose that $n = 1$, $A(z,t) = A(z)$, but A is not necessarily symmetric. Then $\omega_L \in A_\infty(dz)$, and this is the best possible conclusion. (In particular, examples are exhibited where $k_L \notin B_2(dz)$.)

Theorem 3. [27]

If L is defined on a Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$, and $|\nabla A(X)| \leq C/dist\,(X)$, $dist\,(X)|\nabla A(X)|^2$ is a Carleson measure, then $\omega_L \in A_\infty(d\sigma)$.

We will now sketch some of the new ideas in the proofs of the above results, as a series of observations.

Observation 1: Suppose that the Fatou theorem is true for all bounded solutions to $Lu = 0$, with an exceptional set having measure zero $d\sigma$. Then,

$$\lim_{r\downarrow 0} \frac{1}{\omega_L(B(Q,r)\cap\partial\Omega)}\int_{B(Q,r)\cap\partial\Omega} f\,d\omega_L$$

exists for almost all Q $(d\sigma)$ on $\partial\Omega$, and all $f \in L^\infty(\partial\Omega, d\omega_L)$. (This is nothing but the estimate $N(u)(Q) \simeq M_{\omega_L}(f)(Q)$.) Moreover, if ω_L is any doubling measure, and $\omega_L(E) = 0$, then the following "converse to the Lebesgue differentiation theorem" holds ([18]): There exists $0 \leq f \leq 1$ such that, for all $Q \in E$,

$$\lim_{r\downarrow 0} \frac{1}{\omega_L(B(2,r)\cap\partial\Omega)}\int_{B(2,r)\cap\partial\Omega} f\,d\omega_L$$

fails to exist. Thus, in view of the Fatou theorem, $\sigma(E) = 0$. All of this can be done quantitatively, and so if "a quantitative Fatou ($d\sigma$) holds for bounded solutions," then $\sigma \in A_\infty(d\omega_L)$. (Again see [28]).

Observation 2: If for all solutions to $Lu = 0$ we have $\int_{\partial\Omega} S^2(u)\, d\sigma \simeq \int_{\partial\Omega} N(u)^2\, d\sigma$ for all Lipschitz domains, then the "quantitative Fatou ($d\sigma$) holds, for bounded solutions." This is a consequence of ideas due to Varopoulos [36], Garnett [22], and Dahlberg [14] in the late 1970s. In fact, the assumed estimate gives an approximation result for bounded solutions, which in turn gives the "quantitative Fatou $d\sigma$." (Again see [28] for details.)

Observation 3: If for all solutions to $Lu = 0$ in Ω and all Lipschitz domains Ω we have

(1) $\int_{\partial\Omega} S^2(u)\, d\sigma \le C \int_{\partial\Omega} N(u)^2\, d\sigma$, and

(2) $\int_{\partial\Omega} u^2\, d\sigma \le C \left(\int_{\partial\Omega} N(u)^2\, d\sigma\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} S^2(u)\, d\sigma\right)^{\frac{1}{2}} + C \left(\int_{\partial\Omega} S^2(u)\, d\sigma\right)$,

then (2) can be strengthened to

(2′) $\int_{\partial\Omega} N(u)^2\, d\sigma \le C \int_{\partial\Omega} S^2(u)\, d\sigma$.

The idea for this is a "stopping-time" argument. For each j, we let

$$E_j = \left\{ z : Nu(z, \varphi(z)) > 2^j,\ S(u)(z, \varphi(z)) \le \varrho 2^j \right\} .$$

Let $h_j(z) = \sup\{t \ge \varphi(z) : \sup_{(x,s)\in\Gamma(z)+(0,t)} |u(x,s)| > 2^j\}$. Then h_j is Lipschitz, independently of j, and, for ϱ small, $M_{h_j} u(z, h_j(z)) \ge C2^j$ for all $z \in E_j$, where M_{h_j} denotes the Hardy-Littlewood maximal operator with respect to surface measure on the graph of h_j. One then obtains (2′) from (2) on the graph of h_j, via a "good λ inequality" (see [28]).

Finally in [28] and in [27] we adopt the ideas in [18] to obtain (1) and (2) above in the framework of Theorems 2 and 3, respectively. The argument for this in Theorem 2 is in fact quite intricate, and depends on some changes of variables that are available only when $n = 1$. Moreover, it is further complicated by the fact that (1) and (2) are only obtained for domains Ω with "small Lipschitz constant", and one then needs to use. David's "build up scheme" [15], in combination with the ideas sketched above. Whether Theorem 2 remains valid for $n > 1$ is a challenging problem.

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