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Sobolev Type Embeddings in the Limiting Case

Michael Cwikel and Evgeniy Pustylnik

ABSTRACT. We use interpolation methods to prove a new version of the limiting case of the Sobolev embedding theorem, which includes the result of Hansson and Brezis–Wainger for $W^k_{n/k}$ as a special case. We deal with generalized Sobolev spaces W^k_A, where instead of requiring the functions and their derivatives to be in $L_{n/k}$, they are required to be in a rearrangement invariant space A which belongs to a certain class of spaces "close" to $L_{n/k}$.

We also show that the embeddings given by our theorem are optimal, i.e., the target spaces into which the above Sobolev spaces are shown to embed cannot be replaced by smaller rearrangement invariant spaces. This slightly sharpens and generalizes an earlier optimality result obtained by Hansson with respect to the Riesz potential operator.

In memory of Gene Fabes.

לזכר חברנו ישעל בעריל פייבס.

1. Introduction

The theory of Sobolev type embeddings has its origins in classical inequalities from which integrability properties of a real function can be deduced from those of its derivatives (see, e.g., [2, chapter V]). In his fundamental work [17] Sobolev was the first to apply methods of functional analysis to this topic. His results deal with functions defined on a suitable domain in $\mathbb{R}^n$.

In the 1960s it was discovered (for systematic treatments see, e.g., [5, 16]) that the original rather complicated proofs of embedding theorems given by Sobolev and his followers could be simplified by using interpolation of linear operators, in particular the Marcinkiewicz interpolation theorem.

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Subsequently a series of papers appeared, treating the so-called "limiting case" of the Sobolev embedding theorem, and culminating in the result of Hansson [11] and Brezis-Wainger [7]. As shown by Hansson, this result is best possible in some sense.

In this paper we shall use interpolation methods to provide an alternative proof of the Hansson–Brezis–Wainger limiting case embedding theorem. Our proof is considerably simpler, at least from our point of view, than those in [11] and [7]. We shall also show that this embedding theorem is best possible, in a slightly stronger sense than that considered by Hansson.

In fact our methods will also give analogous optimal embedding theorems for generalized Sobolev spaces where the classical L_p conditions on the function and its derivative are replaced by requirements in terms of some other rearrangement invariant space.

2. Some Terminology and some History

Throughout this paper Ω will denote a domain in $\mathbb{R}^n$. For one special application (in a proof in the last section) we will take $\Omega = \mathbb{R}^n$. Apart from this particular case, we will always suppose that Ω is bounded and also that it is star shaped with respect to every point of some euclidean ball contained in Ω. (Quite possibly our results here can be extended to more general domains, such as those studied in [15] in a context similar to ours.)

We shall denote n-dimensional Lebesgue measure by μ, but we will also use the usual notation $\int f(x)dx$ or $\int f(y)dy$ for integration with respect to μ on $\mathbb{R}^n$.

We shall use the standard notation $|x| = \sqrt{\sum_{j=1}^n x_j^2}$ and $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$ for norms and inner products of elements $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ of $\mathbb{R}^n$. We shall also use the standard notation

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_n^{\alpha_n}} f$$

for (generalized) partial derivatives, for each multi–index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of order $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n$.

Let $W_p^k(\Omega)$ denote, as usual, the Sobolev space of (equivalence classes of) functions $f \in L_p(\Omega, \mu)$ whose generalized partial derivatives of order k are all in $L_p(\Omega, \mu)$. This space may be normed by

$$\|f\|_{W_p^k(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=k} \left\|\partial^\alpha f\right\|_{L_p(\Omega)} . \tag{2.1}$$

Sobolev's classical embedding theorem [17] states that if $\Omega \subset \mathbb{R}^n$ is a bounded domain which is star shaped with respect to every point of some euclidean ball contained in Ω, then

$$W_p^k(\Omega) \subset L_q(\Omega) \quad \text{continuously, whenever } p < \frac{n}{k} \text{ and } q \le \frac{np}{n-pk} . \tag{2.2}$$

In the limiting case, i.e., when $p = \frac{n}{k}$, this inclusion does not hold for $q = \infty$. However, we do have that

$$W_{n/k}^k(\Omega) \subset L_q(\Omega, \mu) \qquad \text{for all } q < \infty . \tag{2.3}$$

In other words, the optimal integrability conditions satisfied by functions in $W_{n/k}^k$ cannot be specified as simple L_p conditions.

In the late 1960s Peetre [16] and Trudinger [22] independently found refinements of (2.3) expressed in terms of Orlicz spaces of exponential type. Using methods which are rooted in the work of Yano [23] (see also [24, p. 119]), they were each able to prove that a continuous embedding of the form

$$W_{n/k}^k(\Omega) \subset L_\Phi(\Omega, \mu) \tag{2.4}$$

holds for some Orlicz space $L_\Phi(\Omega, \mu)$ generated by the function $\Phi(u) = e^{u^\lambda}$ for some $\lambda > 1$. Such Orlicz spaces are clearly contained in $L_q(\Omega)$ for every $q < \infty$. In Trudinger's result $\lambda = \frac{n}{n-1}$ for all $k = 1, 2, \ldots$ and he also showed that this value of λ is best possible when $k = 1$. Subsequently Strichartz [20] strengthened this result by obtaining the above embedding when $\lambda = \frac{n}{n-k}$. He also observed that $\frac{n}{n-k}$ is the best possible value of λ for any choice of $k \le n-1$. In fact, this is the same value of λ as was found by Peetre using interpolation properties of Sobolev spaces. Somehow Peetre's contribution has gone largely unnoticed by many authors, probably because it is rather inconspicuously embedded in a long article dealing with other topics.

Remark 1.

We wish to thank Igor Verbitsky for drawing our attention to some additional historical information given in the monograph [1, p. 81] concerning the embedding (2.4). Furthermore he informed us of the connection between (2.4) and a capacitary inequality of Maz'ja [1, (7.6.1) on p. 209], which can also be exploited to obtain optimal constants in (2.4).

To obtain further refinements of the limiting case of the Sobolev embedding theorem, it is necessary to work with a wider class of function spaces. The rearrangement invariant spaces constitute a natural class to consider if one wishes to study integrability-like conditions of functions in $W^k_{n/k}(\Omega)$. This class contains the L_p and Orlicz spaces appearing in the results mentioned thus far, and furthermore, although of course Sobolev spaces are not rearrangement invariant, the Sobolev norm of a function depends on the size of the function and of its derivatives, rather than on the particular location of the places where they are large (e.g., the Sobolev norm of a function having compact support in Ω is invariant under translations of the function which keep its support within Ω).

One rearrangement invariant space which turns out to be of very great interest in the present context is the space, which we shall denote here by V or $V(\Omega, \mu)$, which consists of all (equivalence classes of) measurable functions $f : \Omega \to \mathbb{R}$ for which

$$\|f\|_V := \left(\int_0^1 \left(\frac{f^*(t)}{\ln\frac{e}{t}}\right)^{n/k} \frac{dt}{t}\right)^{k/n} \tag{2.5}$$

is finite. Here f^* denotes the nonincreasing rearrangement of f, i.e., the right continuous nonincreasing function $f^* : (0, \infty) \to [0, \infty)$ such that the linear Lebesgue measure of $\{t \in (0, \infty) : f^*(t) > \alpha\}$ equals $\mu(\{x \in \Omega : |f(x)| > \alpha\})$ for all $\alpha > 0$. Of course $f^*(t) = 0$ for all $t \ge \mu(\Omega)$. Although it is not clear that the homogeneous non-negative functional $f \mapsto \|f\|_V$ satisfies the triangle inequality, it can be shown to be equivalent to a norm, and so it is a quasinorm, i.e., for some absolute constant C it satisfies

$$\|u + v\|_V \le C\,(\|u\|_V + \|v\|_V) \quad \text{for all } u, v \in V\,. \tag{2.6}$$

Remark 2.

We can also consider a variant of (2.5) where the interval of integration $(0, 1)$ *is replaced by* $(0, \delta)$ *for some positive number* $\delta \ne 1$. *Of course if* $\delta > 1$ *the function* $\left(\frac{1}{\ln\frac{e}{t}}\right)^{n/k}$ *may not be non-negative nor even real valued on* $(1, \delta)$ *and so, on that part of the range of integration, it should be replaced by some bounded non-negative function. It is easy to see that any new quasinorm obtained by such modifications will be equivalent to* $\|f\|_V$.

In 1979 Hansson [11, pp. 96–101] and, independently, in 1980 Brezis and Wainger [7, Theorem 2, p. 781] proved the continuous embedding

$$W^k_{n/k}(\Omega) \subset V\,. \tag{2.7}$$

As pointed out in both [11] and [7], the space V is strictly smaller than the various versions of the space L_Φ which appear in (2.4). The proofs of (2.7) in these two papers are very different from each other, and they each use rather elaborate techniques (related to capacities, convolution inequalities, etc.). The main step in both proofs is to show that the *Riesz potential operator*, i.e., the operator $J_{n,k}$ defined by

$$J_{n,k}f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-k}}\,dy\,, \tag{2.8}$$

satisfies

$$J_{n,k} : L_{n/k}(\Omega, \mu) \to V \quad \text{boundedly}\,. \tag{2.9}$$

This will also be the main step in the alternative proof of (2.7) which we will present in the next section.

To explain why (2.9) is sufficient to imply (2.7), we need to recall an integral representation formula from the classical work of Sobolev which, in some approximate sense, enables a function to be reconstructed from its partial derivatives of order k. More precisely, for any given fixed domain Ω satisfying the above-mentioned star shape condition, Sobolev constructs special bounded functions $Q_\alpha : \Omega \times \Omega \to \mathbb{R}$ for each multi–index α such that either $\alpha = 0$ or $|\alpha| = k$, with the property (cf. [18, equation (7.12), p. 55] or [15, Theorem 1, p. 20] that

$$f(x) = \int_\Omega Q_0(x,y)f(y)dy + \sum_{|\alpha|=k}\int_\Omega \frac{Q_\alpha(x,y)}{|x-y|^{n-k}}\partial^\alpha f(y)dy \quad \text{for a.e. } x \in \Omega \tag{2.10}$$

for each $f : \Omega \to \mathbb{R}$ whose generalized partial derivatives of order k are all integrable on Ω. (In fact $Q_0 : \Omega \times \Omega \to \mathbb{R}$ is continuous and, for each α with $|\alpha| = k$, $Q_\alpha : \Omega \times \Omega \to \mathbb{R}$ is continuous at all points (x, y) such that $x \neq y$.)

In view of the easily checked continuous inclusions

$$L_\infty(\Omega, \mu) \subset V \quad \text{and} \quad L_{n/k}(\Omega, \mu) \subset L_1(\Omega, \mu)\,, \tag{2.11}$$

it is obvious that, for every $f \in W^k_{n/k}(\Omega)$, the first term on the right of (2.10), i.e., the function $u(x) = \int_\Omega Q_0(x,y)f(y)dy$ satisfies

$$\|u\|_V \le C_1\|u\|_{L_\infty(\Omega)} \le C_2\|f\|_{L_1(\Omega)} \le C_3\|f\|_{W^k_{n/k}(\Omega)} \tag{2.12}$$

for suitable absolute constants C_i depending only on Ω, n, and k. Furthermore it is clear that the absolute value of the remaining sum of terms is dominated by a constant multiple of $\sum_{|\alpha|=k} J_{n,k}(|\partial^\alpha f|)$. Thus, we obtain from (2.9), (2.12), and (2.6) that

$$\|f\|_V \le C_4\|f\|_{W^k_{n/k}(\Omega)}\,,$$

which is of course equivalent to (2.7).

Remark 3.

(i) In fact, Sobolev followed a similar line of reasoning in his original proof of (2.2). He obtained (2.2) as a consequence of the boundedness of the mapping $J_{n,k} : L_p(\Omega, \mu) \to L_q(\Omega, \mu)$. Some subsequent alternative proofs use Fourier transform methods instead of the integral representation (2.10).

(ii) Igor Verbitsky has pointed out to us that the embedding (2.7) can also be deduced from the same capacitary inequality of Maz'ja mentioned above in Remark 1 and the standard isoperimetric estimate for capacity from below. (See [15, pp. 109, 105].)

(iii) Another result rather similar to (2.7) was obtained by Brudnyi [8], also in 1979. Rather than working with rearrangement invariant spaces, he considered spaces which are variants of the

John–Nirenberg space BMO. For $p \in [1, \infty)$ let $BMO^p(\Omega)$ be the space of all locally integrable functions $f : \Omega \to \mathbb{R}$ for which the seminorm

$$\|f\| = \sup_K \left(\int_0^{\mu(K)} \left(\frac{(f - f_K)^*(t)}{\ln\left(\frac{2\mu(K)}{t}\right)} \right)^p \frac{dt}{t} \right)^{1/p}$$

is finite. Here the supremum is taken over all cubes $K \subset \Omega$ with sides parallel to the axes, and $f_K = \frac{1}{\mu(K)} \int_K f(x)dx$ for each such K. When $p = n/k$ one can consider $BMO^p(\Omega)$ as a sort of "oscillatory" analogue of the space V, i.e., it measures the local oscillation rather than the absolute size of the function f, with respect to a similar weighted L_p norm. The embedding theorems in [8] immediately imply the continuous inclusion

$$W^k_{n/k}(\Omega) \subset BMO^{n/k}(\Omega)$$

whenever $n \geq 2k$.

3. A New Proof of the Hansson-Brezis-Wainger Embedding Theorem, and some Generalizations

The starting point of our investigation of this topic is the observation that the space V is a member of the class of *Lorentz–Zygmund spaces* $L^{p,r}(\log L)^\alpha$, which were introduced and studied by Bennett and Rudnick [3] (see also [4, p. 253]). We have learned that some other authors have also noticed and used this fact about V, e.g., in [10] and in other papers referred to in [10]. In our case, as we shall see in this section, this observation enables us to give an alternative and perhaps easier proof of (2.9) and (2.7), by applying a general interpolation theorem from [3] to the operator $J_{n,k}$.

For any underlying measure space $(\mathcal{M}, \nu)$ such that $\nu(\mathcal{M}) < \infty$, the space $L^{p,r}(\log L)^\alpha$ on $(\mathcal{M}, \nu)$ is defined to consist of all (equivalence classes of) ν-measurable functions $f : \mathcal{M} \to \mathbb{R}$ for which the quasinorm

$$\|f\|_{p,r;\alpha} = \left(\int_0^1 \left[t^{1/p} f^*(t) \left(\ln \frac{e}{t} \right)^\alpha \right]^r \frac{dt}{t} \right)^{1/r} \tag{3.1}$$

is finite. As usual, the integral is replaced by a supremum when $r = \infty$.

Remark 4.

In fact these spaces can be defined for an arbitrary underlying measure space and it is possible to use a norm that takes into account the behavior of $f^*(t)$ as t tends to infinity. But here we have restricted attention to the case where $\nu(\mathcal{M}) < \infty$ because this simplifies the formulæ for the norms and is sufficient for our applications here.

The Lorentz–Zygmund spaces include as particular cases some previously known and important spaces, such as $L_p = L^{p,p}(\log L)^0$ and also the *Lorentz space* $\Lambda_p = L^{p,1}(\log L)^0$ whose norm is

$$\|f\|_{\Lambda_p} = \int_0^\delta t^{\frac{1}{p}-1} f^*(t)dt \quad (\text{where } \delta = \nu(\mathcal{M})) . \tag{3.2}$$

Another example is the *Marcinkiewicz space* or *weak L_p space* $M_p = L^{p,\infty}(\log L)^0$ whose norm is

$$\|f\|_{M_p} = \sup_{0<t<\delta} t^{\frac{1}{p}} f^{**}(t) = \sup_{0<t<\delta} t^{\frac{1}{p}-1} \int_0^t f^*(s)ds . \tag{3.3}$$

Let $(\mathcal{M}_j, \nu_j)$ for $j = 1, 2$ be two (possibly different) measure spaces. Again, for our purposes here it is sufficient to consider the cases where $\nu_j(\mathcal{M}_j) < \infty$ for $j = 1, 2$. The interpolation theorems which are relevant for our purposes here deal with linear operators T which map some class of measurable functions on $(\mathcal{M}_1, \nu_1)$ into some (possibly different) class of measurable functions on $(\mathcal{M}_2, \nu_2)$.

If $T : L_p(\nu_1) \to L_q(\nu_2)$ boundedly, then we say that T is of *strong type* (p, q). If T satisfies the weaker condition that $T : \Lambda_p(\nu_1) \to M_q(\nu_2)$ boundedly, then we say that it is of *weak type* (p, q).

Remark 5.

This terminology is not always used with exactly the same meaning. What we call "weak type" here is often referred to as "restricted weak type." The terminology "weak type (p, q)" is classically taken to mean that $T : L_p(\nu_1) \to M_q(\nu_2)$ is bounded. See, e.g., [4, p. 230] and [13, p. 130] for definitions of various variants of this notion.

The following theorem is part of the statement of [3, Theorem C1 (b) on page 10]. There, by using a suitably generalized definition of weak type operators, the result is also obtained for $p = \infty$. We do not need this case for our application here. In the statement of Theorem C1 the underlying measure spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are both taken to be $\mathbb{T}$, but exactly the same proof works for general finite measure spaces (cf. also [9, Section 7]).

Theorem 1. (Bennett-Rudnick)

Every operator T of weak types (a, b) and (p, q) for $0 < a < p < \infty$, $0 < b < q \le \infty$, acts continuously from the space $L^{p,r}(\log L)^{\alpha+1}$ defined on $(\mathcal{M}_1, \nu_1)$ into $L^{q,s}(\log L)^{\beta}$ on $(\mathcal{M}_2, \nu_2)$ for all $1 \le r \le s \le \infty$ and all $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \frac{1}{r} = \beta + \frac{1}{s} < 0$.

From here the proof of the Hansson–Brezis–Wainger embedding theorem is almost immediate: Let us choose $(\mathcal{M}_1, \nu_1) = (\mathcal{M}_2, \nu_2) = (\Omega, \mu)$. Then it is easy to check that the operator $J_{n,k}$ maps $\Lambda_{n/k}$ into L_∞ and also maps L_1 into $M_{n/(n-k)}$, i.e., $J_{n,k}$ is of weak types $(\frac{n}{k}, \infty)$ and $(1, \frac{n}{n-k})$ (see, e.g., [12, p. 153]). From Theorem 1 we then deduce that $J_{n,k}$ maps $L_{n/k}$ boundedly into $L^{\infty,n/k}(\log L)^{-1} = V$, proving (2.9) and consequently also (2.7).

Recently [9] we have obtained a new interpolation theorem which includes and sharpens certain cases of Theorem 1 and which applies to more general classes of rearrangement invariant spaces. We shall now describe this theorem and use it to obtain a generalization of (2.7) to a broader class of Sobolev spaces.

For our purposes here, as in [9], we shall define the *rearrangement invariant spaces* (or r.i. spaces) on a given measure space $(\mathcal{M}, \nu)$ to be those Banach spaces E of ν–measurable functions on $f : \mathcal{M} \to \mathbb{R}$ which are exact interpolation spaces with respect to the Banach pair $(L_1(\nu), L_\infty(\nu))$.

Various authors use this terminology with slightly different meanings than ours, and others refer to similar spaces as "symmetric spaces." See, e.g., [13, p. 122], for an example of a symmetric space which is not rearrangement invariant in our sense. Where necessary we can use the more explicit notation $E(\mathcal{M}, \nu)$ or $E(\mathcal{M})$ or $E(\nu)$ to indicate the underlying measure space.

For each r.i. space E on $(0, \infty)$ with Lebesgue measure dt, we define the space $\widetilde{E}$ to be the collection of (equivalence classes of) all functions $f : (0, 1) \to \mathbb{R}$, which are of the form $f(t) = \varphi\left(\ln \frac{1}{t}\right)$ for some $\varphi \in E$. We norm $\widetilde{E}$ by

$$\|f\|_{\widetilde{E}} = \|\varphi\|_E, \quad \text{where } f(t) = \varphi\left(\ln \frac{1}{t}\right).$$

Given any such r.i. space E and numbers $p \in (1, \infty)$, $q \in (1, \infty]$ we shall define the r.i. spaces $A(\mathcal{M}, \nu)$ and $B(\mathcal{M}, \nu)$ to be the collections of all measurable real valued functions f on an arbitrary given measure space $(\mathcal{M}, \nu)$ for which the norm

$$\|f\|_{A(\mathcal{M},\nu)} = \left\| t^{\frac{1}{p}} f^{**}(t) \right\|_{\widetilde{E}} \tag{3.4}$$

or, respectively, the norm

$$\|f\|_{B(\mathcal{M},\nu)} = \left\| \sup_{0<s<t} s^{\frac{1}{q}} f^{**}(s) / \ln \frac{e}{s} \right\|_{\widetilde{E}} \tag{3.5}$$

is finite. Note that although f^{**} is defined on the whole interval $(0, \infty)$, we only take its values on $(0, 1)$ into account for calculating these norms.

Some important examples of the space $A(\mathcal{M}, \nu)$ are obtained when we choose E to be L_r for some $r \in [1, \infty]$. Then the norm of the space $\widetilde{L}_r$ is

$$\|f\|_{\widetilde{L}_r} = \left(\int_0^1 |f(t)|^r \frac{dt}{t} \right)^{1/r} \quad \text{for } r < \infty, \text{ or } \quad \|f\|_{\widetilde{L}_\infty} = \operatorname*{ess\,sup}_{0<t<1} |f(t)| .$$

Consequently, if $r < \infty$,

$$\|f\|_{A(\mathcal{M},\nu)} = \left(\int_0^1 \left(t^{\frac{1}{p}} f^{**}(t) \right)^r \frac{dt}{t} \right)^{1/r}$$

and, when $r = \infty$,

$$\|f\|_{A(\mathcal{M},\nu)} = \sup_{0<t<1} t^{\frac{1}{p}} f^{**}(t) = \|f\|_{M_p} .$$

If $\nu(\mathcal{M}) < \infty$, then $A(\mathcal{M}, \nu)$ coincides with the two-parameter Lorentz space $L^{p,r}$ on $(\mathcal{M}, \nu)$ [14]. (Since $p > 1$, the preceding norm is equivalent to the quasinorm for $L^{p,r}$ obtained by replacing f^{**} by f^*.)

Remark 6.

In fact, for all choices of E, again assuming that $\nu(\mathcal{M}) < \infty$, the space $A(\mathcal{M}, \nu)$ is rather "close" to $L_p(\mathcal{M}, \nu)$. More precisely (see [9, Theorems 4.6, 4.8, and 7.7]), $A(\mathcal{M}, \nu)$ lies between the two spaces $\Lambda_p(\mathcal{M}, \nu) = L^{p,1}$ and $M_p(\mathcal{M}, \nu) = L^{p,\infty}$ and is an interpolation space with respect to them.

We will be interested here in those spaces $A(\mathcal{M}, \nu)$ and $B(\mathcal{M}, \nu)$ which are obtained using an r.i. space E on which the Hardy operator H

$$Hf(t) = \frac{1}{t} \int_0^t f(s)\, ds \tag{3.6}$$

is a bounded mapping. In fact, the boundedness of H on E is equivalent to requiring q_E, the upper Boyd index of E, to satisfy

$$q_E < 1 .$$

A proof of this equivalence can be found in [13]. (It follows from the discussion on pp. 127–128 together with Theorem 6.6 on p. 138.)

To simplify our presentation in [9], many results, including our main interpolation theorem in Section 5, are stated for r.i. spaces where the underlying measure space is $(0, 1)$ with Lebesgue measure dt. However, for our applications here, we need a version of the corollary of this theorem where $(0, 1)$ is replaced by the bounded domain $\Omega \subset \mathbb{R}^n$. In fact, it is rather obvious that the proof of the interpolation theorem works, with trivial modifications, for arbitrary measure spaces. In any case it can be also extended to those measure spaces by a simple method described in [9, Section 7].

More specifically, if we apply Theorem 7.6 and Examples 7.5 to Corollary 5.3 of Theorem 5.1 (all of these are in [9]), then we immediately obtain the following result:

Theorem 2.

Let $(\mathcal{M}_1, \nu_1)$ and $(\mathcal{M}_2, \nu_2)$ be arbitrary measure spaces such that $\nu_1(\mathcal{M}_1)$ and $\nu_2(\mathcal{M}_2)$ are finite. Let T be a linear operator mapping measurable functions on $\mathcal{M}_1$ to measurable functions on $\mathcal{M}_2$. Suppose that T is of weak types (a, b) and (p, q) where $1 \leq a < p < \infty$ and $1 \leq b < q \leq \infty$. Let E be an r.i. space on $((0, \infty), dt)$ satisfying $q_E < 1$ and let $A(\mathcal{M}_1, \nu_1)$ and $B(\mathcal{M}_2, \nu_2)$ be defined by (3.4) and (3.5), respectively. Then T maps $A(\mathcal{M}_1, \nu_1)$ boundedly into $B(\mathcal{M}_2, \nu_2)$.

Remark 7.

As explained in [9], Theorem 2 implies and even strengthens the conclusion of Theorem 1 in the case where $\alpha = \beta = -1$. We also remark that, again by [9, Examples 7.5], it follows that, in the statement of Theorem 2, we can omit the requirement that $\nu_1(\mathcal{M}_1)$ and $\nu_2(\mathcal{M}_2)$ are finite, provided that we replace the weak type conditions on T by the requirements that T is a bounded map from $\Lambda_a(\nu_1) + L_\infty(\nu_1)$ into $M_b(\nu_2) + L_\infty(\nu_2)$ and from $\Lambda_p(\nu_1) + L_\infty(\nu_1)$ into $M_q(\nu_2) + L_\infty(\nu_2)$.

We are now ready to define the generalized Sobolev spaces for which we can obtain an analogue of the Hansson–Brezis–Wainger embedding theorem. We choose the underlying measure space to be (Ω, μ), and for some $p \in (1, \infty)$ and some r.i. space E we define the space $A(\Omega, \mu)$ as above. Then we take $W_A^k(\Omega)$ to be the (generalized Sobolev) space of (equivalence classes of) functions $f \in A(\Omega, \mu)$ whose generalized partial derivatives of order k are all in $A(\Omega, \mu)$ (cf. [21, p. 39]). This space is normed by

$$\|f\|_{W_A^k(\Omega)} = \|f\|_{A(\Omega,\mu)} + \sum_{|\alpha|=k} \|\partial^\alpha f\|_{A(\Omega,\mu)} .$$

If we choose $E = L_p((0, \infty), dt)$, then $A(\Omega, \mu) = L^{p,p}(\Omega, \mu) = L_p(\Omega, \mu)$ and so $W_A^k(\Omega) = W_p^k(\Omega)$. More specifically, if we choose $p = n/k$, then we obtain the Sobolev space $W_{n/k}^k(\Omega)$ which appears in the limiting case of the embedding theorem.

If, for this same choice of p, we allow E to be an arbitrary r.i. space on $((0, \infty), dt)$, then, since $A(\Omega, \mu)$ is "close" to $L_{n/k}$ (cf. Remark 6), we get a Sobolev space "close" to $W_{n/k}^k(\Omega)$. Here is our embedding theorem for such spaces. It contains (2.7) as a special case (i.e., for $E = L_{n/k}$).

Theorem 3.

Let E be an r.i. space on $((0, \infty), dt)$ satisfying $q_E < 1$. Let the spaces $A(\Omega, \mu)$ and $G(\Omega, \mu)$ be defined by

$$\begin{aligned} A(\Omega, \mu) &:= \left\{ f(x) : \|f\|_{A(\Omega,\mu)} = \left\| t^{\frac{k}{n}} f^{**}(t) \right\|_{\widetilde{E}} < \infty \right\} \quad \textit{and} \\ G(\Omega, \mu) &:= \left\{ f(x) : \|f\|_{G(\Omega,\mu)} = \left\| f^{**}(t) / \ln \frac{e}{t} \right\|_{\widetilde{E}} < \infty \right\} . \end{aligned}$$

Then

$$W_A^k(\Omega) \subset G(\Omega, \mu) . \tag{3.7}$$

Proof. This is analogous to our proof of (2.7) except that we now use Theorem 2 instead of Theorem 1. Again we use the fact that the Riesz potential operator $J_{n,k}$ is of weak types $(\frac{n}{k}, \infty)$ and $(1, \frac{n}{n-k})$. So we take $p = \frac{n}{k}$, $q = \infty$, $a = 1$ and $b = \frac{n}{n-k}$ and apply Theorem 2 with $(\mathcal{M}_1, \nu_1) = (\mathcal{M}_2, \nu_2) = (\Omega, \mu)$ to obtain that $J_{n,k}$ is a bounded map of $A(\Omega, \mu)$ into $B(\Omega, \mu)$. From the definition of this latter space ((3.5) with $q = \infty$) it is clear that $B(\Omega, \mu)$ is continuously embedded (with norm one) in $G(\Omega, \mu)$. Thus, analogously to (2.9), we have that

$$J_{n,k} : A(\Omega, \mu) \to G(\Omega, \mu) \text{ boundedly} . \tag{3.8}$$

To complete the proof we have to show that (3.8) implies (3.7). Here the argument is almost exactly the same as the proof given before that (2.9) implies (2.7). There are only a few small changes. First, in view (cf. Remark 6) of the continuous inclusion $A(\Omega, \mu) \subset M_{n/k}(\Omega, \mu)$, we have that $A(\Omega, \mu) \subset L_1(\Omega, \mu)$. (This can also be easily proved directly from the definition of $A(\Omega, \mu)$.) Consequently all the generalized partial derivatives of order k of functions $f \in W_A^k(\Omega)$ are integrable. This enables us to apply (2.10). Then, in place of the inclusions (2.11), we now need to use the two continuous inclusions $L_\infty(\Omega, \mu) \subset G(\Omega, \mu)$ and $A(\Omega, \mu) \subset L_1(\Omega, \mu)$. The second of these has just been discussed. To obtain the first, we need to use the property $q_E < 1$, i.e., the fact that the Hardy operator H (3.6) is bounded on E. This ensures that $H(\chi_{(0,1)})(t) = \min\{1, 1/t\} \in E$. Consequently the smaller function $\varphi(t) = \frac{1}{1+t}$ is also in E or, equivalently, $\varphi\left(\ln \frac{1}{t}\right) = \frac{1}{\ln \frac{e}{t}} \in \widetilde{E}$. Then $\|f\|_{G(\Omega,\mu)} \leq \|\frac{1}{\ln \frac{e}{t}}\|_{\widetilde{E}} \cdot \|f\|_{L_\infty}$ for each $f \in G(\Omega, \mu)$, which is equivalent to the required inclusion. All other parts of the proof are exactly analogous to before. □

Remark 8.

In the preceding proof we used the obvious fact that $B(\Omega, \mu) \subset G(\Omega, \mu)$. Our proof also implies the seemingly stronger result $W_A^k(\Omega) \subset B(\Omega, \mu)$. However, since here $q = \infty$, the two spaces $B(\Omega, \mu)$ and $G(\Omega, \mu)$ coincide to within equivalence of norms. This is proved in [9, Theorem 6.27] when the underlying measure space is $(0, 1)$ and the result extends obviously to our measure space, e.g., by the methods of [9, Section 7].

4. The Optimality of the Hansson-Brezis-Wainger Theorem and its Generalizations

Among the results in his paper [11], Hansson also shows that the property (2.9) of the operator $J_{n,k}$ is best possible, in the sense that $J_{n,k}$ does not map $L_{n/k}(\Omega, \mu)$ into any r.i. space which is strictly smaller than V. This suggests that perhaps (2.7) is also the best possible, in the analogous sense that it too does not hold if V is replaced by a smaller r.i. space.

On the one hand this is not entirely obvious; in the transition from (2.9) to (2.7) the methods used to estimate the size of the right-hand side of (2.10) are quite crude. So it is conceivable that a more delicate argument could show that f is significantly smaller. On the other hand, the optimality of (2.7) seems very likely in the light of various results in [19, pp. 130–138], which show a close relationship between the Riesz potential operator and the Bessel potential operator, which is relevant for defining Sobolev spaces when the bounded domain Ω is replaced by the whole of $\mathbb{R}^n$.

In this section we shall show that indeed V is the optimal space for the embedding (2.7). In fact, we will obtain this as a special case of a more general result, that *all* of the embeddings obtained in Theorem 3 are optimal. Before formally stating and proving this (as Theorem 5), we shall discuss a number of auxiliary results which will be needed for the proof.

One key fact is a certain property of the special operator T of "weighted Hardy type" defined on measurable functions $f : (0, 1) \to \mathbb{R}$ by

$$Tf(t) = t^{-\frac{1}{q}} \int_{\left(\frac{t}{e}\right)^m}^{1} s^{\frac{1}{p}-1} f(s)\, ds \quad \text{for all} \quad t \in (0, 1), \quad \text{where} \quad m = \frac{1/b - 1/q}{1/a - 1/p}. \tag{4.1}$$

This operator has already been studied in [9]. It is very easy to check (cf. the proof of Theorem 3.4 of [9]) that it satisfies all the conditions of Theorem 2 in the case where both of the measure spaces $(\mathcal{M}_1, \nu_1)$ and $(\mathcal{M}_2, \nu_2)$ are chosen to be $((0, 1), dt)$. But it has the following additional property (cf. [9]):

Theorem 4.

Let T be the operator defined by (4.1) for some choice of p, q, a, and b satisfying $1 \leq a < p < \infty$ and $1 \leq b < q \leq \infty$. Let E be an r.i. space on $((0,\infty),dt)$ which satisfies $q_E < 1$ and define the spaces $A((0,1),dt)$ and $B((0,1),dt)$ by (3.4) and (3.5). Then, for every function $g \in B((0,1),dt)$, there exists a positive non-increasing function $h \in A((0,1),dt)$ such that $Th(t) \geq g^*(t)$ for all $t \in (0,1)$.

In fact the function h is a constant multiple of the function f whose construction and description constitute the main part of the proof of Theorem 5.7 of [9]. The purpose of this result in [9] is to show that, at least in the case where $(\mathcal{M}_1,\nu_1) = (\mathcal{M}_2,\nu_2) = ((0,1),dt)$, the space $B = B((0,1),dt)$ is optimal, i.e., the conclusion of Theorem 2 does not hold for any space smaller than B. It is perhaps rather surprising that this optimality can be established by considering just this one special operator, and we will exploit this fact further here.

The simple results to be discussed in the following paragraphs will be used later for calculations with the Riesz potential operator.

For any fixed $u \in \mathbb{R}^n$ with $|u| = 1$, let $\mathcal{K}_u \subset \mathbb{R}^n$ be the cone $\mathcal{K}_u = \{y \in \mathbb{R}^n : 2\langle y,u\rangle \geq |y|\}$. For $0 \leq a \leq b < \infty$ we let $\mathcal{K}_u(a,b) = \{y \in \mathcal{K}_u : a \leq |y| \leq b\}$. We shall need a rather standard integration formula for radial functions in $\mathbb{R}^n$, namely that

$$\int_{\mathcal{K}_u} \phi(|y|)dy = c_n \int_0^\infty \phi(r) r^{n-1} dr \tag{4.2}$$

for all measurable $\phi : [0,\infty) \to [0,\infty]$, where, as before, μ denotes n-dimensional Lebesgue measure and where the constant $c_n = n \cdot \mu(\mathcal{K}_u(0,1))$. It is important to note that c_n is independent of our choice of the "unit vector" u. The formula (4.2) can be obtained using n-dimensional spherical coordinates. Alternatively, we can use the fact that $\mu(\mathcal{K}_u(0,b)) = b^n \mu(\mathcal{K}_u(0,1))$ and therefore $\mu(\mathcal{K}_u(a,b)) = (b^n - a^n)\mu(\mathcal{K}_u(0,1))$ to first obtain (4.2) when $\phi = \chi_{[a,b]}$ for all a, b with $0 \leq a \leq b$. It can then be extended to all non-negative measurable ϕ by standard arguments.

We will also need to use the fact that

$$|y - x| \leq |y| \quad \text{for all } y \in \mathcal{K}_u(|x|,b) \text{ if } x = |x|u \text{ and } |x| \leq b\,. \tag{4.3}$$

To check this, observe that each such y satisfies $|y-x|^2 = \langle y-x, y-x\rangle = |y|^2 + |x|^2 - 2\langle y,x\rangle = |y|^2 + |x|^2 - |x|\,2\langle y,u\rangle \leq |y|^2 + |x|^2 - |x|\,|y| \leq |y|^2$, as required.

We are now ready to state our main result:

Theorem 5.

Let E be any r.i. space on $((0,\infty),dt)$ satisfying $q_E < 1$ and let the spaces $W_A^k(\Omega)$ and $G(\Omega,\mu)$ be defined using E as in Theorem 3. Then every r.i. space X on (Ω,μ) which satisfies $W_A^k(\Omega) \subset X$ must also satisfy $G(\Omega,\mu) \subset X$.

Remark 9.

This theorem also implies of course that $J_{n,k}$ cannot map $A(\Omega,\mu)$ into any rearrangement invariant space which is smaller than $G(\Omega,\mu)$, i.e., we also have a generalization of Hansson's optimality result to this context.

Proof of Theorem 5. For each positive number ρ, let $D(\rho)$ denote the open euclidean ball of radius ρ in $\mathbb{R}^n$ centered at the origin. Let us denote the volume of the unit ball, i.e., $\mu(D(1))$ by ω_n. We can suppose without loss of generality that our bounded domain Ω contains the origin. Therefore, there exists some $\epsilon > 0$ such that

$$D(2\epsilon) \subset \Omega \text{ and } \mu(D(2\epsilon)) = \omega_n(2\epsilon)^n < 1\,. \tag{4.4}$$

Suppose the theorem is false, i.e., suppose that there exists an r.i. space X on (Ω,μ) such that $W_A^k(\Omega) \subset X$ and a function $w : \Omega \to \mathbb{R}$ such that $w \in G(\Omega,\mu)\backslash X$. We shall show that this leads to a contradiction because it implies the existence of a function $\Theta \in W_A^k(\Omega)$ which is not in X.

Since $G(\Omega, \mu) \subset L_1(\Omega, \mu)$, it follows that there exists a number $\beta > 0$ which is sufficiently large to ensure that

$$\mu(\{x \in \Omega : |w(x)| > \beta\}) \leq \mu(D(\epsilon)) \ . \tag{4.5}$$

Since $L_1(\Omega, \mu) \cap L_\infty(\Omega, \mu)$ is contained in both $G(\Omega, \mu)$ and in X, we obtain that the function $w \cdot \chi_{\{x\in\Omega, |w(x)|\leq\beta\}}$ is an element of $G(\Omega, \mu) \cap X$. Consequently, the function $v = w \cdot \chi_{\{x\in\Omega, |w(x)|>\beta)\}} = w - w \cdot \chi_{\{x\in\Omega, |w(x)|\leq\beta\}}$ is also in $G(\Omega, \mu)\backslash X$. By the definition of non-increasing rearrangements, for each positive s, the set $\{t \in (0, \infty) : v^*(t) > s\}$ is the interval $(0, \lambda(s))$ where $\lambda(s) = \mu(\{x \in \Omega : |v(x)| > s\})$. Because of (4.5) we have

$$\lambda(s) \leq \mu(D(\epsilon)) \quad \text{for all } \ s > 0 \ . \tag{4.6}$$

Let $g : \mathbb{R}^n\backslash\{0\} \to [0, \infty)$ be the function defined by $g(x) = v^*(\omega_n|x|^n)$. Then, for each positive s, the set $\{x \in \mathbb{R}^n\backslash\{0\} : g(x) > s\} = \{x \in \mathbb{R}^n\backslash\{0\} : \omega_n|x|^n \in (0, \lambda(s))\} = D\left(\sqrt[n]{\frac{\lambda(s)}{\omega_n}}\right)\backslash\{0\}$. Since $\mu\left(D\left(\sqrt[n]{\frac{\lambda(s)}{\omega_n}}\right)\backslash\{0\}\right) = \lambda(s)$ we see from (4.6) that

$$\left\{x \in \mathbb{R}^n : g(x) > 0\right\} \subset D(\epsilon) \ . \tag{4.7}$$

So $g = g \cdot \chi_\Omega$ and its non-increasing rearrangement g^* is the same function, whether we calculate it with respect to the measure space (Ω, μ) or with respect to $(\mathbb{R}^n, \mu)$. Furthermore, from the previous calculation, $g^*(t) = v^*(t)$ for all $t \in (0, \infty)$ and so $g \in G(\Omega, \mu)\backslash X$. By (4.7) and (4.4) it follows that $\{t \in (0, \infty) : g^*(t) > 0\} \subset (0, \mu(D(\epsilon))) \subset (0, 1)$ and so g^*, when considered as a function on $(0, 1)$, is an element of $G((0, 1), dt) = B((0, 1), dt)$ (cf. Remark 8). Consequently, by Theorem 4, there exists a positive non-decreasing function $h : (0, 1) \to [0, \infty)$ with $h \in A((0, 1), dt)$ such that the weighted Hardy operator (4.1) satisfies

$$Th(t) \geq g^*(t) \quad \text{for all } \ t \in (0, 1) \ . \tag{4.8}$$

In our case here we have to choose $p = n/k$, $q = \infty$, $a = 1$, and $b = \frac{n}{n-k}$, so that this operator is given by $Tf(t) = \int_{t/e}^1 s^{k/n-1} f(s)ds$.

Let us now introduce the non-increasing function $h_1 = (h + M) \cdot \chi_{(0,\mu(D(2\epsilon)))}$ where the number $M \geq 0$ is chosen to satisfy

$$\int_{\mu(D(\epsilon))}^{\mu(D(2\epsilon))} M \cdot s^{k/n-1} ds = \int_{\mu(D(\epsilon))}^{1} s^{k/n-1} h(s) ds \ .$$

Clearly $h_1 \in A((0, 1), dt)$. Furthermore, for all $t \in (0, \mu(D(\epsilon)))$, we have

$$\begin{aligned} \int_{t/e}^1 s^{k/n-1} h_1(s) ds &= \int_{t/e}^{\mu(D(2\epsilon))} s^{k/n-1} h_1(s) ds \\ &\geq \int_{t/e}^{\mu(D(\epsilon))} s^{k/n-1} h(s) ds + \int_{\mu(D(\epsilon))}^{\mu(D(2\epsilon))} M \cdot s^{k/n-1} ds \\ &= \int_{t/e}^1 s^{k/n-1} h(s) ds = Th(t) \ . \end{aligned}$$

Combining this with (4.8) shows that

$$\int_{t/e}^1 s^{k/n-1} h_1(s) ds \geq g^*(t) \ . \tag{4.9}$$

In fact this inequality holds for all $t \in (0, 1)$ since $g^*(t) = 0$ for $t \in [\mu(D(\epsilon)), 1)$.

We have made the transition from h to h_1 because we need to have a function in $A((0,1),dt)$ which is equimeasurable with a function ψ supported on $D(2\epsilon) \subset \Omega$. Since h_1 is equal a.e. to its non-increasing arrangement h_1^* we can in fact suppose that $h_1 = h_1^*$. Let us define $\psi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by setting $\psi(x) = h_1(\omega_n|x|^n)$. Then we can show that $\psi^* = h_1$ and that ψ is supported in $D(2\epsilon)$ by exactly the same reasoning as was used above to show that $g^* = v^*$ and that the support of g is contained in $D(\epsilon)$. Since h_1, like h, is an element of $A((0,1),dt)$, we also obtain that $\psi \in A(\Omega,\mu)$.

For each $x \in D(\epsilon)$, we choose $u = \frac{1}{|x|}x$ and apply the Riesz potential operator to ψ to obtain, using (4.3) and (4.2), that

$$\begin{aligned} J_{n,k}\psi(x) &= \int_{D(2\epsilon)} \frac{h_1(\omega_n|y|^n)}{|x-y|^{n-k}}dy \\ &\geq \int_{\mathcal{K}_u(|x|,2\epsilon)} \frac{h_1(\omega_n|y|^n)}{|y|^{n-k}}dy = c_n \int_{|x|}^{2\epsilon} \frac{h_1(\omega_n r^n)}{r^{n-k}} r^{n-1}dr \\ &= \frac{c_n}{n\omega_n^{k/n}} \int_{\omega_n|x|^n}^{\omega_n(2\epsilon)^n} s^{k/n-1}h_1(s)ds\,. \end{aligned} \tag{4.10}$$

Since $h_1(s) = 0$ for all $s \geq \mu(D(2\epsilon)) = \omega_n(2\epsilon)^n$ this last integral equals

$$C\int_{\omega_n|x|^n}^1 s^{k/n-1}h_1(s)ds\,,$$

where $C = \frac{c_n}{n\omega_n^{k/n}}$. So, using (4.9), we deduce that

$$J_{n,k}\psi(x) \geq Cg^*\left(e\omega_n|x|^n\right) = Cv^*\left(e\omega_n|x|^n\right) = Cg\left(e^{1/n}x\right) = Cg_1(x) \tag{4.11}$$

for all $x \in D(\epsilon)$. But in fact this inequality holds for all $x \in \mathbb{R}^n$ since the function $g_1(x) = g(e^{1/n}x)$ on the right-hand side is supported in the smaller ball $D(e^{-1/n}\epsilon)$. It is rather obvious that g_1, like g, is in $G(\Omega,\mu)$ but not in X. (Both $G(\Omega,\mu)$ and X are r.i. spaces on (Ω,μ) and the two truncated dilation maps $S_\pm$ given by $Sf(x) = f(e^{\pm 1/n}x)\cdot\chi_{D(\epsilon)}(x)$, which map between g and g_1, are both bounded on $L_1(\Omega,\mu)$ and $L_\infty(\Omega,\mu)$.) So (4.11) shows that $J_{n,k}$ does not map $A(\Omega,\mu)$ into X.

We note that this last fact establishes that Hansson's version of the optimality result extends to our more general context (as already mentioned in Remark 9).

We shall now take this line of argument a little further and construct the function $\Theta \in W_A^k(\Omega)$ referred to at the beginning of the proof. We will show that it is not in X by comparing it with $J_{n,k}\psi$. It will provide the contradiction required to complete the proof of our theorem. It is given by the formula

$$\Theta(x) = \int_{D(2\epsilon)} G_k(x-y)h_1\left(\omega_n|y|^n\right)dy\,, \tag{4.12}$$

where the function $G_k \in L_1(\mathbb{R}^n,\mu)$ is strictly positive and continuous, and is defined by [19, Equation (26), p. 132] i.e.,

$$G_k(x) = C_1\int_0^\infty e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\delta^{(-n+k)/2}\frac{d\delta}{\delta} \tag{4.13}$$

for all values of $k \in (0,n)$ including the integer value of interest to us here. Here, C_1, like the constants C_j to follow, stands for a strictly positive absolute constant whose value depends only on k and/or n. The formula (4.12) is an analogue of the formula in (4.10) for $J_{n,k}\psi$, where convolution with the function $\frac{1}{|x|^{n-k}}$ is replaced by convolution with $G_k(x)$. But the ratio of these two functions can be controlled for small x. Using the formula

$$\frac{1}{|x|^{n-k}} = C_2\int_0^\infty e^{-\pi|x|^2/\delta}\delta^{(-n+k)/2}\frac{d\delta}{\delta} \tag{4.14}$$

(cf. [19, (28), p. 132]) it is easy to show (e.g., via the change of variables $s = \delta/|x|^2$) that

$$\lim_{x\to 0} |x|^{k-n} \cdot G_k(x) = C_3.$$

Consequently $G_k(x-y) \geq \frac{C}{|x|^{k-n}}$ for all $x \in \Omega$ and all $y \in D(2\epsilon)$, where this time the strictly positive constant C may depend on ϵ and the diameter of Ω as well as on n and k. This gives that $\Theta(x) \geq C J_{n,k}\psi(x)$ for all $x \in \Omega$ and shows that $\Theta \notin X$.

The only remaining step required to complete our proof is to show that $\Theta \in W_A^k(\Omega)$. In order to do this, we use the fact that Θ is defined by (4.12) on all of $\mathbb{R}^n$ and also some additional properties of the function G_k related to Sobolev spaces on the whole of $\mathbb{R}^n$, which are presented in [19].

We recall that in [19] the notation $L_k^p(\mathbb{R}^n)$ is used for the intersection of Sobolev spaces $\bigcap_{j=1}^k W_p^j(\mathbb{R}^n)$. (Although we do not explicitly need to use it here, we may remark in passing that in fact this intersection equals $W_p^k(\mathbb{R}^n)$. More generally, by the corollary on page 23 of [15], $\bigcap_{j=1}^k W_p^j(\Omega) = W_p^k(\Omega)$ for a large class of domains, including $\Omega = \mathbb{R}^n$ and also all bounded domains having the star-shaped property specified in Section 2.) In [19, Theorem 3, pp. 135–136], it is proved that for all $p \in (1,\infty)$, $f \in L_k^p(\mathbb{R}^n)$ if and only if it can be expressed as a convolution $f = G_k * g$ for some $g \in L_p(\mathbb{R}^n)$. Furthermore $\|f\|_{L_k^p(\mathbb{R}^n)} := \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L_p(\mathbb{R}^n,\mu)}$ is equivalent to $\|g\|_{L_p(\mathbb{R}^n,\mu)}$. As a consequence of all this we have that for each multi-index α with $|\alpha| \leq k$ the linear operator T_α defined by $T_\alpha h = \partial^\alpha (G_k * h)$ is a bounded operator from $L_p(\mathbb{R}^n,\mu)$ to $L_p(\mathbb{R}^n,\mu)$. (Here of course ∂^α denotes the operation of taking a *generalized* partial derivative.) This operator can be modified in an obvious way to give an operator S_α which maps boundedly from $L_p(\Omega,\mu)$ into $L_p(\Omega,\mu)$, namely we take $S_\alpha f(x) = \partial^\alpha \left(\int_\Omega G_k(x-y)f(y)dy\right)$ and restrict our attention to points x which are in Ω.

Now we apply the real interpolation method to the operator S_α. Given any $p \in (1,\infty)$, we choose p_0 and p_1 such that $1 < p_0 < p < p_1 < \infty$. Then, since $S_\alpha : L_{p_j}(\Omega,\mu) \to L_{p_j}(\Omega,\mu)$ is bounded for $j = 0, 1$, we deduce (see, e.g., [5, Theorem 5.3.2, p. 113]) that $S_\alpha : L^{p,r}(\Omega,\mu) \to L^{p,r}(\Omega,\mu)$ boundedly for each $r \in (0,\infty]$. In particular, by setting $r = 1$ and $r = \infty$, we obtain that S_α is bounded on $\Lambda_p(\Omega,\mu)$ and also on $M_p(\Omega,\mu)$. We can of course choose p to be n/k in the preceding argument. Then, since (cf. Remark 6) our space $A(\Omega,\mu)$ is an interpolation space with respect to the couple $\left(\Lambda_{n/k}(\Omega,\mu), M_{n/k}(\Omega,\mu)\right)$, we obtain that $S_\alpha : A(\Omega,\mu) \to A(\Omega,\mu)$ is bounded.

Using the definition and properties of ψ established above, we can rewrite (4.12) as

$$\Theta(x) = \int_\Omega G_k(x-y)\psi(y)dy\,. \tag{4.15}$$

Let α be any multi-index of order $|\alpha| \leq k$. We can see from (4.15) that $\partial^\alpha\Theta = S_\alpha\psi$. Since $\psi \in A(\Omega,\mu)$, the boundedness of S_α on this space now shows that $\partial^\alpha\Theta \in A(\Omega,\mu)$. In particular, applying this to the multi-indices for which $|\alpha| = 0$ or $|\alpha| = k$, we have that $\Theta \in W_A^k(\Omega)$. This completes our proof. □

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Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel
e-mail: mcwikel@math.technion.ac.il

Department of Mathematics, Technion, Israel Institute of Technology, Haifa 32000, Israel
e-mail: evg@techunix.technion.ac.il