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Balls and Quasi-Metrics: A Space of Homogeneous Type Modeling the Real Analysis Related to the Monge-Ampère Equation

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ABSTRACT. We prove that having a quasi-metric on a given set X is essentially equivalent to have a family of subsets $\mathcal{S}(x,r)$ of X for which $y \in \mathcal{S}(x,r)$ implies both $\mathcal{S}(y,r) \subset \mathcal{S}(x,Kr)$ and $\mathcal{S}(x,r) \subset \mathcal{S}(y,Kr)$ for some constant K. As an application, starting from the Monge-Ampère setting introduced in [3], we get a space of homogeneous type modeling the real analysis for such an equation.

1. Introduction

It is well known that such real analytic problems as boundedness and type for the Hardy-Littlewood maximal operator, the John-Nirenberg estimate for the distribution of BMO functions, its parabolic extensions, weight theory and so on, can be solved in the setting of spaces of homogeneous type, where Wiener-type covering lemmas hold true.

In a recent paper of Caffarelli and Gutiérrez [3], a Besicovitch type covering lemma is proved for the family of sections $\mathcal{S}(x,r)$ coming from the Monge-Ampère equation where the Monge-Ampère measure

$$\det D^2\phi = \mu$$

satisfies a doubling condition.

Moreover, they assume certain natural properties for the family of sections $\{ \mathcal{S}(x,r) : x \in \mathbb{R}^n,\ r > 0 \}$ (see Section 3), which allow them to prove the "engulfing property of the sections" (see [4]):

(1.1) *There exists a constant $K > 0$ such that if $y \in \mathcal{S}(x,r)$ then $\mathcal{S}(x,r) \subset \mathcal{S}(y,Kr)$.*

As we show in Section 3, it is also possible to prove the following property:

(1.2) *There exists a constant $K > 0$ such that if $y \in \mathcal{S}(x,r)$ then $\mathcal{S}(y,r) \subset \mathcal{S}(x,Kr)$.*

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In Section 2 we prove an abstract result showing essentially that if a family $\{\mathcal{S}(x, r)\}$ satisfies the engulfing properties (1.1) and (1.2), then the function

$$d(x, y) = \inf\{r /\ x \in \mathcal{S}(y, r),\ \ y \in \mathcal{S}(x, r)\}$$

is a quasi-metric, with triangular constant K, whose d-balls are equivalent to the family $\{\mathcal{S}(x, r)\}$ in the sense that

$$\mathcal{S}\left(x, \frac{r}{2K}\right) \subset B_d(x, r) \subset \mathcal{S}\left(x, \frac{r}{K}\right)$$

holds for every $r > 0$ and $x \in X$.

As a consequence of that result $(\mathbb{R}^n, d, \mu)$ becomes a space of homogeneous type, where μ is the Monge-Ampère measure.

Finally, the equivalence of sections with d-balls shows that BMO spaces, parabolic BMO spaces, and Hardy-Littlewood maximal function, defined by the class of sections, are all equivalent to the corresponding concepts on the space $(\mathbb{R}^n, d, \mu)$ where the results are known. See, for example, [1, 2, 5, 6].

2.

Given a set X, let us consider a function $\mathcal{S} : X \times \mathbb{R}^+ \cup \{0\} \longrightarrow \mathcal{P}(X)$ such that $\mathcal{S}(x, r)$ has the following properties

(2.1) $\bigcap_{r>0} \mathcal{S}(x, r) = \{x\}$, for every $x \in X$;

(2.2) $\bigcup_{r>0} \mathcal{S}(x, r) = \mathbb{R}^n$, for every $x \in X$;

(2.3) for each $x \in X$, $\mathcal{S}(x, r)$ is a non-decreasing function in r;

(2.4) there exists a constant K such that for all $y \in \mathcal{S}(x, r)$

$$\mathcal{S}(x, r) \subset \mathcal{S}(y, Kr)$$

and

$$\mathcal{S}(y, r) \subset \mathcal{S}(x, Kr)\ .$$

Lemma 1.

Let X be a set and $\mathcal{S}$ a function satisfying properties **(2.1)** *to* **(2.4)** *above. Then, the function $d : X \times X \longrightarrow \mathbb{R}^+ \cup \{0\}$ defined by*

$$d(x, y) = \inf\{r /\ x \in \mathcal{S}(y, r),\ \ y \in \mathcal{S}(x, r)\}$$

is a quasi-metric. On the other hand, given a quasi-metric d defined on X, the family of d-balls in X satisfies the above four properties.

Lemma 2.

Let d be the quasi-metric associated by the first part of the previous lemma to the given family $\mathcal{S}$ satisfying **(2.1)** *to* **(2.4)**. *Let $B_d(x, r)$ be a d-ball of center x and radius r, then*

$$\mathcal{S}\left(x, \frac{r}{2K}\right) \subset B_d(x, r) \subset \mathcal{S}(x, r)\ .$$

Proof of Lemma 1. The function d is defined on all of $X \times X$ because of **(2.2)**, the symmetry is a direct consequence of the definition of d. It is clear that $d(x, x) = 0$. On the other hand, if $d(x, y) = 0$, by definition we have, for all $n \in \mathbb{N}$, that $x \in \mathcal{S}(y, \frac{1}{n})$. Hence, using **(2.1)**, we see that

$x = y$. Let us now prove the quasi-triangular property. For $x, y, z \in X$, given $\epsilon > 0$ there exist positive numbers r_1 and r_2 such that

$$d(x,z) + \frac{\epsilon}{2} > r_1, \quad x \in S(z, r_1), \quad z \in S(x, r_1)$$

and

$$d(z,y) + \frac{\epsilon}{2} > r_2, \quad y \in S(z, r_2), \quad z \in S(y, r_2) \; .$$

Since $z \in \mathcal{S}(y, r_2)$, using **(2.4)**, we have that $\mathcal{S}(z,r) \subset \mathcal{S}(y, Kr)$ with $r = \max\{r_1, r_2\}$, but $x \in \mathcal{S}(z, r_1) \subset \mathcal{S}(z, r)$, then $x \in \mathcal{S}(y, Kr)$. The same argument is used to prove that $y \in \mathcal{S}(x, Kr)$. Therefore,

$$\begin{aligned} d(x,y) &\le Kr \\ &\le K\left(d(x,z) + \frac{\epsilon}{2} + d(z,y) + \frac{\epsilon}{2}\right) \\ &\le K(d(x,z) + d(z,y) + \epsilon) \; . \end{aligned}$$

Hence, $d(x, y) \le K(d(x,z) + d(z, y))$.

Suppose now we have a quasi-metric d on X with constant C in the quasi-triangular inequality. Then the function $\mathcal{S} : X \times \mathbb{R}^+ \cup \{0\} \longrightarrow \mathcal{P}(X)$ defined by

$$\mathcal{S}(x,r) = \left\{ y \in \mathbb{R}^n / \; d(x,y) < r \right\}$$

easily verifies **(2.1)**, **(2.2)**, and **(2.3)**. To show **(2.4)** we have to prove that there exists a constant K such that if $y \in \mathcal{S}(x,r)$; then $\mathcal{S}(x,r) \subset \mathcal{S}(y, Kr)$. Let $z \in \mathcal{S}(x,r)$, since $y \in \mathcal{S}(x,r)$ we have

$$d(z,y) \le C(d(z,x) + d(z,y)) \le 3Cr \; .$$

Hence, $z \in \mathcal{S}(y, 3Cr)$. Taking $K = 3C$ we have the thesis. The other inclusion holds even with a smaller K, namely $K = 2C$. □

Proof of Lemma 2. The second inclusion is nothing but the definition of d. In order to prove the first one, we observe that if $y \in \mathcal{S}(x, \frac{r}{2K})$, then, because of **(2.4)**, we have that $x \in \mathcal{S}(x, \frac{r}{2K}) \subset \mathcal{S}(y, \frac{r}{2})$. Hence $d(x,y) < r$. □

3.

Following [3], let ϕ be a real convex function defined on $\mathbb{R}^n$. Given a point $x \in \mathbb{R}^n$ let $\mathcal{L}$ be a supporting hyperplane of ϕ at the point $(x, \phi(x))$, and given $r > 0$ we shall write $S(x,r)$ for the set

$$S(x,r) = S_\phi(x,r) = \left\{ y \in \mathbb{R}^n : \; \phi(y) < \mathcal{L} + r \right\} \; .$$

They are convex sets obtained by projecting on $\mathbb{R}^n$ the points on the graph of ϕ that are below a supporting hyperplane lifted in r. These sets will be called *sections.* So, for each $x \in \mathbb{R}^n$ a family of convex sets $\mathcal{F}_x = \{S(x,r) : \; r \in (0, +\infty)\}$ is considered. Moreover, given a section $S(x,r)$ we shall consider an affine transformation T that "normalizes" $S(x,r)$, i.e.,

$$B(0, \alpha(n)) \subset T(S(x,r)) \subset B(0,1) \; ,$$

where $\alpha(n)$ is a constant depending only on the dimension and B denotes the usual Euclidean ball.

In order to obtain a Besicovitch type covering lemma, Caffarelli and Gutiérrez start working with the following set of conditions on the family of all sections:

(a) There exist constants k_1, k_2, k_3, ϵ_1 and ϵ_2, all positive, and with the following property: given two sections $S(x_0, r_0)$, $S(x, r)$ with $r \leq r_0$, such that

$$S(x_0, r_0) \bigcap S(x, r) \neq \emptyset$$

and given T a transformation that normalizes $S(x_0, r_0)$, there exists $z \in B(0, k_3)$, depending on $S(x_0, r_0)$ and $S(x, r)$, such that

$$B\left(z, k_2\left(\frac{r}{r_0}\right)^{\epsilon_2}\right) \subset T(S(x, r)) \subset B\left(z, k_1\left(\frac{r}{r_0}\right)^{\epsilon_1}\right)$$

and

$$T(x) \in B\left(z, \frac{1}{2}k_2\left(\frac{r}{r_0}\right)^{\epsilon_2}\right).$$

(b) There exists $\delta > 0$, such that given a section $S(x, r)$ and $y \notin S(x, r)$, if T is an affine transformation that normalizes $S(x, r)$ then

$$B\left(T(y), \epsilon^{\delta}\right) \bigcap T\left(S(x, (1-\epsilon)r)\right) = \emptyset$$

for any $0 < \epsilon < 1$.

(c) $\bigcap_{r>0} S(x, r) = \{x\}$.

The purpose of this section is to show that such a family $\mathcal{F} = \bigcup_{x \in \mathbb{R}^n} \mathcal{F}_x$ satisfies **(2.1)** to **(2.4)** of Section 2, hence Lemmas 1 and 2 hold. It is clear that property **(2.1)** is precisely condition **(c)**. Properties **(2.2)** and **(2.3)** follow easily from the definition of $S(x, r)$ as sections of a convex function defined on all of $\mathbb{R}^n$. In the following lemma we shall prove that the sections verify **(2.4)**.

Lemma 3.

There exists a constant K, depending only on δ, k_1, and ϵ_1, such that for $y \in S(x, r)$ we have

(i) *$S(y, r) \subset S(x, Kr)$, and*

(ii) *$S(x, r) \subset S(y, Kr)$.*

Proof of Lemma 3. (i) Take $K > \sup\{2, (2^{\delta+1}k_1)^{\frac{1}{\epsilon_1}}\}$. Suppose there exists $w \in S(y, r)$ such that $w \notin S(x, Kr)$. Then, because of **(b)**, taking $\epsilon = \frac{1}{2}$, we have

$$B\left(T(w), 2^{-\delta}\right) \bigcap T\left(S\left(x, \frac{Kr}{2}\right)\right) = \emptyset.$$

Then, since $K > 2$, we have that $T(y) \in T(S\left(x, \frac{Kr}{2}\right))$ and

$$|T(w) - T(y)| > 2^{-\delta}. \tag{3.1}$$

On the other hand, since $S(y, r)$ and $S(x, Kr)$ have a non-empty intersection, from **(a)** we get

$$T(S(y, r)) \subset B\left(z, k_1\left(\frac{1}{K}\right)^{\epsilon_1}\right),$$

where T normalizes $S(x, Kr)$ and $z \in B(0, k_3)$.

Then $|T(y) - z| < k_1\left(\frac{1}{K}\right)^{\epsilon_1}$ and $|T(w) - z| < k_1\left(\frac{1}{K}\right)^{\epsilon_1}$. As a consequence $|T(y) - T(w)| \leq |T(y) - z| + |z - T(w)| < 2k_1\left(\frac{1}{K}\right)^{\epsilon_1}$. But, because of the choice of K, $|T(w) - T(y)| < 2^{-\delta}$. This is in contradiction with (3.1).

(ii) Now, let us take $w \in S(x, r)$ such that $w \notin S(y, Kr)$ and consider K as in (i). Then, because of **(b)**, taking $\epsilon = \frac{1}{2}$, we have that

$$B\left(T(w), 2^{-\delta}\right) \bigcap T\left(S\left(y, \frac{Kr}{2}\right)\right) = \emptyset ,$$

hence $|T(w) - T(y)| > 2^{-\delta}$ which is again in contradiction with the fact that $|T(w) - T(y)| < 2k_1 \left(\frac{1}{K}\right)^{\epsilon_1} \leq 2^{-\delta}$. $\square$

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