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LINEAR/NONLINEAR FORMS AND THE NORMAL LAW: CHARACTERIZATION BY HIGH ORDER CORRELATIONS

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Summary

The problem of characterizing the normal law associated with linear forms and processes, as well as with quadratic forms, is considered. The classical condition of constancy of regression is replaced by a distinct condition of high-order uncorrelatedness.

1. Introduction

Let $\{U_i\}$ be a finite or infinite sequence of independent random variables and let X and Y be a pair of linear forms in the U_i's. Various stochastic properties of X and Y have been used to characterize the normality of the U_i's. Among these are independence (the Darmois-Skitovich theorem [1], [5] and [8] and its extensions ([2], Chap. 3)), the property of identical distributions (Marcinkiewicz theorem [5] and [6], and its extension ([2], Chap. 4)) and the property of constancy of regression (the Kagan-Linnik-Rao theorem and its extensions ([2], Chap. 5)). The characterization of normality of the U_i's by independence or constancy of regression has also been extended in several directions to various special cases of linear and nonlinear polynomial statistics (see [5], Chap. 5 and [2], Chaps. 4 and 6). For stochastic integrals with respect to processes of independent increments, characterization of the Brownian motion has been considered by Lugannani and Thomas [4]. For stationary time series, Slud [9] provided a characterization for certain linear autoregressive processes.

In this note we consider characterization problems associated with linear and quadratic forms as well as with linear processes in identically distributed $\{U_i\}$. The principal feature of the characterization results

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obtained here is the fact that the classical condition of constancy of regression $E[L_2|L_1]=0$ of one statistic L_2 on another L_1 is replaced here by the condition of high-order uncorrelatedness, i.e., by $\operatorname{cov}\{L_1^l, L_2^k\}=0$ for certain values of the positive integers l, k. For linear forms and processes $E[L_2|L_1]=0$ implies $\operatorname{cov}\{L_1^l, L_2\}=0$ so that the new conditions are generally weaker than constancy of regression. On the other hand, the new characterization presupposes the existence of moments. A by-product of the approach taken here is the fact that given an integer $p\geqq 3$, one can characterize the distribution F of the U_i's as being approximately normal with moments identical to those of a normal distribution up to the given order p. For the special case of two linear forms satisfying a constancy of regression condition, a result of this type was given in [7], Theorem 5.

Characterization results associated with linear forms are presented in Section 2, those for quadratic forms in Section 3. Characterization results associated with, possibly nonstationary, linear processes of the form

$$(1.1) \qquad Y_n=\sum_{i=-\infty}^{\infty} h_{n,i}U_i\,, \qquad n=0, \pm 1, \cdots$$

are given in Section 4. The derivations are collected in Section 5; a unified approach is adopted and all results are shown to follow from a simple recursive relationship (Lemma 5.1) concerning the covariance structure of products of the U_i's.

2. Linear forms

Let $\{U_i\}_{i=1}^N$ be a finite sequence of independent identically distributed random variables with $E|U_i|^p<\infty$, for some integer $p\geqq 3$, such that $E[U_i]=0$, and $\operatorname{Var}[U_i]=\sigma^2>0$. Define the two linear forms

$$(2.1) \qquad X=\sum_{i=1}^{N} a_iU_i\,, \qquad Y=\sum_{i=1}^{N} b_iU_i\,.$$

For every integer $k\geqq 2$, define

$$(2.2) \qquad \gamma_k(l)=\sum_{i=1}^{N}(a_i)^l(b_i)^{k-l}\,, \qquad l=1,\cdots,k-1\,,$$

and note that when $\gamma_2(1)=0$ the random variables X and Y are uncorrelated. We have the following result.

THEOREM 2.1. *Let $\gamma_2(1)=0$ and $p\geqq 3$ be a fixed integer. If for every integer $k=3,\cdots,p$ there exists an integer l_k, $1\leqq l_k\leqq k-1$, such that $\gamma_k(l_k)\neq 0$ then the condition $\operatorname{cov}\{X^{l_k}, Y^{k-l_k}\}=0$, for $k=3,\cdots,p$, implies* *that the distribution F of the U_i's has the same moments structure up to order p as that of a normal variate $\mathcal{N}(0, \sigma^2)$.*

If we choose $l_k=1$ in Theorem 2.1, then the condition $\operatorname{cov}\{X, Y^{k-1}\}=0$ for $k=3,\cdots,p$ is implied by $\mathrm{E}[X|Y]=0$. Thus the characterization result of Rao ([7], Theorem 5), valid under a constant regression condition, is seen to be a special case of Theorem 2.1. Next we have the characterization of the normal law as follows (compare with the corresponding characterization ([2], Theorem 5.5.2) under a constant regression condition).

COROLLARY 2.1. *If, in addition,* $\mathrm{E}|U_i|^n<\infty$ *for all* $n=1,2,\cdots$, *and the conditions in Theorem* 2.1 *are satisfied for all* $k=3,4,\cdots$, (*i.e.,* $p=\infty$) *then the* U_i'*s are normally distributed,* $\mathcal{N}(0, \sigma^2)$.

By imposing stronger conditions on the vectors $\boldsymbol{a}=(a_1,\cdots,a_N)'$ and $\boldsymbol{b}=(b_1,\cdots,b_N)'$, the normality of the U_i's can be deduced under a weaker requirement on the covariance of the powers of X and Y. As $\gamma_2(1)=0$ is identical to the orthogonality of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the notion of strict orthogonality is defined by the additional requirement that for every integer $k\geqq 3$,

$$\gamma_k(l)\neq 0 \qquad \text{for all} \quad l=1,\cdots,k-1\,. \tag{2.3}$$

The smallest dimension strictly orthogonal vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can have is $N=3$. An example is the vectors $\boldsymbol{a}=(1,1,-2/\alpha)'$ and $\boldsymbol{b}=(1,1,\alpha)'$, $1<\alpha<2$. We then have from Theorem 2.1 the following result.

COROLLARY 2.2. *Let* $\mathrm{E}|U_i|^n<\infty$ *for all* $n=1,2,\cdots$ *and assume that the vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are strictly orthogonal. If for every integer* $k\geqq 3$, *we have* $\operatorname{cov}\{X^{l_k}, Y^{k-l_k}\}=0$ *for some integer* l_k, $1\leqq l_k\leqq k-1$, *then the* U_i'*s are normally distributed,* $\mathcal{N}(0, \sigma^2)$.

The extension of Theorem 2.1 and its corollaries to linear forms with a denumerable number of variables $\{U_i\}_{i=1}^{\infty}$ is immediate. For example, the counterpart to Corollary 2.1 is as follows: Let

$$X=\sum_{i=1}^{\infty} a_i U_i\,, \qquad Y=\sum_{i=1}^{\infty} b_i U_i \tag{2.4}$$

where $\mathrm{E}|U_i|^p<\infty$ for all $p=1,2,\cdots$. Then X and Y have finite moments of all orders provided

$$\sum_{i=1}^{\infty}(a_i)^{2p}<\infty\,, \quad \sum_{i=1}^{\infty}(b_i)^{2p}<\infty \qquad \text{for all} \quad p=1,2,\cdots\,. \tag{2.5}$$

A sufficient condition for (2.5) to hold is clearly

$$\sum_{i=1}^{\infty} a_i^2 < \infty \,, \qquad \sum_{i=1}^{\infty} b_i^2 < \infty \,. \tag{2.5$'$}$$

Now define for every integer $k \geqq 2$,

$$\bar{\gamma}_k(l) = \sum_{i=1}^{\infty} (a_i)^l (b_i)^{k-l} \,, \qquad l = 1, \cdots, k-1 \,. \tag{2.6}$$

We then have the following version of Corollary 2.1 (compare with the corresponding characterization of the normal law under a constant regression condition ([2], Theorems 5.6.4 and 5.6.5)).

COROLLARY 2.3. *Let* $\bar{\gamma}_2(1)=0$. *If for every integer* $k \geqq 3$ *there exists an integer* l_k, $1 \leqq l_k \leqq k-1$ *such that* $\bar{\gamma}_k(l_k) \neq 0$, *then the condition* $\mathrm{cov}\,\{X^{l_k}, Y^{k-l_k}\}=0$ *for all* $k=3, 4, \cdots$ *implies that the* U_i*'s are normally distributed* $\mathcal{N}(0, \sigma^2)$.

3. Linear and quadratic forms

The characterization of the normal law by the independence of linear and quadratic forms is well known ([5], Chap. 5 and [2], Chap. 4). Some characterization results, associated with such forms, through the property of constancy of regression are given in [2], Section 6.3. As in Section 2 we replace the condition of constancy of regression by the condition of high-order uncorrelatedness of the linear and quadratic forms.

Let $\boldsymbol{U}=(U_1, \cdots, U_N)'$ be a vector of independent identically distributed random variables such that $\mathrm{E}|U_i|^p < \infty$ for all $p=1, 2, \cdots$ and $\mathrm{E}[U_i]=0$, $\mathrm{Var}[U_i]=\sigma^2>0$. Define the linear form

$$X = \boldsymbol{a}'\boldsymbol{U} \tag{3.1}$$

and the two quadratic forms

$$T = \boldsymbol{U}'\underset{\sim}{B}\boldsymbol{U} \,, \qquad S = \boldsymbol{U}'\underset{\sim}{Q}\boldsymbol{U} \tag{3.2}$$

where $\underset{\sim}{B}=[b_{i,j}]$ and $\underset{\sim}{Q}=[q_{i,j}]$ are two real $N \times N$ symmetric matrices. It is well known that when $\boldsymbol{U}$ is normally distributed, X and T (respectively T and S) are independent if and only if $\underset{\sim}{B}\boldsymbol{a}=0$ (respectively $\underset{\sim}{B}\underset{\sim}{Q}=\underset{\sim}{0}$) (see [5], p. 70). As in Section 2, we define

$$\tilde{\gamma}_k(l) = \sum_{i=1}^{N} (a_i)^l (b_{ii})^{(k-l)/2} \,, \qquad k \geqq 3, \quad l=1, \cdots, k-1, \quad k-l \text{ even}$$

and

$$\Gamma_k(l) = \sum_{i=1}^{N} (b_{ii})^l (q_{ii})^{k-l} \,, \qquad k \geqq 1, \quad l=1, \cdots, k-1 \,.$$

The normality of the U_i's is deduced from the uncorrelatedness of certain powers of the linear and quadratic statistics X and T as follows (compare with the corresponding result through constant regression of T on X ([2], Theorem 6.3.1)).

THEOREM 3.1. *Let* $\underset{\sim}{B}\boldsymbol{a}=0$. *If for every integer* $k \geqq 3$ *there exists an integer* l_k, *with* $1 \leqq l_k < k$ *and* $k-l_k$ *is even, such that* $\tilde{\gamma}_k(l_k) \neq 0$, *then the condition* $\operatorname{cov}\{X^{l_k}, T^{(k-l_k)/2}\}=0$ *for all integers* $k \geqq 3$ *implies that the* U_i*'s are normally distributed,* $\mathcal{N}(0, \sigma^2)$.

Evidently it suffices that the hypothesis in Theorem 3.1 be satisfied for $l_k=1$ when k is odd and $l_k=2$ when k is even.

The best known example related to Theorem 3.1 is that the constant regression of the sample variance T on the sample mean X of a random sample from a population implies the normality of the underlying distribution ([2], Theorem 6.3.1). On the other hand, specializing Theorem 3.1 to this case, we have

$$\boldsymbol{a}'=\frac{1}{N}(1,\cdots,1)\ , \qquad \underset{\sim}{B}=\frac{1}{N}[I-N\boldsymbol{a}\boldsymbol{a}']$$

so that $\underset{\sim}{B}\boldsymbol{a}=0$ and $\tilde{\gamma}_k(l) \neq 0$ for $k \geqq 3$ and $l=1,\cdots,k-1$ are automatically satisfied. Hence if the U_i's have finite moments of all orders then their normality is implied by the uncorrelatedness of certain powers of the sample mean and variance as specified in Theorem 3.1.

Next we characterize the distribution F of the U_i's by uncorrelatedness of certain powers of the two quadratic statistics T and S of (3.2). In this case it is necessary to assume the symmetry of the distribution F since the covariance of two quadratic forms provides no information on odd moments of F.

THEOREM 3.2. *Let* F *be a symmetric distribution and let* $\underset{\sim}{B}\underset{\sim}{Q}=\underset{\sim}{0}$. *If for every integer* $k \geqq 2$ *there exists an integer* l_k, $1 \leqq l_k < k$ *such that* $\Gamma_k(l_k) \neq 0$, *then the condition* $\operatorname{cov}\{T^{l_k}, S^{k-l_k}\}=0$ *for all* $k \geqq 2$ *implies the normality of the distribution* F.

In view of the recursive nature of the proof it is possible, as in Theorem 2.1, to state versions of Theorems 3.1 and 3.2 under which F is characterized as being approximately normal with moments matching those of a normal variate $\mathcal{N}(0, \sigma^2)$ up to a given order p. This is the case, for example, when the conditions in Theorem 3.1 are satisfied only for $k=3,\cdots,p$.

4. Linear processes

In this section we consider a characterization problem associated with linear processes defined as follows. Let $\{U_i\}_{i=-\infty}^{\infty}$ be an infinite sequence of independent identically distributed random variables such that $\mathrm{E}|U_i|^p<\infty$, for all $p=1, 2, \cdots$, and $\mathrm{E}[U_i]=0$, $\mathrm{Var}[U_i]=\sigma^2>0$. Define the output process

$$Y_n=\sum_{i=-\infty}^{\infty} h_{n,i}U_i\ , \qquad n=0, \pm1, \cdots \tag{4.1}$$

where, for each integer n, $\{h_{n,i}\}_{i=-\infty}^{\infty}$ is a sequence of real numbers satisfying

$$\sum_{i=-\infty}^{\infty} h_{n,i}^2<\infty\ . \tag{4.2}$$

The process $\{Y_n\}$ is well defined in the mean-square sense with mean zero and covariance function

$$r(n, m)=\sigma^2 \sum_{i=-\infty}^{\infty} h_{n,i}h_{m,i}\ . \tag{4.3}$$

The process $\{Y_n\}$ may be viewed as the output of a, possibly time-varying, linear filter with "white noise" input $\{U_i\}$. Our concern is the characterization of the distribution F of the input process $\{U_i\}$ as normal, or approximately normal, by high-order covariance properties of the output process $\{Y_n\}$. We limit ourselves to the characterization of normality; the other case of approximate normality, with normal moments up to a given order, can be obtained by a truncation argument as in Theorem 2.1.

The process $\{Y_n\}$ has finite moments of all orders provided

$$\sum_{i=-\infty}^{\infty} h_{n,i}^{2k}<\infty\ , \qquad k=1, 2, \cdots, \tag{4.4}$$

for all integers n; a sufficient condition for (4.4) to hold is clearly

$$\sum_{i=-\infty}^{\infty} h_{n,i}^2<\infty \tag{4.4$'$}$$

for all integers n—a condition assumed henceforth. For each integer $k\geqq 2$, define

$$\gamma_k(l; n, m)=\sum_{i=-\infty}^{\infty} h_{n,i}^l h_{m,i}^{k-l}\ , \qquad l=1, \cdots, k-1\ , \tag{4.5}$$

and note that $\gamma_2(1; n, m)\equiv r(n, m)/\sigma^2=0$ for all $n\neq m$ implies that the process $\{Y_n\}$ is uncorrelated. As in previous sections we characterize the normality of the U_i's by the property of high-order uncorrelatedness of the output process $\{Y_n\}$. The proof of the following result is identical to that of Theorem 2.1 and Corollary 2.1 and is omitted.

THEOREM 4.1. *Let* $\gamma_2(1; n, m)=0$ *for all* $n\neq m$. *If for every integer* $k\geqq 3$ *there exists a triplet of integers* (l_k, n_k, m_k) *with* $1\leqq l_k\leqq k-1$ *and* $n_k\neq m_k$ *such that* $\gamma_k(l_k; n_k, m_k)\neq 0$, *then the condition* $\operatorname{cov}\{Y_{n_k}^{l_k}, Y_{m_k}^{k-l_k}\}=0$ *for all integers* $k\geqq 3$ *implies that the* U_i*'s are normally distributed,* $\mathcal{N}(0, \sigma^2)$.

Evidently the results of Section 2 (Corollary 2.3) can also be applied to characterize normality of the U_i's if one identifies the two variates X and Y of Section 2 with the values of the process $\{Y_n\}$ at two *fixed* instants of time; no such a restriction is imposed in Theorem 4.1. Loosely speaking, Theorem 4.1 deduces normality of the input process $\{U_i\}$ from uncorrelatedness of powers of the output process $\{Y_n\}$ at *some* instants of time. A more clear picture emerges in case the process $\{Y_n\}$ is also stationary, which is considered below.

In the stationary case, we have

$$Y_n=\sum_{i=-\infty}^{\infty} h_{n-i}U_i\ , \qquad n=0, \pm 1, \cdots , \tag{4.6}$$

where $\{h_i\}$ represents the impulse response of a time-invariant filter satisfying

$$\sum_{i=-\infty}^{\infty} h_i^2<\infty\ . \tag{4.7}$$

The covariance function of the process (4.6) is given by

$$r(n)=\sigma^2\sum_{i=-\infty}^{\infty} h_i h_{n+i}\ , \tag{4.8}$$

and its spectral density $\phi(\lambda)$ by

$$\phi(\lambda)=\frac{\sigma^2}{2\pi}|H(e^{i\lambda})|^2\ , \qquad -\pi\leqq\lambda\leqq\pi\ , \tag{4.9}$$

where the transfer function $H(e^{i\lambda})$ is defined in the L_2 sense by

$$H(e^{i\lambda})=\sum_{n=-\infty}^{\infty} h_n e^{-in\lambda}\ . \tag{4.10}$$

It is clear that when $|H(e^{i\lambda})|=\text{Const.}$ (a "white" filter) then $\{Y_n\}$ is a discrete-time white noise process. The condition $|H(e^{i\lambda})|=\text{Const.}$ is equivalent to $\gamma_2(n)=0$ for all $n\neq 0$ where for every integer $k\geqq 2$, $\gamma_k(n)$ is defined by

$$\gamma_k(n)=\sum_{i=-\infty}^{\infty} h_i h_{n+i}^{k-1}\ , \qquad n=0, \pm 1, \cdots\ . \tag{4.11}$$

The following simple result follows immediately from Theorem 4.1.

COROLLARY 4.1. *Assume the linear time-invariant filter* $\{h_i\}$ *is white and for every integer* $k\geqq 3$, $\gamma_k(n)\neq 0$ *for* **some** $n=n_k\neq 0$. *If for every integer* $k\geqq 3$, $\operatorname{cov}\{Y_0, Y_n^{k-1}\}=0$ *for* **all** $n\neq 0$, *then the input process* $\{U_i\}$ *is normal.*

Note that the above characterization by high-order uncorrelatedness is implied by the constant regression condition $\mathrm{E}[Y_0|Y_n]=0$ for all $n\neq 0$.

The condition in Corollary 4.1 on the uncorrelatedness of the powers of the process $\{Y_n\}$ can be considerably weakened provided stronger conditions are imposed on the filter $\{h_i\}$. We shall call a filter satisfying (4.7) strictly white if it is white ($\gamma_2(n)=0$ for $n\neq 0$) and, in addition, for every integer $k\geqq 3$,

$$\gamma_k(n)\neq 0 \qquad \text{for all} \quad n\neq 0\,. \tag{4.12}$$

An example of a strictly white filter is given by

$$h_n=\begin{cases} 0\,, & n<0 \\ a\,, & n=0,\ \sqrt{2}/2\leqq a<1 \\ -(1-a^2)a^{n-1}\,, & 0<n \end{cases}$$

for which the condition (4.12) is easily verified. We then have the following result.

COROLLARY 4.2. *Assume the linear time-invariant filter* $\{h_i\}$ *to be strictly white. If for every integer* $k\geqq 3$, $\operatorname{cov}\{Y_0, Y_n^{k-1}\}=0$ *at* **some** *instant* $n=n_k\neq 0$, *then the input process* $\{U_i\}$ *is normal.*

It would be of interest to generalize the results of this section to the case where the input process $\{U_i\}$ is correlated. In this case one seeks the characterization of the input $\{U_i\}$ as a normal *process*, based on appropriate stochastic properties of the output process $\{Y_n\}$.

5. Derivations

Let $\{U_i\}$ be a finite or infinite sequence of independent identically distributed random variables having finite absolute moments up to order p, where $p\geqq 2$ is a fixed integer. Let

$$c_k=\operatorname{cum}_k\{U_i\}\,, \qquad k=1, 2,\cdots, p\,, \tag{5.1}$$

be the k-th cumulant of U_i. When U_i is $\mathcal{N}(m, \sigma^2)$ we have $c_k\equiv c_k^G$ with

$$c_1^G = m\,, \quad c_2^G = \sigma^2\,, \quad c_k^G = 0 \qquad \text{for} \quad k \geqq 3\,. \tag{5.2}$$

For any integer $k \leqq p$ define the covariance function

$$\begin{aligned} C_{k,l}[\boldsymbol{i}^{(k)}] &= C_{k,l}[i_1, \cdots, i_k] \\ &= \operatorname{cov}\left\{\prod_{j=1}^{l} U_{i_j},\ \prod_{j=l+1}^{k} U_{i_j}\right\}\,, \qquad l = 1, \cdots, k-1\,, \end{aligned} \tag{5.3}$$

where $i_1, \cdots, i_k$ are arbitrary integers. When the U_i's are normal, we denote the covariance (5.3) by $C_{k,l}^G$. The properties of $C_{k,l}[\boldsymbol{i}^{(k)}]$ are given by the following simple lemma.

LEMMA 5.1. *Let k be a positive integer, $k \leqq p$.*

(i) *If $c_j = c_j^G$ for $j = 1, \cdots, k$, then*

$$C_{k,l}[\boldsymbol{i}^{(k)}] = C_{k,l}^G[\boldsymbol{i}^{(k)}]\,.$$

(ii) *If $c_j = c_j^G$ for $j = 1, \cdots, k-1$, then*

$$C_{k,l}[\boldsymbol{i}^{(k)}] = C_{k,l}^G[\boldsymbol{i}^{(k)}] + (c_k - c_k^G)\delta[\boldsymbol{i}^{(k)}]$$

where

$$\delta[\boldsymbol{i}^{(k)}] = \begin{cases} 1\,, & i_1 = i_2 = \cdots = i_k \\ 0\,, & \textit{otherwise}\,. \end{cases}$$

PROOF. Since the U_i's are i.i.d. random variables, $C_{k,l}[\boldsymbol{i}^{(k)}]$ is given by a finite sum of products of the marginal cumulants $c_1, \cdots, c_k$. Moreover, the cumulant c_k contributes to $C_{k,l}[\boldsymbol{i}^{(k)}]$ only when $i_1 = i_2 = \cdots = i_k$. Hence one can write

$$C_{k,l}[\boldsymbol{i}^{(k)}] = \delta[\boldsymbol{i}^{(k)}]c_k + (1 - \delta[\boldsymbol{i}^{(k)}])F_{k,l}[\{c_j\}_{j=1}^{k-1}; \boldsymbol{i}^{(k)}]\,, \tag{5.4}$$

where the precise structure of $F_{k,l}[\,\cdot\,;\,\cdot\,]$ can be obtained from the rules of Leonov and Shirayev [3] but is not needed here. In particular, when the U_i's are normal $\mathcal{N}(m, \sigma^2)$ we have

$$C_{k,l}^G[\boldsymbol{i}^{(k)}] = \delta[\boldsymbol{i}^{(k)}]c_k^G + \{1 - \delta[\boldsymbol{i}^{(k)}]\}F_{k,l}[\{c_j^G\}_{j=1}^{k-1}; \boldsymbol{i}^{(k)}]\,. \tag{5.5}$$

The lemma follows immediately from (5.4) and (5.5).

PROOF OF THEOREM 2.1. We first note that $\mathrm{E}[X] = \mathrm{E}[Y] = 0$ and $\operatorname{cov}\{X, Y\} = \sigma^2\gamma_2(1) = 0$. Using Lemma 5.1 we show, in a recursive manner, that the cumulants $\{c_k\}$ of the random variable U_i satisfy $c_k = c_k^G = 0$ for $k = 3, \cdots, p$ from which the result follows. Fix k, $3 \leqq k \leqq p$, and let

$$R_k(l) \stackrel{\Delta}{=} \operatorname{cov}\{X^l, Y^{k-l}\}\,, \qquad l = 1, \cdots, k-1\,,$$

and by (2.1) and (5.3),

(5.6) $$R_k(l)=\sum_{(i_1,\cdots,i_k)}\cdots\sum\left[\prod_{j=1}^{l} a_{i_j}\right]\left[\prod_{j=l+1}^{k} b_{i_j}\right]C_{k,l}[\boldsymbol{i}^{(k)}]\,.$$

As the first step in the recursion we show $c_3=0$: Since the U_i's are independent with zero means, it is seen directly from (5.3) that

$$C_{3,l}[\boldsymbol{i}^{(3)}]=\mathrm{E}\,[U_1^3]\delta[\boldsymbol{i}^{(3)}]=c_3\delta[\boldsymbol{i}^{(3)}]\,.$$

Thus, by (5.6),

$$R_3(l)=c_3\sum_{i=1}^{N}(a_i)^l(b_i)^{3-l}=c_3\gamma_3(l)\,,\qquad l=1,2\,.$$

The condition $R_3(l_3)=0$ now implies $c_3=0=c_3^G$. Suppose we have shown that $c_j=c_j^G$ for $j=1,\cdots,k-1$. We prove that $R_k(l_k)=0$ implies $c_k=c_k^G$. Using Lemma 5.1 (ii) and (2.2) in (5.6), we obtain

(5.7) $$R_k(l)=\sum_{(i_1,\cdots,i_k)}\cdots\sum\left[\prod_{j=1}^{l} a_{i_j}\right]\left[\prod_{j=l+1}^{k} b_{i_j}\right]C_{k,l}^G[\boldsymbol{i}^{(k)}]+(c_k-c_k^G)\gamma_k(l)\,,\qquad l=1,\cdots,k-1\,.$$

The multiple sum in (5.7) is clearly identical to $R_k^G(l)\stackrel{\Delta}{=}\mathrm{cov}\,\{X_G^l, Y_G^{k-l}\}$ where X_G and Y_G denote the (normal) linear forms when the U_i's are $\mathcal{N}(0, \sigma^2)$. Thus

(5.8) $$R_k(l)=R_k^G(l)+(c_k-c_k^G)\gamma_k(l)\,,\qquad l=1,\cdots,k-1\,.$$

Since $\gamma_2(1)=0$, the random variables X_G and Y_G are uncorrelated and, being normal, are also independent. Hence $R_k^G(l)\equiv 0$ so that

(5.9) $$R_k(l)=(c_k-c_k^G)\gamma_k(l)\,,\qquad l=1,\cdots,k-1\,.$$

Finally the condition $R_k(l_k)=0$ implies $c_k=c_k^G$ since $\gamma_k(l_k)\neq 0$.

Proof of Corollary 2.1. By Theorem 2.1 we have $c_k=c_k^G$ for all $k=1,2,\cdots$ and the result follows.

Proof of Theorem 3.1. It is similar to that of Theorem 2.1. Define for every integer $k\geqq 3$,

(5.10) $$\tilde{R}_k(l)=\mathrm{cov}\,\{X^l, T^{(k-l)/2}\}\,,\qquad l=1,\cdots,k-1,\quad k-l \text{ even}\,,$$

and by (3.1)–(3.2), we have

(5.11) $$\tilde{R}_k(l)=\sum_{(i_1,\cdots,i_k)}\cdots\sum\left[\prod_{j=1}^{l} a_{i_j}\right]\left[\prod_{j=l+1}^{(k+l)/2} b_{i_j,i_{(k-l)/2+j}}\right]C_{k,l}[\boldsymbol{i}^{(k)}]\,,$$

where $C_{k,l}[\boldsymbol{i}^{(k)}]$ is defined in (5.3). Using Lemma 5.1 one obtains, as in the proof of Theorem 2.1,

$$\tilde{R}_k(l)=\tilde{R}_k^G(l)+(c_k-c_k^G)\tilde{\gamma}_k(l)\ , \tag{5.12}$$

where $\tilde{R}_k^G(l)=\text{cov}\,\{X_G^l, T_G^{(k-l)/2}\}$ and X_G and T_G are the linear and quadratic forms (3.1) and (3.2) when the U_i's are $\mathcal{N}(0, \sigma^2)$. Since $\underset{\sim}{B}\boldsymbol{a}=\mathbf{0}$, X_G and T_G are independent ([5], Theorem 4.1.2). Hence $\tilde{R}_k^G(l)=0$ and (5.12) becomes

$$\tilde{R}_k(l)=(c_k-c_k^G)\tilde{\gamma}_k(l)\ , \qquad l=1,\cdots,k-1 \quad \text{with} \quad k-l \text{ even}\,,$$

so that $c_k=c_k^G$ under the condition $\tilde{R}_k(l_k)=0$.

PROOF OF THEOREM 3.2. We first note that the cumulants $c_{2k-1}=0$ for $k=1, 2,\cdots$ so it suffices to show $c_{2k}=0$ for $k=2, 3,\cdots$. Define for every integer $k\geqq 2$,

$$\bar{R}_k(l)=\text{cov}\,\{T^l, S^{k-l}\}\ , \qquad l=1,\cdots,k-1\,. \tag{5.13}$$

Then by (3.2),

$$\bar{R}_k(l)=\sum_{(i_1,\cdots,i_{2k})}\cdots\sum\left[\prod_{j=1}^{l} b_{i_j,i_{l+j}}\right]\left[\prod_{j=2l+1}^{k+l} q_{i_j,i_{k-l+j}}\right]C_{2k,2l}[\boldsymbol{i}^{(2k)}] \tag{5.14}$$

where $C_{2k,2l}[\boldsymbol{i}^{(2k)}]$ is defined in (5.3). Using Lemma 5.1 (ii), as in the proof of Theorem 3.1, and noting that $\underset{\sim}{B}\underset{\sim}{Q}=\underset{\sim}{0}$ implies $\bar{R}_k^G(l)=\text{cov}\,\{T_G^l, S_G^{k-l}\}=0$ ([5], Theorem 4.1.4), we have

$$\bar{R}_k(l)=(c_{2k}-c_{2k}^G)\Gamma_k(l)\ , \qquad k\geqq 2, \quad l=1,\cdots,k-1\,.$$

Thus $c_{2k}=c_{2k}^G$ since $\bar{R}_k(l_k)=0$ and $\Gamma_k(l_k)\neq 0$.

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