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ASYMPTOTIC EFFICIENCY OF THE SPEARMAN ESTIMATOR AND CHARACTERIZATIONS OF DISTRIBUTIONS

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Summary

The Spearman estimator is designed to be a nonparametric estimator for the expectation of a tolerance distribution. We characterize the one-parameter families of distributions (the parameter being the mean of the distribution) for which the Spearman estimator has asymptotic efficiency one. In particular, when the parameter indexes the location, the characterizing distribution is the logistic distribution. In any other case of efficiency one, the family of distributions is given by certain transformations of a logistic distribution.

1. Introduction

Let $F(x)$ denote the (unknown) tolerance distribution of a quantal assay model. The objective is to estimate the expectation of F by performing an experiment as follows. Specify a $d>0$ (dose-interval), an $x_0 \in [-d/2, d/2]$ and let $x_i = x_0 + id$ for $i=0, \pm 1, \cdots, \pm k$ be the dose levels. Let n_i experimental units be subjected to dose level x_i and suppose only R_i, the proportion of the n_i units responding to dose level x_i, be observed. We assume that the R_i are independent and that $n_i R_i$ is binomially distributed with parameters n_i and $F(x_i)$. Quantal assay models are also used in reliability testing (Weaver [14]) and in industry, for example, in munitions testing (Epstein and Churchman [5]).

In applications it is a common practice to specify a parametric representation for F and estimate the unknown parameters from the data (Finney [8] and Berkson [1]). Spearman [13] introduced a nonparametric

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estimator for the expectation μ of F, namely

$$(1.1) \quad \hat{\mu}=R_{-k}(x_{-k}-d/2)+\sum_{i=-k}^{k-1}(x_i+d/2)(R_{i+1}-R_i)+(1-R_k)(x_k+d/2)$$
$$\equiv x_k+d/2-d\sum_{i=-k}^{k}R_i\ .$$

The estimator $\hat{\mu}$ has been studied, for example, by Irwin ([10]), Finney ([6], [7]) and Brown ([2], [3]). In order to study the effect of the location of x_0 and the size of the dose interval d, excluding the possibility of significant misplacement of the range (x_{-k}, x_k) relative to the central part of F, Irwin [10] and Finney [6] introduced an infinite experiment by letting $k\to\infty$. This can be viewed as a formalization of the assumption that the experiment always covers an appropriate range of doses. It follows from Brown ([2], [3]) that in the infinite experiment, $\hat{\mu}$ (obtained by letting $k\to\infty$ in (1.1)) is unbiased when x_0 is randomly chosen in $(-d/2, d/2)$. Moreover, he computed the variance of $\hat{\mu}$ and arrived at the following expression for the asymptotic efficiency of $\hat{\mu}$ relative to a parametric family $\{F_\mu\colon \mu\in\Theta\}$, μ being the expectation of F_μ,

$$(1.2) \qquad e(\mu)=\left[\int_{-\infty}^{\infty}F_\mu(x)(1-F_\mu(x))dx\int_{-\infty}^{\infty}\frac{(\partial F_\mu(x)/\partial\mu)^2}{F_\mu(x)(1-F_\mu(x))}dx\right]^{-1}.$$

Finney [6] computed $e(\mu)=0.9813$ for a normal location family and $e(\mu)=1.000$ for a logistic location family. Brown ([2], [3]) has extensively studied $e(\mu)$ and evaluated it for various families of tolerance distributions. Brown [3] also constructs a class of Student's distributions for which $e(\mu)$ can be made arbitrarily small. Brown [2] enquires whether there are some symmetric distributions other than the logistic for which the Spearman estimator has asymptotic efficiency equal to one. He asserts that there are no other distributions and provides a proof which is not quite correct. The same proof also appears in Rustagi ([12], Section 8.2).

In this paper we will prove that Brown's assertion is still true by giving a more general characterization which covers both location and scale families of distributions. Our main result is that the Spearman estimator has efficiency equal to one for all μ if and only if $F_\mu(x)=G(\alpha(x)-\gamma(\mu))$ where G is the distribution function of a standard logistic distribution and $\alpha(x)$, $\gamma(\mu)$ satisfy certain conditions.

In Sections 2 and 3 we show how definition (1.2) still can be justified as an asymptotic efficiency when, instead of dealing with the somewhat artificial notion of an infinite experiment, one considers the finite experiment described at the beginning of this section, taking limits in appropriate ways. Miller [11] suggests three types of asymptotic approaches and in our paper we follow the most practical approach, namely $k \to \infty$. Also, throughout our paper we implicitly assume that $d \to 0$ as $k \to \infty$.

2. The finite experiment

We will consider the experiment described in Section 1, assuming now that d $(=d_k)$ and hence also the x_i depend on k. The index k will, however, be mostly suppressed in the following. Furthermore, we shall assume that (for given k) x_0 is uniformly distributed on $(-d/2, d/2)$, independent of the R_i.

We shall assume that the expectation μ of the tolerance distribution F exists. Then, as is well-known (see e.g., Chung [4], p. 49),

$$\mu = \int_{-\infty}^{\infty} x dF(x) = -\int_{-\infty}^{0} F(x)dx + \int_{0}^{\infty} (1-F(x))dx \tag{2.1}$$

where both the infinite integrals on the right hand side exist.

Throughout the paper we will assume that the n_i are all equal and fixed, $n_i = n$.

LEMMA 1.

$$\mathrm{E}\hat{\mu} - \mu = \int_{-\infty}^{-k'd} F(x)dx - \int_{k'd}^{\infty} (1-F(x))dx$$

where $k' \equiv k+1/2$ (*here and in the sequel*).

PROOF. By the last equation in (1.1) we have

$$\mathrm{E}(\hat{\mu} \mid x_0) = x_0 + k'd - d \sum_{-k}^{k} F(x_i) \tag{2.2}$$

which, by taking expectation w.r.t. x_0, becomes

$$\begin{aligned} \mathrm{E}\,\hat{\mu} &= k'd - \sum_{-k}^{k} \int_{-d/2}^{d/2} F(y+id)dy \\ &= k'd - \sum_{-k}^{k} \int_{(i-1/2)d}^{(i+1/2)d} F(x)dx = k'd - \int_{-k'd}^{k'd} F(x)dx \\ &= -\int_{-k'd}^{0} F(x)dx + \int_{0}^{k'd} (1-F(x))dx\ . \end{aligned}$$

So the lemma follows by (2.1).

LMEMA 2.

$$\mathrm{Var}\ \hat{\mu} = (d/n) \int_{-k'd}^{k'd} F(x)(1-F(x))dx + O(d^2)\ .$$

PROOF. We have

$$\mathrm{Var}\ \hat{\mu} = \mathrm{E}[\mathrm{Var}\ (\hat{\mu}\,|\,x_0)] + \mathrm{Var}\ [\mathrm{E}(\hat{\mu}\,|\,x_0)]\ .$$

Now, by the last equality of (1.1) we have

$$\mathrm{Var}\ (\hat{\mu}\,|\,x_0) = (d^2/n) \sum_{-k}^{k} F(x_i)(1-F(x_i))$$

so that

$$\begin{aligned}\mathrm{E}[\mathrm{Var}\ (\hat{\mu}\,|\,x_0)] &= (d/n) \sum_{-k}^{k} \int_{-d/2}^{d/2} F(y+id)(1-F(y+id))dy \\ &= (d/n) \int_{-k'd}^{k'd} F(x)(1-F(x))dx\ .\end{aligned}$$

Next, note that

$$\mathrm{Var}\ [\mathrm{E}(\hat{\mu}\,|\,x_0)] = \mathrm{E}[\mathrm{E}(\hat{\mu}\,|\,x_0) - \mathrm{E}\hat{\mu}]^2$$

where, by (2.2),

$$\begin{aligned}|\mathrm{E}(\hat{\mu}\,|\,x_0) - \mathrm{E}(\hat{\mu})| &= \left| x_0 - d\sum_{-k}^{k} F(x_i) + \sum_{-k}^{k} \int_{-d/2}^{d/2} F(y+id)dy \right| \\ &= \left| x_0 + \sum_{-k}^{k} \int_{-d/2}^{d/2} (F(y+id) - F(x_0+id))dy \right| \\ &\leqq \frac{d}{2} + d\sum_{-k}^{k} [F((i+1/2)d) - F((i-1/2)d)] \\ &= \frac{d}{2} + d[F(k'd) - F(-k'd)] \leqq 3d/2\end{aligned}$$

where we have used the fact that F is non-increasing. Hence, combining the above results, we get the lemma.

LEMMA 3.

$$\mathrm{E}(\hat{\mu}-\mu)^2 = (d/n) \int_{-k'd}^{k'd} F(x)(1-F(x))dx + o(d)$$

provided there exists a non-negative and non-decreasing function ϕ on $[0, \infty]$ with $\phi(x) \to \infty$ as $x \to \infty$, satisfying

$$\int_{-\infty}^{0} \phi(x)F(x)dx < \infty \qquad \text{and} \qquad \int_{0}^{\infty} \phi(x)(1-F(x))dx < \infty$$

and $kd \to \infty$ as $k \to \infty$ in such a way that

$$\phi(kd)d^{1/2} \to \infty\ .$$

Remark 3.1. The condition on F is seen to be equivalent to the condition that

$$\mathrm{E}\left[\int_0^{|X|} \phi(x)dx\right] < \infty\ .$$

where X is a r.v. with d.f. F. Thus if F, for example, has a second moment, then $\phi(x)=x$ will do in the lemma and the condition on d is that $kd^{3/2}\to\infty$ as $k\to\infty$.

PROOF. We have $E(\hat{\mu}-\mu)^2=\text{Var}\,\hat{\mu}+(E\hat{\mu}-\mu)^2$, so by Lemma 2 we need only prove that $(E\hat{\mu}-\mu)^2=o(d)$ as $k\to\infty$. By the assumptions we have

$$\infty>a_1\equiv\int_0^\infty\phi(x)(1-F(x))dx\geqq\int_{k'd}^\infty\phi(x)(1-F(x))dx$$
$$\geqq\phi(k'd)\int_{k'd}^\infty(1-F(x))dx$$

so that

$$\int_{k'd}^\infty(1-F(x))dx\leqq a_1/\phi(k'd)\ .$$

Similarly we can show that for $a_2\equiv\int_{-\infty}^0\phi(x)F(x)dx$ we have

$$\int_{-\infty}^{-k'd}F(x)dx\leqq a_2/\phi(k'd)\ .$$

Thus, by Lemma 1,

$$(E\hat{\mu}-\mu)^2=O(1/\phi^2(kd))\qquad\text{as}\quad k\to\infty\ ,$$

which is $o(d)$ provided $1/\phi(kd)d^{1/2}=o(1)$.

3. Efficiency of the Spearman estimator for a parametric family

Let $\{F_\mu:\mu\in\Theta\}$ be a family of distribution functions on R. In the rest of the paper we will make the following assumptions.

(A1) Θ is an open (finite or infinite) interval on R.

(A2) $\int xdF_\mu(x)=\mu$ for all $\mu\in\Theta$ $\left(\text{so}\int|x|dF_\mu(x)<\infty\right)$.

(A3) $x\to F_\mu(x)$ is continuous on R for all $\mu\in\Theta$.

The following assumption is needed in order that the Fisher-information (see below) has meaning:

(A4) For each fixed $x\in R$, $\dfrac{\partial F_\mu(x)}{\partial\mu}$ exists on all of Θ.
(So $\mu\to F_\mu(x)$ is, in particular, continuous on Θ).

Let $H=\{(x,\mu):x\in R\}$, $\mu\in\Theta$, $0<F_\mu(x)<1$, $A(\mu)=\{x:(x,\mu)\in H\}$, $\mu\in$

Θ, $B(x)=\{\mu: (x, \mu) \in H, x \in R\}$. By (A3), $A(\mu)$ is an open interval on R. By (A4), $B(x)$ is an open subset of Θ for each $x \in R$. We shall make the following additional assumptions.

(A5) For each $\mu \in \Theta$, $x \to F_\mu(x)$ is strictly increasing on $A(\mu)$ (so $A(\mu)$ is the support of F_μ).

(A6) For each $x \in R$, $B(x)$ is either empty or is an open *interval.*

(A7) $x \to \dfrac{\partial F_\mu(x)}{\partial \mu}$ is continuous on $A(\mu)$ for each $\mu \in \Theta$.

(A8) $\mu \to \dfrac{\partial F_\mu(x)}{\partial \mu}$ is continuous on $B(x)$ for each $x \in R$ with $B(x) \neq \emptyset$.

(A9) For each $\mu \in \Theta$ there is an $x \in A(\mu)$ with $\dfrac{\partial F_\mu(x)}{\partial \mu} \neq 0$.

(A10) For each $\mu' \in \Theta$ there is a function $K_{\mu'}$ integrable on R and an $\varepsilon > 0$ such that $|\partial F_\mu(x)/d\mu| \leqq K_{\mu'}(x)$ for all $\mu \in (\mu' - \varepsilon, \mu' + \varepsilon)$ and all $x \in R$.

The log-likelihood, given x_0, of the experiment is

$$\log l(\mu \mid x_0) = \sum_{\substack{i=-k \\ x_i \in A(\mu)}}^{k} \{r_i \log F_\mu(x_i) + (n - r_i) \log (1 - F_\mu(x_i))\}$$

and the conditional Fisher-information is easily computed to be

$$\mathrm{E}\left[\left(\frac{\partial \log l}{\partial \mu}\right)^2 \middle| x_0\right] = n \sum_{-k}^{k} \left(\frac{\partial F_\mu(x_i)}{\partial \mu}\right)^2 \Big/ [F_\mu(x_i)(1 - F_\mu(x_i))] \, .$$

Thus the Fisher-information of the randomized experiment is

$$\begin{aligned} I_k(\mu) \equiv \mathrm{E}\left[\left(\frac{\partial \log l}{\partial \mu}\right)^2\right] &= n \sum_{-k}^{k} d^{-1} \int_{-d/2}^{d/2} \left(\frac{\partial F_\mu(x_i)}{\partial \mu}\right)^2 \Big/ [F_\mu(x_i)(1 - F_\mu(x_i)] \\ &= (n/d) \int_{-k'd}^{k'd} \left(\frac{\partial F_\mu(x)}{\partial \mu}\right)^2 \Big/ [F_\mu(x)(1 - F_\mu(x))] dx \, . \end{aligned}$$

This motivates the following definition of asymptotic efficiency of the Spearman estimator:

$$\mathrm{E}(\mu) \equiv \lim_{k \to \infty} [\mathrm{Var}\, \hat{\mu}_k]^{-1} [I_k(\mu)]^{-1} = [V(\mu) I(\mu)]^{-1} \, , \qquad \mu \in \Theta$$

where

$$V(\mu) = \int_{A(\mu)} F_\mu(x)(1 - F_\mu(x)) dx \, ,$$

$$I(\mu)=\int_{A(\mu)}\left\{\left(\frac{\partial F_\mu(x)}{\partial\mu}\right)^2\Big/[F_\mu(x)(1-F_\mu(x))]\right\}dx\ ,\qquad \mu\in\Theta\ .$$

Note that $V(\mu)>0$ by (A3) and $V(\mu)<\infty$ by (A2). Moreover, (A7) and (A9) imply that $I(\mu)>0$.

If there is a function ϕ satisfying the requirements of Lemma 3 for all F_μ, $\mu\in\Theta$, then provided $\phi(kd)d^{1/2}\to\infty$ as $k\to\infty$ we have

$$e(\mu)=\lim_{k\to\infty}[\mathrm{E}(\hat\mu_k-\mu)^2]^{-1}[I_k(\mu)]^{-1}\ .$$

4. The main results

In the following, let $R^-=(-\infty,0)$, $R^+=(0,\infty)$.

THEOREM 1. *If* $\{F_\mu\colon\mu\in\Theta\}$ *satisfies* (A1)-(A9) *and in addition* $e(\mu)=1$ *for all* $\mu\in\Theta$, *then there is an open interval* $A=(a,b)$ $(-\infty\leqq a<b\leqq\infty)$, *a real function* α *on* A *satisfying*

(α0) *α is continuous and strictly increasing on A.*

(α1) $\alpha(x)\to-\infty$ *as* $x\downarrow a$, $\alpha(x)\to+\infty$ *as* $x\uparrow b$.

(α2) $\int_{R^-\cap A}(1+e^{-\alpha(x)})^{-1}dx<\infty$, $\int_{R^+\cap A}(1+e^{\alpha(x)})^{-1}dx<\infty$ *and a real function* γ *on* Θ *satisfying*

(γ) *γ is continuously differentiable on Θ with $\gamma'(\mu)>0$ for all $\mu\in\Theta$.* *such that*

$$F_\mu(x)=(1+e^{-\alpha(x)+\gamma(\mu)})^{-1}\ ,\qquad x\in A,\ \mu\in\Theta\ . \tag{4.1}$$

PROOF. Put $F=F_\mu$ in (2.1) and differentiate both sides with respect to μ. This gives, using Assumption (A10) and the dominated convergence theorem

$$\int_{A(\mu)}\{\partial F_\mu(x)/\partial\mu\}dx=-1\ . \tag{4.2}$$

From Cauchy-Schwarz's inequality we get for each $\mu\in\Theta$,

$$e(\mu)^{-1}=V(\mu)I(\mu)\geqq\left[\int_{A(\mu)}\left|\frac{\partial F_\mu(x)}{\partial\mu}\right|dx\right]^2\geqq-\int_{A(\mu)}\frac{\partial F_\mu(x)}{\partial\mu}dx=1\ . \tag{4.3}$$

Thus, as we assume $e(\mu)=1$, equality holds at both inequality signs of (4.3). From the first of them we conclude that there is a $\beta(\mu)$ (not depending on x) such that

$$\frac{|\partial F_\mu(x)/\partial\mu|}{F_\mu(x)(1-F_\mu(x))}=\beta(\mu)\qquad\text{for all}\quad x\in A(\mu),\ \mu\in\Theta \tag{4.4}$$

("for all" is equivalent to "a.e." because of (A3) and (A7)). From the second, we conclude that

$$\frac{\partial F_\mu(x)}{\partial \mu} \leqq 0 \qquad \text{for all} \quad x \in A(\mu),\ \mu \in \Theta\,.$$

Moreover, from (A8) and (A9) we conclude that $\beta(\cdot)$ is continuous on Θ and that $\beta(\mu) > 0$ for all $\mu \in \Theta$.

Now by (4.4) we have, for any fixed x with $B(x) \neq \emptyset$

$$\frac{\partial F_\mu(x)/\partial \mu}{F_\mu(x)(1-F_\mu(x))} = -\beta(\mu) \qquad \text{for all} \quad \mu \in B(x) \tag{4.5}$$

which implies by integration that for some $a \in \Theta$ and a constant $\alpha(x)$ independent of μ we have

$$F_\mu(x) = \left(1 + \exp\left(-\alpha(x) + \int_a^\mu \beta(t)dt\right)\right)^{-1}, \qquad \mu \in B(x)\,. \tag{4.6}$$

Now fix $\mu \in \Theta$. Then (4.6) and Assumptions (A3) and (A5) imply that α is continuous and strictly increasing on $A(\mu)$, so (α0) is satisfied. Also, (A3) implies that (α1) must hold with $a = \inf A(\mu)$, $b = \sup A(\mu)$. But then $A(\mu)$ must be independent of μ. To see that (α2) holds, note that $\int_{-\infty}^0 F_\mu(x)dx < \infty$ and $\int_0^\infty (1-F_\mu(x))dx < \infty$ (see Section 2). Now (α2) follows from this by putting $\mu = a$ in (4.6). If we finally put $\gamma(\mu) = \int_a^\mu \beta(t)dt$ it is seen that (γ) holds and we are done.

The theorem below implies the sufficiency of the conditions in Theorem 1. In fact we obtain a slightly more general result.

THEOREM 2. *Let A be an open interval on R and let α be a real function on A satisfying* (α0), (α1) *and* (α2). *Then there is an open interval Θ and a real function γ on Θ satisfying* (γ) *such that*:

(i) *The family $\{F_\mu : \mu \in \Theta\}$ given by* (4.1) *satisfies* (A1)–(A9) *and $e(\mu) = 1$ for all $\mu \in \Theta$.*

(ii) *If $G_\mu(x) = (1+e^{-\alpha(x)+\delta(\mu)})^{-1}$, $x \in A$, $\mu \in \Theta'$ and $\int x dG_\mu(x) = \mu$ for all $\mu \in \Theta'$, then $\Theta' \subseteq \Theta$ and $\delta = \gamma$ on Θ'.*

PROOF. (i) It is clear that (A1) and (A3)–(A9) are satisfied. Let α be given as in the theorem and consider first the family $\{\tilde{F}_\lambda : \lambda \in R\}$ where $\tilde{F}_\lambda(x) = (1+e^{-\alpha(x)+\lambda})^{-1}$, $x \in A$, $\lambda \in R$. Then (α2) clearly implies that

$$\int_{R^- \cap A} \tilde{F}_\lambda(x)dx < \infty\,, \qquad \int_{R^+ \cap A} (1-\tilde{F}_\lambda(x))dx < \infty$$

for all $\lambda \in R$, so we have $\int |x| d\tilde{F}_\lambda(x) < \infty$ and can write

$$\phi(\lambda)\equiv\int x d\tilde{F}_\lambda(x)=-\int_{R^-\cap A}\tilde{F}_\lambda(x)dx+\int_{R^+\cap A}(1-\tilde{F}_\lambda(x))dx\,. \tag{4.7}$$

Now we have

$$\left|\frac{\partial\tilde{F}_\lambda(x)}{\partial\lambda}\right|=\frac{e^{-\alpha(x)}}{(1+e^{-\alpha(x)+\lambda})^2}\leqq\frac{e^{2|\lambda|}}{(1+e^{\alpha(x)})(1+e^{-\alpha(x)})}\leqq\frac{e^{2|\lambda|}}{1+e^{-\alpha(x)}} \tag{4.8}$$

so that in a neighborhood of each $\lambda\in R$ we have $|\partial\tilde{F}_\lambda(x)/\partial\lambda|$ bounded on $R^-\cap A$ by the integrable function, const $(1+e^{-\alpha(x)})^{-1}$. This implies, using the dominated convergence theorem, that

$$\frac{\partial}{\partial\lambda}\int_{R^-\cap A}\tilde{F}_\lambda(x)dx=\int_{R^-\cap A}\frac{\partial\tilde{F}_\lambda(x)}{\partial\lambda}dx\,.$$

By a similar argument on $R^+\cap A$ we obtain from (4.7) that ϕ is differentiable with

$$\phi'(\lambda)=-\int_A\frac{\partial\tilde{F}_\lambda(x)}{\partial\lambda}dx>0 \qquad \text{for all} \quad \lambda\in R\,. \tag{4.9}$$

Moreover, using (4.8) and the dominated convergence theorem once more we get that ϕ' is continuous on R. Let Θ be the range of ϕ, which is an open interval since ϕ is continuous and strictly increasing. Now reparametrize the family $\{\tilde{F}_\lambda:\lambda\in R\}$ by putting $F_\mu=\tilde{F}_{\phi^{-1}(\mu)}$, $\mu\in\Theta$ and let $\gamma(\mu)=\phi^{-1}(\mu)$. Then $\int xdF_\mu(x)=\int xd\tilde{F}_{\phi^{-1}(\mu)}=\mu$ (so (A2) holds) and γ clearly satisfies (γ). Moreover, (4.8) implies that (A10) holds. To see that $e(\mu)=1$ for all $\mu\in\Theta$, note first that $\partial F_\mu(x)/\partial\mu<0$ which follows by differentiating (4.1). Then substitute (4.1) into the left hand side of (4.4), to get an expression independent of x. Thus equality holds at both inequality signs in (4.3), and thus $e(\mu)\equiv1$.

For part (ii) of the theorem, suppose $\mu\in\Theta'$. Then we have $G_\mu=\tilde{F}_{\delta(\mu)}=F_{\phi(\delta(\mu))}$ so that

$$\mu=\int xdG_\mu(x)=\int xdF_{\phi(\delta(\mu))}=\phi(\delta(\mu))\,. \tag{4.10}$$

Hence $\mu\in\Theta$ and application of ϕ^{-1} to (4.10) yields $\delta(\mu)=\phi^{-1}(\mu)=\gamma(\mu)$.

5. Special cases

5.1 *The translation case*

Let F_0 be a continuous distribution function on R, strictly increasing on $A(0)=\{x:0<F_0(x)<1\}$ and with $\int xdF_0(x)=0$. Suppose further that F_0 is differentiable everywhere on R with $f_0(x)=F_0'(x)$ and suppose that f_0 is continuous on $A(0)$. Let Θ be an open interval on R and let $\{F_\mu : \mu \in \Theta\}$ be the family given by

$$F_\mu(x) = F_0(x-\mu) , \qquad x \in R, \ \mu \in \Theta . \tag{5.1}$$

Then Assumptions (A1)-(A10) are satisfied with $A(\mu) = A(0)+\mu$, $\partial F_\mu(x)/\partial\mu = -f_0(x-\mu)$. Furthermore, $e(\mu)$ is independent of $\mu \in \Theta$, given by

$$e = \left[\int_{A(0)} F_0(x)(1-F_0(x))dx \int_{A(0)} \frac{f_0^2(x)}{F_0(x)(1-F_0(x))} dx\right]^{-1} . \tag{5.2}$$

COROLLARY 1. *For a translation family given by* (5.1) *we have* $e=1$ *if and only if* $A(0)=R$ *and for some* $\beta>0$, $F_0(x)=(1+e^{-\beta x})^{-1}$, $x \in R$.

PROOF. Sufficiency is well-known (Section 1), so we need to prove necessity. By Theorem 1 we have $A(\mu)$ independent of μ. Since $A(\mu) = A(0)+\mu$ this implies that $A(0)=R$. Next, since Θ is open we have for any $\mu \in \Theta$, $x \in R$ and $|h|$ sufficiently small,

$$F_{\mu+h}(x) = F_0(x-\mu-h) = F_\mu(x-h) ,$$

so substitution into (4.1) yields $-\alpha(x)+\gamma(\mu+h) = -\alpha(x-h)+\gamma(\mu)$. Reordering, dividing by h and letting $h \to 0$ we get $\alpha'(x) = \gamma'(\mu)$ for all x and $\mu \in \Theta$ and hence they must be equal to a constant which is positive by (γ). Thus for some $\beta>0$, $-\alpha(x)+\gamma(\mu) = -\beta(x-\mu)+\delta$ where δ is a constant. This gives $F_0(x)=(1+e^{-\beta x+\delta})^{-1}$ from which it follows that $\delta=0$ since $\int x dF_0(x)=0$.

As remarked in Section 1, Brown [2] stated a weaker result than Corollary 1, namely that the logistic family is the only location family of *symmetric* distributions for which $e=1$. Corollary 1 implies that Brown's assertion holds with "symmetric" deleted from the statement.

The asymptotic efficiency e given by (5.2) has meaning whenever (A4) is satisfied, i.e. when we assume that $f_0(x) = F_0'(x)$ exists for all $x \in R$, without assuming continuity of f_0 on $A(0)$. We see that a direct proof of Corollary 1 can be given in this case, using (5.2). In fact, applying Cauchy-Schwarz' inequality to (5.2) we get

$$e^{-1} \geqq \left[\int f_0(x)dx\right]^2 = 1$$

with equality if and only if there is a constant $\beta>0$ with

$$\frac{f_0(x)}{F_0(x)(1-F_0(x))} = \beta \qquad \text{for a.e. } x \in A(0) .$$

By integration and by the continuity of F_0 this implies that for some α,

$$\ln \frac{F_0(x)}{1-F_0(x)} = \beta x + \alpha \qquad \text{for all} \quad x \in A(0)$$

or $F_0(x)=(1+e^{-\beta x+\alpha})^{-1}$, $x \in A(0)$. As F_0 is required to be continuous on R we must have $A(0)=R$ and finally $\int x dF_0 = 0$ implies $\alpha=0$. Finally, we remark that the condition of Lemma 3 is satisfied here with, for example, $\phi(x)=x^r$ for any $r>0$.

5.2 *The scale case*

Let F_1 be a continuous distribution function strictly increasing on $A(1)=\{x: 0<F_1(x)<1\}$, with $\int x dF_1(x)=1$. Suppose that F_1 is differentiable everywhere on R, with $f_1(x)=F_1'(x)$ continuous on $A(1)$. Let Θ be an open interval contained in R^+ and let $\{F_\mu: \mu \in \Theta\}$ be the family ("scale family") given by

(5.3) $$F_\mu(x)=F_1(x/\mu) \; , \qquad x \in R, \; \mu \in \Theta \; .$$

The Assumptions (A1)–(A10) are now satisfied, with $A(\mu)=\mu A(0)$, $\partial F_\mu(x)/\partial\mu = -x\mu^{-2}f_1(x/\mu)$, and $e(\mu)=e$ is given by

(5.4) $$e=\left[\int_{A(1)} F_1(x)(1-F_1(x))dx \int_{A(1)} \frac{x^2 f_1^2(x)}{F_1(x)(1-F_1(x))}dx\right]^{-1} .$$

Brown [2] has computed e for $F_1(x)=e^{-x}$, $x>0$ to be 0.8319. The following result follows from our Theorems 1 and 2.

COROLLARY 2. *For a scale family given by* (5.3) *we have* $e<1$ *whenever* $A(1)\not\subseteq R^+$. *We have* $e=1$ *if and only if* $A(1)=R^+$ *and for some* $\beta>1$,

$$F_1(x)=(1+C(\beta)x^{-\beta})^{-1} \; , \qquad x \in R^+$$

where $C(\beta)$ *is a positive constant determined by*

(5.5) $$\beta^{-1}C(\beta)^{1/\beta}\int_0^\infty (1+y)^{-1}\, y^{-1/\beta}\, dy = 1 \; .$$

PROOF. We have

$$\int_{A(\mu)} \left|\frac{\partial F_\mu(x)}{\partial\mu}\right| dx = \int \frac{|x|}{\mu^2} f_1(x/\mu) = \int |x| f_1(x)dx \geqq \int x f_1(x)dx = 1$$

so that (4.2) implies that $e \leqq 1$ with equality only when $\int |x| f_1(x)dx=1$, i.e. when $A(1)\subseteq R^+$. This proves the first part of the theorem. For the rest, sufficiency follows from Theorem 2 with $A=R^+$, $\alpha(x)=\beta \ln x$. To prove necessity, note first that we must have $A(1)\subseteq R^+$ and so since

$A(\mu)=\mu A(1)$ by Theorem 1 is independent of μ we must have $A(\mu)=R^+$. Next, for each fixed $x \in R^+$, $\mu \in \Theta$ and $|h|$ sufficiently small we have $F_\mu(x)=F_1(x/\mu)=F_{\mu(x+h)/x}(x+h)$ which by (4.1) gives $-\alpha(x)+\gamma(\mu)=-\alpha(x+h)+\gamma(\mu(x+h)/x)$ so that

$$\frac{\alpha(x+h)-\alpha(x)}{h}=\frac{\gamma(\mu(x+h)/x)-\gamma(\mu)}{h}.$$

Letting $h \to 0$ and using (γ) we obtain

$$\alpha'(x)=\frac{\mu\gamma'(\mu)}{x},$$

for all x in R^+ and $\mu \in \Theta$. As observed in the proof of Corollary 1, $\mu\gamma'(\mu)\equiv\beta>0$ is independent of μ and so $\alpha(x)=\beta \ln x+\delta$, $\gamma(\mu)=\beta \ln \mu+\rho$. That is, if we put $\mu=1$ in (4.1), we get

$$F_1(x)=(1+Cx^{-\beta})^{-1}, \qquad x \in R^+$$

for some $C>0$. We require $\int_0^\infty x dF_1(x)=\int_0^\infty F_1(x)dx=1$. So in order that the last integral converge, we must have $\beta>1$ and in order that it equals 1, a suitable substitution shows that $C=C(\beta)$ must satisfy (5.5).

As in the translation case, a direct proof using (5.4) will show that Corollary 2 holds true if the continuity assumption of f_1 on (A1) is dropped.

It can be shown that $C(2)=(2/\pi)^2$, $C(3)=(3\sqrt{3}/\pi)^3$ (see Gradshteyn and Ryzhik [9], formulae 2.211 and 2.235).

The condition of Lemma 3 is seen to be satisfied here with, for example, $\phi(x)=x^r$ for $0<r<1-\beta$.

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REFERENCES

[1] Berkson, J. (1953). A statistically precise and relatively simple method of estimating the bioassay with quantal response, based on the logistic function, *J. Amer. Statist. Ass.*, 48, 565-599.

* On leave from the University of Kentucky.
** On leave from the Norwegian Institute of Technology, Trondheim, Norway.

[2] Brown, B. W. Jr. (1959). Some properties of the Spearman estimator in bioassay, *Tech. Report No. 6*, Department of Statistics, University of Minnesota, Minneapolis, Minnesota.

[3] Brown, B. W. Jr. (1961). Some properties of the Spearman estimator in bioassay, *Biometrika*, **48**, 293-302.

[4] Chung, K. L. (1974). *A Course in Probability Theory*, Academic Press, New York.

[5] Epstein, B. and Churchman, C. W. (1944). On the statistics of sensitivity data, *Ann. Math. Statist.*, **15**, 90-96.

[6] Finney, D. J. (1950). The estimation of the mean of a normal tolerance distribution, *Sankhyā*, **10**, 341-360.

[7] Finney, D. J. (1952). The estimation of the E.D. 50 for logistic response curve, *Sankhyā*, **12**, 121-136.

[8] Finney, D. J. (1971). *Probit Analysis*, 3rd ed., Cambridge University Press.

[9] Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Table of Integrals and Products*, Academic Press, New York.

[10] Irwin, J. O. (1937). Statistical method applied to biological assays, *J. R. Statist. Soc.* (Suppl.), **4**, 1-60.

[11] Miller, R. G. (1973). Nonparametric estimators of the mean tolerance in bioassay, *Biometrika*, **60**, 535-542.

[12] Rustagi, J. S. (1976). *Variational Methods in Statistics*, Academic Press, New York.

[13] Spearman, C. (1908). The method of 'right and wrong cases' ('constant stimuli') without Gauss' formula, *Brit. J. Psychol.*, **2**, 227-242.

[14] Weaver, O. R. (1957). Using attributes for measurement of continuous variables in the rocket industry, *Proc. 12th Midwest Quality Control Conference*, American Society for Quality Control.