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NONPARAMETRIC INFERENCE ON THE DIFFERENCE OF LOCATION PARAMETERS OF CORRELATED VARIABLES FROM FRAGMENTARY SAMPLES

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Summary

In this paper, two types of robust estimators and approximate confidence intervals for the difference of location parameters of correlated random variables are proposed and investigated when some observations are missing. It is shown that the suggested estimators are consistent and asymptotically normally distributed. In addition, the proposed approximate confidence intervals are also shown to enjoy some nice asymptotic properties.

1. Introduction

The problem of estimation of the difference of location parameters of two correlated random variables with incomplete paired observations can be described as follows: Let (X, Y) be a random vector with continuous joint distribution $H(x, y)$ and marginal distributions $F_1(x)$ and $F_2(y)$ for X and Y respectively such that $F_2(x)=F_1(x-\theta)$, $\theta \in R$. Assume that we have a fragmentary random sample $\{(X_1, Y_1), \cdots, (X_n, Y_n), (X_{n+1}, \cdot), \cdots, (X_{n+m}, \cdot), (\cdot, Y_{n+1}), \cdots, (\cdot, Y_{n+l})\}$ observed from (X, Y), where "$\cdot$" denotes a missing observation. Then the problem is how to use these data wisely to make an inference on the shift parameter θ.

In the situation that $H(x, y)$ is a bivariate normal distribution, the problem of estimation of the mean difference θ has been extensively studied; see, in particular, papers by Anderson [1], Lin [10], [11], Lin and Stivers [12], Mehta and Gurland [13] and Wilks [16]. Assuming that X and Y are linearly related, Gupta and Rohatgi [5] proposed estimators expressed as linear combinations of fragmentary sample means. Recently Wei [15] proposed a nonparametric approach using the median of all possible differences Y_j-X_i, $i=1, \cdots, n+m$, $j=1, \cdots, n+l$ and

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$(Y_i+Y_j-X_i-X_j)/2$, $1\leqq i\leqq j\leqq n$. He compared this median estimator with the estimators suggested by Lin and Stivers [12], Gupta and Rohatgi [5] and a naive estimator $\sum_{j=1}^{n+l} Y_j/(n+l)-\sum_{i=1}^{n+m} X_i/(n+m)$. The results show that the median estimator performs quite satisfactory under bivariate normal distribution and other estimators perform very poorly under Gumbel's bivariate exponential distribution.

In this article, two large classes of nonparametric estimators of θ are studied in detail. Each class contains the median estimator as a special case. The first class of estimators is the class of linear combinations of sample quantiles derived from the empirical distributions $\hat{G}(t)=M^{-1}\left[\alpha \sum_{i=1}^{n+m}\sum_{j=1}^{n+l} I(Y_j-X_i\leqq t)+\beta \sum_{1\leqq i\leqq j\leqq n} I(Y_i+Y_j-X_i-X_j\leqq 2t)\right]$, where α and β are two nonnegative integers with $\alpha+\beta>0$, $M=\alpha(n+m)(n+l)+(\beta n(n+1)/2)$, and $I(\cdot)$ is the usual indicator function. More specifically, estimators θ^* from this class can be expressed as

$$\theta^*=\sum_{i=1}^{k} c_i\hat{\xi}_{p_i} ,$$

where c_i are nonnegative constants, $0<p_i<1$, $i=1,\cdots,k$ and $\hat{\xi}_{p_i}$ are defined as

$$\hat{\xi}_{p_i}=\inf\{t: \hat{G}(t)\geqq p_i\} .$$

Note that the estimators θ^* considered here and below depend on α and β. If we are unable to identify the pairing of X_i with Y_i, $i=1,\cdots,n$, then we simply choose $\beta=0$. In Section 2, we see that if we choose c_i and p_i properly, then θ^* will be a consistent estimator of θ. It is believed that suitable members of this class, such as $0.3\hat{\xi}_{1/3}+0.4\hat{\xi}_{1/2}+0.3\hat{\xi}_{2/3}$ and $0.25\hat{\xi}_{1/4}+0.5\hat{\xi}_{1/2}+0.25\hat{\xi}_{3/4}$ are more distributionally robust and insensitive to spurious observations. The second group of estimators considered here is the class of M-estimators. An M-estimator for θ is a solution $\hat{\theta}$ of the equation

$$\alpha \sum_{i=1}^{n+m}\sum_{j=1}^{n+l} \Psi(Y_j-X_i-c)+\beta \sum_{1\leqq i\leqq j\leqq n} \Psi\left(\frac{W_i+W_j}{2}-c\right)=0 ,$$

where $W_i=Y_i-X_i$, $i=1,\cdots,n$. The function Ψ is usually skew-symmetric about 0. Typical Ψ considered in robust estimation are the Huber [9] family $\Psi(x)=\min(k, \max(-k, x))$ and Hampel's "redescenders" [6] and [7] etc. In Section 2, sufficient conditions are given to ensure that $\hat{\theta}$ is a consistent estimator of θ with other good asymptotic properties.

In Section 3, we also study the important problem of constructing approximate confidence intervals for θ. Two approaches are examined. The first approach is based on the sample quantiles. The second method is to derive approximate confidence intervals from M-estimates; see Boos [2].

2. Nonparametric estimators of θ

Let $K(t)$ and $L(t)$ be the distribution functions of $Y_2 - X_1$ and $(W_1 + W_2)/2$, respectively. Define $N = 2n + m + l$ and $\lambda_{1N} = m/N$, $\lambda_{2N} = l/N$ and $\lambda_{3N} = n/N$. We assume that there exists a constant λ_0 such that $0 < \lambda_0 \leqq \lambda_{3N}$ for all large N. In addition, we let $\lambda_{iN} \to \lambda_i$, $i = 1, 2, 3$. Throughout this paper we assume that both K and L are symmetric with symmetry point θ. This implies that the distribution function $G(t) = \gamma K(t) + \delta L(t)$, where $\gamma = 1 - \delta = \alpha(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)/(\alpha(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3) + \beta(\lambda_3^2/2))$, is also symmetric with symmetry point θ. The definitions of the estimators θ^* and $\hat{\theta}$ are motivated by the fact that the empirical distribution $\hat{G}(t)$ is a reliable estimator of the distribution $G(t)$. Thus if we choose constants $c_i > 0$ and $0 < p_i < 1$ $(i = 1, 2, \cdots, k)$ such that $\sum_{i=1}^{k} c_i = 1$ and $\sum_{i=1}^{k} c_i \xi_{p_i} = \theta$, where $\xi_{p_i} = \inf\{t : G(t) \geqq p_i\}$, then $\theta^* = \sum_{i=1}^{k} c_i \hat{\xi}_{p_i}$ will be a consistent estimator of θ. For examples, $\{c_1 = 1, p_1 = 1/2\}$ or $\{c_1 = 0.3, c_2 = 0.4, c_3 = 0.3, p_1 = 1/3, p_2 = 1/2, p_3 = 2/3\}$ or $\{c_1 = 0.25, c_2 = 0.5, c_3 = 0.25, p_1 = 1/4, p_2 = 1/2, p_3 = 3/4\}$. On the other hand, if, in addition to some mild conditions, we let Ψ be a skew-symmetric function such that there is only one solution θ for $\int_{-\infty}^{\infty} \Psi(x - t)dG(t) = 0$, then $\hat{\theta}$ can also be shown to be a consistent estimator of θ.

In what follows, we first establish an almost sure representation for sample quantiles $\hat{\xi}_p$ (Theorem 2.1). Using this representation we are able to obtain some useful asymptotic properties for θ^* (Corollary 2.1). Asymptotic normality and other properties of the M-estimators $\hat{\theta}$ are stated in Theorems 2.2 and 2.3.

THEOREM 2.1. *Let* $0 < p < 1$ *and* $p_N = p + O(N^{-1/2}(\log N)^{-1/2})$, $N \to \infty$. *Assume that the distributions* $K(t)$ *and* $L(t)$ *are twice differentiable at* ξ_p *with* $K'(\xi_p)L'(\xi_p) > 0$, *and* F_1 *satisfies a Lipschitz condition of order* 1, *and* $\lambda_{iN} = \lambda_i + O(N^{-1/2}(\log N)^{1/2})$, $i = 1, 2, 3$, $N \to \infty$. *Then with probability* 1,

$$\hat{\xi}_{p_N} - \xi_p = \frac{p_N - \hat{G}(\xi_p)}{\gamma_N K'(\xi_p) + \delta_N L'(\xi_p)} + O(N^{-3/4}(\log N)^{3/4}), \qquad N \to \infty,$$

where

$$\gamma_N = 1 - \delta_N = \alpha(n+m)(n+l)/[\alpha(n+m)(n+l) + (\beta n(n+1)/2)] \quad (\to \gamma,\ N \to \infty).$$

PROOF. See Appendix.

Using this almost sure representation and asymptotic properties of $\hat{G}(t)$ we can easily establish the following

COROLLARY 2.1. (a) *Let* $\lambda_{jN}=\lambda_j+O(N^{-1/2}(\log N)^{1/2})$, $N\to\infty$, *for* $j=1,2,3$. *Assume that for each* $i=1,\cdots,k$, $K(t)$ *and* $L(t)$ *are both twice differentiable at* ξ_{p_i} *with* $K'(\xi_{p_i})L'(\xi_{p_i})>0$ *and* F_1 *satisfies a Lipschitz condition of order* 1. *Then with probability* 1,

$$\frac{\sqrt{N}}{\sqrt{\log\log N}}(\theta^*-\theta)=O(1)\ , \qquad N\to\infty\ .$$

(b) *If, in addition to the conditions assumed in* (a), *we let* $\lambda_{jN}=\lambda_j+O(N^{-1/2})$, $N\to\infty$, $j=1,2,3$. *Then*

$$\sqrt{N}(\theta^*-\theta)\to N(0,\sigma^2)\ , \qquad N\to\infty\ ,$$

where $0<\sigma^2<\infty$ *is the asymptotic variance of* $\sqrt{N}\theta^*$.

Remark. If $c_1=1$ and $p_1=1/2$, then the value of σ^2 is given in Wei [15]. In general, the asymptotic variances of other linear combinations of sample quantiles are quite complicated. Fortunately, in Section 3, we are able to suggest two types of approximate confidence intervals without requiring the knowledge of the value of σ^2.

We now focus our attention on the class of M-estimators. For a given function $\Psi(x)$, we put $\lambda_F(t)=\int_{-\infty}^{\infty}\Psi(x-t)dF(x)$ for any distribution F. It is clear that if Ψ is skew-symmetric about 0, then θ is a solution of $\lambda_K(t)=0$ and $\lambda_L(t)=0$, and hence also a solution of $\lambda_G(t)=0$. Let the distribution function of $W_1=Y_1-X_1$ be denoted as F_3, and we defined:

$$\Psi_1(t)=\int_{-\infty}^{\infty}\Psi(y-t)dF_2(y)\ ,$$

$$\Psi_2(t)=\int_{-\infty}^{\infty}\Psi(t-x)dF_1(x)\ ,$$

and

$$\Psi_3(t)=\int_{-\infty}^{\infty}\Psi\left(\frac{w}{2}+t\right)dF_3(w)\ .$$

Throughout this paper, we assume $\Psi_i(t)$, $i=1,2,3$, are measurable functions. Also, we define the following functions involved in the expression of the asymptotic variance of $\sqrt{N}\hat{\theta}$:

$$q_1(s)=\int_{-\infty}^{\infty}\Psi_2^2(y-s)dF_2(y)\ ,$$

$$q_2(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi_3^2\left(\frac{y-x}{2}-s\right)dH(x,y)\ ,$$

$$q_3(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi_1(x-s)\Psi_2(y-s)dH(x, y) ,$$

$$q_4(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi_1(x-s)\Psi_3\left(\frac{y-x}{2}-s\right)dH(x, y) ,$$

$$q_5(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi_2(y-s)\Psi_3\left(\frac{y-x}{2}-s\right)dH(x, y) .$$

In Theorem 2.2, two sets of sufficient conditions are assumed in order to establish asymptotic normality of the M-estimators. The method of the proof is based on that given by Huber ([9], Lemmas 4 and 5) in conjunction with the projection technique.

THEOREM 2.2. (a) *Assume the following conditions*:

(i) Ψ *is bounded, nondecreasing and skew-symmetric about* 0;

(ii) $\lambda_G(t)$ *is differentiable at* $t=\theta$ *with* $\lambda'_G(\theta)<0$;

(iii) $\lambda_{iN}=\lambda_i+O\left(\frac{1}{\sqrt{N}}\right)$, $N\to\infty$, $i=1, 2, 3$;

(iv) $q_j(s)$, $j=1,\cdots,5$, *are continuous at* $s=\theta$.

Then for any solution sequence $\hat{\theta}$ *of the equation* $\lambda_{\hat{G}}(t)=0$,

$$\sqrt{N}(\hat{\theta}-\theta)\xrightarrow{d}N(0, \sigma_1^2) , \qquad N\to\infty ,$$

where

$$\sigma_1^2=\frac{\sigma_0^2}{(\lambda'_G(\theta))^2}$$

and

$$\sigma_0^2=\gamma^2\left(\frac{1}{(\lambda_1+\lambda_3)}+\frac{1}{(\lambda_2+\lambda_3)}\right)q_1(\theta)+\frac{4\delta^2}{\lambda_3}q_2(\theta)$$
$$+\frac{2\gamma^2\lambda_3}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}q_3(\theta)+4\gamma\delta\left(\frac{q_4(\theta)}{\lambda_1+\lambda_3}+\frac{q_5(\theta)}{\lambda_2+\lambda_3}\right) .$$

(b) *Let* $\hat{\theta}$ *be a solution sequence of* $\lambda_{\hat{G}}(t)=0$ *satisfying* $\hat{\theta}\xrightarrow{p}\theta$, $N\to\infty$ *and assume the following conditions*:

(i) θ *is a root of* $\lambda_K(t)=0$ *and* $\lambda_L(t)=0$;

(ii) Ψ *has a bounded, uniformly continuous derivative* Ψ';

(iii) $\int_{-\infty}^{\infty}\frac{\partial\Psi(x-t)}{\partial t}\Big|_{t=\theta}dK(x)$ *and* $\int_{-\infty}^{\infty}\frac{\partial\Psi(x-t)}{\partial t}\Big|_{t=\theta}dL(x)$ *are finite and* $\int_{-\infty}^{\infty}\frac{\partial\Psi(x-t)}{\partial t}\Big|_{t=\theta}dK(x)+\int_{-\infty}^{\infty}\frac{\partial\Psi(x-t)}{\partial t}\Big|_{t=\theta}dL(x)\neq 0$.

Then

$$\sqrt{N}(\hat{\theta}-\theta)\xrightarrow{d}N(0, \sigma_2^2) , \qquad N\to\infty ,$$

where

$$\sigma_2^2=\sigma_0^2\Big/\left[\gamma\int_{-\infty}^{\infty}\frac{\partial\Psi(x-t)}{\partial t}\Big|_{t=\theta}dK(x)+\delta\int_{-\infty}^{\infty}\frac{\partial\Psi(x-t)}{\partial t}\Big|_{t=\theta}dL(x)\right]^2 .$$

PROOF. See Appendix.

Remarks. (1) In (a), condition (iii) can be relaxed if we assume that $\lambda_K(t)$ and $\lambda_L(t)$ are both differentiable at $t=\theta$ with $\lambda'_G(\theta)<0$.

(2) Regarding the condition imposed on $\hat{\theta}$ in (b), we note that if Ψ is a continuous and bounded function, $\lambda_G(\theta)=0$ and that at θ, $\lambda_G(t)$ changes sign only once in a neighborhood of θ. Then there is a sequence of solution $\hat{\theta}$ such that with probability 1,

$$\hat{\theta}\to\theta\,,\qquad N\to\infty\,;$$

see Boos and Serfling ([3], Theorem 2.1).

(3) In condition (ii) of (b) we assume that Ψ' is bounded. Actually this condition can be replaced by only assuming that

$$\mathrm{E}\left[\int_{-\infty}^{\infty}\frac{\partial\Psi(Y_1-x-t)}{\partial t}\Big|_{t=\theta}dF_1(x)\right]^2<\infty\,,$$

$$\mathrm{E}\left[\int_{-\infty}^{\infty}\frac{\partial\Psi(y-X_1-t)}{\partial t}\Big|_{t=\theta}dF_2(y)\right]^2<\infty\,,$$

and

$$\mathrm{E}\left[\int_{-\infty}^{\infty}\frac{\partial\Psi((W_1+w)/2-t)}{\partial t}\Big|_{t=\theta}dF_3(w)\right]^2<\infty\,.$$

In view of the proof for case (b) of Theorem 2.2 (see Appendix), we easily see that if, with probability 1, $\hat{\theta}\to\theta$, $N\to\infty$, then under regularity conditions similar to (a) of Corollary 2.1, we can obtain a stronger result:

$$\frac{\sqrt{N}}{\sqrt{\log\log N}}(\hat{\theta}-\theta)=O(1)\,,\qquad N\to\infty\,,$$

with probability 1. The same result is also obtainable for case (a) by using the following almost sure representation, similar to that given in Theorem 2.1, for *M*-estimators. This representation is not only useful for the purpose of establishing the LIL type result for $\hat{\theta}$ but also valuable for constructing confidence intervals for θ. A weaker version of a similar representation was derived by Boos [2].

Let $\Psi(t)$ be nondecreasing, left continuous, and strictly positive (negative) for large positive (negative) values of t, and define, for any distribution F,

$$\gamma_F(d) = -\int_{-\infty}^{\infty} \Psi(x-d)dF(x) = -\lambda_F(d)$$

and

$$\gamma_F^{-1}(t) = \inf\,[d : \gamma_F(d) \geqq t]\ , \qquad t \in (\inf_{x\in R} \gamma_F(x), \sup_{x\in R} \gamma_F(x))\ .$$

$\gamma_F(d)$ is nondecreasing and right continuous and thus $\gamma_F^{-1}(d) \leqq s$ if and only if $d \leqq \gamma_F(s)$. Furthermore, if θ is the only root of $\lambda_F(t)=0$, then $\gamma_F^{-1}(0)=\theta$.

THEOREM 2.3. *Assume the following conditions:*

(i) *$\Psi(t)$ is bounded, nondecreasing and satisfies a Lipschitz condition of order* 1;

(ii) *$\gamma_G(d)$ is twice differentiable at $d=\theta$ with $\gamma'_G(\theta)>0$;*

(iii) *θ is the root of $\lambda_G(t)=0$;*

(iv) $\lambda_{iN} = \lambda_i + O(N^{-1/2}(\log N)^{1/2})$, $i=1, 2, 3$.

Then for $p_N = O(N^{-1/2}(\log N)^{1/2})$, $N\to\infty$, we have, with probability 1,

$$\gamma_{\hat{G}}^{-1}(p_N) - \theta = \frac{p_N - \gamma_{\hat{G}}(\theta)}{\gamma'_G(\theta)} + O(\gamma_N - \gamma) + O(\delta_N - \delta) + O(N^{-3/4}(\log N)^{3/4})\ , \qquad N\to\infty\ . \tag{2.1}$$

PROOF. A proof is sketched in the Appendix.

Remarks. (1) It should be noted that if we replace (ii) by assuming that $\gamma_K(d)$ and $\gamma_L(d)$ are both twice differentiable at $d=\theta$ with $\gamma_G(\theta)>0$, then with probability 1,

$$\gamma_{\hat{G}}^{-1}(p_N) - \theta = \frac{p_N - \gamma_{\hat{G}}(\theta)}{\gamma'_{G_N}(\theta)} + O(N^{-3/4}(\log N)^{3/4})\ , \qquad N\to\infty\ , \tag{2.2}$$

where

$$G_N(t) = \gamma_N K(t) + \delta_N L(t)\ .$$

(2) Fix $p_N=0$, then $\hat{\theta} = \gamma_{\hat{G}}^{-1}(\theta)$ is a solution of the equation $\int_{-\infty}^{\infty} \Psi(x-t)d\hat{G}(x)=0$, since Ψ is a continuous function. Using (2.2) or (2.3) we immediately have, with probability 1,

$$\frac{\sqrt{N}}{\sqrt{\log\log N}}(\hat{\theta}-\theta) = O(1)\ , \qquad N\to\infty\ ,$$

if some regularity conditions similar to (a) of Corollary 2.1 are satisfied.

(3) The members in Huber family $\Psi(x) = \min(k, \max(-k, x))$ satisfy condition (i) of Theorem 2.3.

3. Confidence intervals for the difference of locations

Here we consider two methods of determining approximate confidence intervals for θ using Theorems 2.1 and 2.3, respectively.

According to Corollary 2.1, for the median estimator $\theta^*=\hat{\xi}_{1/2}$ we have $\sqrt{N}(\theta^*-\theta)\xrightarrow{d}N(0, \sigma^2)$, $N\to\infty$. If σ^2 is known and let Z_α denote the $100(1-\alpha)$ percentile of the standard normal distribution, then the confidence interval $I_N=(\theta^*-Z_\alpha(\sigma/\sqrt{N}), \theta^*+Z_\alpha(\sigma/\sqrt{N}))$ has confidence coefficient converging to $1-2\alpha$ as $N\to\infty$. Unfortunately, in most cases, σ^2 is not known and therefore the above approach is useless.

To propose a useful confidence interval for θ, we need to estimate the asymptotic variance of $\sqrt{N}\theta^*$.

From (2.1), it is not difficult to see that

$$\sqrt{N}(\theta^*-\theta)/\sigma_N\xrightarrow{d}N(0, 1)\ , \qquad N\to\infty\ ,$$

where

$$\sigma_N^2=\boldsymbol{\sigma}_N^2/(\gamma_N K'(\theta)+\delta_N L'(\theta))^2\ ,$$

where

$$\boldsymbol{\sigma}^2=\left[\frac{1}{12}\left(\frac{\gamma_N^2}{\lambda_{1N}+\lambda_{3N}}+\frac{\gamma_N^2}{\lambda_{3N}+\lambda_{2N}}+\frac{4\delta_N^2}{\lambda_{3N}}\right)+\frac{\gamma_N^2\lambda_{3N}}{(\lambda_{3N}+\lambda_{1N})(\lambda_{3N}+\lambda_{2N})}\left(\frac{1}{2}-2\theta_1\right)\right.$$
$$\left.+\frac{4\delta_N\gamma_N}{\lambda_{3N}+\lambda_{1N}}\left(\theta_2-\frac{1}{4}\right)+\frac{4\delta_N\gamma_N}{\lambda_{3N}+\lambda_{2N}}\left(\frac{1}{4}-\theta_3\right)\right]\ ,$$

$$\theta_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(x)F_1(y-\theta)dH(x, y)\ ,$$

$$\theta_2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(x)F_3(2\theta-(y-x))dH(x, y)\ ,$$

and

$$\theta_3=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(y-\theta)F_3(2\theta-(y-x))dH(x, y)\ .$$

(Note that if $H(x, y)=H(y, x)$ for all $(x, y)\in R^2$ then $\theta_2=1/2-\theta_3$.) Also, in view of the asymptotic variance σ^2 of $\sqrt{N}\theta^*$ derived by Wei [15], we have $\sigma_N^2\to\sigma^2$, $N\to\infty$.

Using our method it is enough to estimate simple parameters θ_i. Let us first define

$$\hat{F}_1(t)=(n+m)^{-1}\sum_{i=1}^{n+m}I(X_j\leqq t) \qquad \text{and} \qquad \hat{F}_3(t)=n^{-1}\sum_{i=1}^{n}I(Y_i-X_i\leqq t)\ .$$

Then natural estimators of θ_i are $\hat{\theta}_i$:

$$\hat{\theta}_1 = n^{-1} \sum_{i=1}^{n} \hat{F}_1(X_i) \cdot \hat{F}_1(Y_i - \theta^*) ,$$

$$\hat{\theta}_2 = n^{-1} \sum_{i=1}^{n} \hat{F}_1(X_i) \cdot \hat{F}_3(2\theta^* - Y_i + X_i)$$

and

$$\hat{\theta}_3 = n^{-1} \sum_{i=1}^{n} \hat{F}_1(Y_i - \theta^*) \cdot \hat{F}_3(2\theta^* - Y_i + X_i) .$$

Their asymptotic properties are described in

COROLLARY 3.1. *If F_1 and F_3 satisfy a Lipschitz condition of order* 1 *and there is a sequence of positive integers $\alpha_N \to \infty$, $N \to \infty$, such that with probability* 1, $\alpha_N(\theta^* - \theta) \to 0$, $N \to \infty$. *Then with probability* 1,

$$\beta_N(\hat{\theta}_i - \theta) \to 0 , \qquad N \to \infty, \quad i = 1, 2, 3 ,$$

where

$$\beta_N = \min(\alpha_N, N^{1/2}(\log N)^{-1/2}) .$$

Define $p_{1N} = -Z_\alpha \boldsymbol{\sigma}_N / \sqrt{N}$ and $p_{2N} = Z_\alpha \boldsymbol{\sigma}_N / \sqrt{N}$, where $\boldsymbol{\sigma}_N^2$ is the strong consistent estimator of $\boldsymbol{\sigma}^2$ obtained by replacing θ_i by $\hat{\theta}_i$ in the definition of $\boldsymbol{\sigma}^2$. Then the suggested distribution-free interval is $I_N^* = (\hat{\xi}_{p_{1N}}, \hat{\xi}_{p_{2N}})$. The advantage of this approach is that we don't need to estimate $K'(\theta)$ and $L'(\theta)$. Furthermore, applying Corollary 3.1 and Theorem 2.1, it can be shown that the confidence coefficient of $I_N^* \to 1 - 2\alpha$ as $N \to \infty$. Also,

$$\sqrt{N} \cdot (\text{length of } I_N^*) \xrightarrow{P} 2\sigma Z_\alpha , \qquad N \to \infty .$$

Next we turn our attention to the second approach using the result of Theorem 2.1. From (2.3) we see that under some suitable conditions,

$$\frac{\sqrt{N}}{\sigma_N'} (\gamma_{\hat{G}}^{-1}(p_N) - \theta) \xrightarrow{d} N(0, 1) , \qquad N \to \infty ,$$

where

$$(\sigma_N')^2 = \frac{\sigma_0^2}{\lambda'_{G_N}(\theta)} ,$$

$$\sigma_0^2 = \left[\gamma_N^2 \left(\frac{1}{\lambda_{1N} + \lambda_{3N}} + \frac{1}{\lambda_{2N} + \lambda_{3N}} \right) q_1(\theta) + \frac{4\delta_N^2}{\lambda_{3N}} q_2(\theta) \right.$$
$$\left. + \frac{2\gamma_N^2 \lambda_{3N} q_3(\theta)}{(\lambda_{1N} + \lambda_{3N})(\lambda_{2N} + \lambda_{3N})} + 4\gamma_N \delta_N \left(\frac{q_4(\theta)}{\lambda_{1N} + \lambda_{3N}} + \frac{q_5(\theta)}{\lambda_{2N} + \lambda_{3N}} \right) \right] .$$

Hence the interval $(\gamma_{\hat{G}}^{-1}(p_N) - Z_\alpha(\sigma_N'/\sqrt{N}), \gamma_{\hat{G}}^{-1}(p_N) + Z_\alpha(\sigma_N'/\sqrt{N}))$ has confidence coefficient converging to $1-2\alpha$ as $N\to\infty$. Again, σ_0 and $\lambda'_{G_N}(\theta)$ are in general not known and thus, as above, we have to find a way to estimate $q_i(\theta)$. (It is not necessary to estimate $\lambda'_{G_N}(\theta)$.)

To estimate $q_i(\theta)$, let us first consider estimation of $\Psi_i(t)$, $i=1, 2, 3$. The estimator of $\Psi_1(t)$ considered here is $\hat{\Psi}_1(t)=(n+l)^{-1}\sum_{j=1}^{n+l}\Psi(Y_j-t)$ and the estimator of $\Psi_2(t)$ is $\hat{\Psi}_2(t)=(n+m)^{-1}\sum_{i=1}^{n+m}\Psi(t-X_i)$ and the estimator of Ψ_3 is $\hat{\Psi}_3(t)=n^{-1}\sum_{i=1}^{n}\Psi\left(\frac{Y_i-X_i}{2}+t\right)$. The asymptotic properties of these estimators are stated in

COROLLARY 3.2. *Let Ψ be continuous and of finite variation $\|\Psi\|_V$. Then with probability* 1,

$$N^{1/2}(\log N)^{-1/2}\sup_{t\in R}|\hat{\Psi}_i(t)-\Psi_i(t)|\to 0\ , \qquad N\to\infty, \quad i=1, 2, 3\ .$$

The results of Corollary 3.2 lead us to consider the following type of estimators for $q_i(\theta)$:

$$\hat{q}_1(\theta)=(n+l)^{-1}\sum_{i=1}^{n+l}\hat{\Psi}_2^2(Y_i-\hat{\theta})\ ,$$

$$\hat{q}_2(\theta)=n^{-1}\sum_{i=1}^{n}\hat{\Psi}_3^2\left(\frac{Y_i-X_i}{2}-\hat{\theta}\right)\ ,$$

$$\hat{q}_3(\theta)=n^{-1}\sum_{i=1}^{n}\hat{\Psi}_1(X_i-\hat{\theta})\hat{\Psi}_2(Y_i-\hat{\theta})\ ,$$

$$\hat{q}_4(\theta)=n^{-1}\sum_{i=1}^{n}\hat{\Psi}_1(X_i-\hat{\theta})\hat{\Psi}_3\left(\frac{Y_i-X_i}{2}-\hat{\theta}\right)\ ,$$

$$\hat{q}_5(\theta)=n^{-1}\sum_{i=1}^{n}\hat{\Psi}_2(Y_i-\hat{\theta})\hat{\Psi}_3\left(\frac{Y_i-X_i}{2}-\hat{\theta}\right)\ ,$$

where $\hat{\theta}=\gamma_{\hat{G}}^{-1}(0)$. Their asymptotic properties are summarized in

COROLLARY 3.3. *Let the conditions of Corollary* 3.2 *be satisfied and Ψ be Lipschitz of order* 1. *If there is a sequence of positive real numbers $\alpha'_N\to\infty$, $N\to\infty$ such that with probability* 1, *$\alpha'_N(\hat{\theta}-\theta)\to 0$, $N\to\infty$, then with probability* 1,

$$\beta'_N|\hat{q}_i(\theta)-q_i(\theta)|\to 0\ , \qquad N\to\infty, \quad i=1,\cdots,5\ ,$$

where $\beta'_N=\min(\alpha'_N, N^{1/2}(\log N)^{-1/2})$.

In view of the properties of $\hat{q}_i(\theta)$ described in Corollary 3.3, a natural estimator $\hat{\sigma}_0^2$ of σ_0^2 can be obtained by replacing $q_i(\theta)$ by $\hat{q}_i(\theta)$, $i=1, \cdots, 5$, in the definition of σ_0^2. If we do so and if the conditions of

Corollary 3.3 are satisfied then with probability 1,

$$\beta'_N|\hat{\sigma}_0^2-\sigma_0^2|\to 0, \qquad N\to\infty .$$

Based on the result, we now propose a second type of confidence intervals for θ. We define two quantities $q_{1N}=-Z_\alpha(\hat{\sigma}_0/\sqrt{N})$ and $q_{2N}=+Z_\alpha(\hat{\sigma}_0/\sqrt{N})$. Then the distribution-free confidence interval $\hat{I}_N=(\gamma_{\hat{G}}^{-1}(q_{1N}), \gamma_{\hat{G}}^{-1}(q_{2N}))$ has some desirable properties. That is, the confidence coefficient of $\hat{I}_N\to 1-2\alpha$, $N\to\infty$ and $\sqrt{N}$(length of $\hat{I}_N)\to 2Z_\alpha\sigma_1$, $N\to\infty$, where σ_1 is defined in part (a) of Theorem 2.2.

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Appendix

The proof of Theorem 2.1 consists of the following two lemmas. In each of these lemmas, suitable conditions of the theorem are in action.

LEMMA A1. *Let* $0<p<1$ *and* $p_N=p+O(N^{-1/2}(\log N)^{1/2})$, $N\to\infty$. *If* $K'(\xi_p)L'(\xi_p)>0$ *and* $\lambda_{iN}=\lambda_i+O(N^{-1/2}(\log N)^{1/2})$, $N\to\infty$, $i=1, 2, 3$, *then with probability* 1,

$$\hat{\xi}_{p_N}=\xi_p+O(N^{-1/2}(\log N)^{1/2}) , \qquad N\to\infty .$$

PROOF. Let $\varepsilon_N=c_0N^{-1/2}(\log N)^{1/2}$, where c_0 is some positive constant whose value will be specified later. Define

$$\hat{K}(t)=[(n+m)(n+l)]^{-1}\sum_{i=1}^{n+m}\sum_{j=1}^{n+l} I(Y_j-X_i\leqq t) ,$$

$$\hat{L}(t)=[n(n+1)/2]^{-1}\sum_{1\leqq i\leqq j\leqq n} I(W_i+W_j\leqq 2t) ,$$

$$\hat{T}_1(t)=(n+m)^{-1}\sum_{i=1}^{n+m}[F_2(X_i+t)-K(t)] ,$$

$$\hat{T}_2(t)=(n+l)^{-1}\sum_{j=1}^{n+l}[\bar{F}_1(Y_j-t)-K(t)] ,$$

where $\bar{F}_1(t)=1-F_1(t)$. Write

$$\hat{S}(t)=[\hat{K}(t)-K(t)]-\hat{T}_1(t)-\hat{T}_2(t)=[(n+m)(n+l)]^{-1}\sum_{i=1}^{n+m}\sum_{j=1}^{n+l} g(X_i, Y_j, t) ,$$

where

$$g(X_i, Y_j, t)=[I(Y_j-X_i\leqq t)-K(t)]-[F_2(X_i+t)-K(t)]-[\bar{F}_1(Y_j-t)-K(t)] ,$$

and realize that for any positive integer d and any sequence of constants $t_N\in R$, $\mathrm{E}\prod_{k=1}^{d} g(X_{i_k}, Y_{j_k}, t_N)=0$, if there is a subscript which appears only once in $\{i_1, j_1, \cdots, i_d, j_d\}$. Consequently,

$$\text{(A1)} \qquad \mathrm{E}\,[\hat{S}(t_N)]^d=O(N^{-d}) , \qquad N\to\infty ,$$

for any sequence of real numbers t_N.

Now we consider

$$\begin{aligned}\text{(A2)} \qquad & \mathrm{P}\,(\hat{\xi}_{p_N}>\xi_p+\varepsilon_N)\\ &\leqq \mathrm{P}\left(-\gamma_N[\hat{K}(\xi_p+\varepsilon_N)-K(\xi_p+\varepsilon_N)]>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{2}\right)\\ &\quad+\mathrm{P}\left(-\delta_N[\hat{L}(\xi_p+\varepsilon_N)-L(\xi_p+\varepsilon_N)]>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{2}\right)\\ &=U_{1N}+U_{2N} , \qquad \text{say} ,\end{aligned}$$

where

$$G_N(t)=\gamma_N K(t)+\delta_N L(t) .$$

It is noted that if $\alpha=0$ or $\beta=0$, then $U_{1N}=0$ or $U_{2N}=0$ for large N. So, in the following we assume $\alpha\neq 0$ and $\beta\neq 0$. Now regarding the first term on the right-hand side of (A2), we have

$$\begin{aligned} U_{1N}\leqq & \mathrm{P}\left(-\gamma_N\hat{S}(\xi_p+\varepsilon_N)>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{6}\right) \\ & +\mathrm{P}\left(-\gamma_N\hat{T}_1(\xi_p+\varepsilon_N)>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{6}\right) \\ & +\mathrm{P}\left(-\gamma_N\hat{T}_2(\xi_p+\varepsilon_N)>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{6}\right) . \end{aligned}$$

Furthermore, according to (A1),

$$\text{(A3)} \quad \mathrm{P}\left(-\gamma_N\hat{S}(\xi_p+\varepsilon_N)>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{6}\right)=O(N^{-2}) , \qquad N\to\infty ,$$

since for large N,

$$G_N(\xi_p+\varepsilon_N)-p_N\geqq\frac{1}{2}G'(\xi_p)\varepsilon_N .$$

Also, using Hoeffding's inequality (Hoeffding [8]), the standard argument establishes that

$$\begin{aligned} \text{(A4)} \quad & \mathrm{P}\left(-\gamma_N\hat{T}_1(\xi_p+\varepsilon_N)>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{6}\right) \\ & +\mathrm{P}\left(-\gamma_N\hat{T}_2(\xi_p+\varepsilon_N)>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{6}\right)=O(N^{-2}) , \qquad N\to\infty . \end{aligned}$$

We now treat the second term on the right-hand side of (A2). Define

$$L^*(t)=\binom{n}{2}^{-1}\sum_{1\leqq i<j\leqq n} I(W_i+W_j\leqq 2t) ,$$

then there exists a constant $c>0$ such that for large N,

$$\begin{aligned} \text{(A5)} \quad & \mathrm{P}\left(-\delta_N\{\hat{L}(\xi_p+\varepsilon_N)-L(\xi_p+\varepsilon_N)\}>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{2}\right) \\ & \leqq\mathrm{P}\left(-L^*(\xi_p+\varepsilon_N)+L(\xi_p+\varepsilon_N)>\frac{G_N(\xi_p+\varepsilon_N)-p_N}{2\delta_N}-cN^{-1}\right) . \end{aligned}$$

Applying Hoeffding's inequality for U-statistics (Hoeffding [8]) we can show that the probability on the right-hand side of (A5) is also bounded by $O(N^{-2})$ if c is properly chosen.

(A2), (A3), (A4) and (A5) imply that

$$P(\hat{\xi}_{p_N} > \xi_p + \varepsilon_N) = O(N^{-2}), \qquad N \to \infty .$$

A similar argument also shows that

$$P(\hat{\xi}_{p_N} < \xi_p - \varepsilon_N) = O(N^{-2}), \qquad N \to \infty .$$

Thus the assertion of the lemma follows immediately by applying the Borel-Cantelli lemma.

LEMMA A2. *Let* $0<p<1$. *If both* $K'(t)$ *and* $L'(t)$ *are bounded in a neighborhood of* ξ_p *and* F_1 *satisfies a Lipschitz condition of order* 1, *then with probability* 1,

$$\sup_{|t| \leqq cN^{-1/2}(\log N)^{1/2}} |[\hat{G}(t+\xi_p) - \hat{G}(\xi_p)] - [G_N(t+\xi_p) - G_N(\xi_p)]| = O(N^{-3/4}(\log N)^{3/4}), \qquad N \to \infty ,$$

where c *is a positive constant.*

PROOF. Write

$$\begin{aligned} A_N &= \sup_{|t| \leqq cN^{-1/2}(\log N)^{1/2}} |[\hat{G}(t+\xi_p) - \hat{G}(\xi_p)] - [G_N(t+\xi_p) - G_N(\xi_p)]| \\ &\leqq \gamma_N \sup_{|t| \leqq cN^{-1/2}(\log N)^{1/2}} |[\hat{K}(t+\xi_p) - \hat{K}(\xi_p)] - [K(t+\xi_p) - K(\xi_p)]| \\ &\quad + \delta_N \sup_{|t| \leqq cN^{-1/2}(\log N)^{1/2}} |[\hat{L}(t+\xi_p) - \hat{L}(\xi_p)] - [L(t+\xi_p) - L(\xi_p)]| . \end{aligned}$$

Define

$$B_{1N}(t) = [\hat{K}(t+\xi_p) - \hat{K}(\xi_p)] - [K(t+\xi_p) - K(\xi_p)] ,$$
$$B_{2N}(t) = [\hat{L}(t+\xi_p) - \hat{L}(\xi_p)] - [L(t+\xi_p) - L(\xi_p)] .$$

Choose d_N as a sequence of positive integers such that $d_N \sim cN^{1/4}(\log N)^{1/2}$ and put $\eta_{r,N} = r(c_N/d_N)$, where $c_N = cN^{-1/2}(\log N)^{1/2}$ and $-d_N \leqq r \leqq d_N$. Define $Q_{r,N} = [\eta_{r,N}, \eta_{r+1,N}]$, then for all $t \in Q_{r,N}$,

$$B_{2N}(t) \leqq B_{2N}(\eta_{r+1,N}) + \alpha'_{r,N} \qquad \text{and} \qquad B_{1N}(t) \leqq B_{1N}(\eta_{r+1,N}) + \alpha_{r,N} ,$$

where $\alpha_{r,N} = K(\eta_{r+1,N}) - K(\eta_{r,N})$ and $\alpha'_{r,N} = L(\eta_{r+1,N}) - L(\eta_{r,N})$. Similarly,

$$B_{1N}(t) \geqq B_{1N}(\eta_{r,N}) - \alpha_{r,N} \qquad \text{and} \qquad B_{2N}(t) \geqq B_{2N}(\eta_{r,N}) - \alpha'_{r,N} \quad \text{for all} \quad t \in Q_{r,N} .$$

Thus

$$A_N \leqq \gamma_N \max \{|B_{1N}(\eta_{r,N})| : -d_N \leqq r \leqq d_N\}$$

$$+\delta_N \max\{|B_{2N}(\eta_{r,N})|: -d_N \leqq r \leqq d_N\}$$
$$+\gamma_N \max\{\alpha_{r,N}: -d_N \leqq r \leqq d_N - 1\}$$
$$+\delta_N \max\{\alpha'_{r,N}: -d_N \leqq r \leqq d_N - 1\} .$$

Now, according to the conditions of Lemma A2, it is easy to see that

$$\gamma_N \max[\alpha_{r,N}: -d_N \leqq r \leqq d_N - 1] + \delta_N \max[\alpha'_{r,N}: -d_N \leqq r \leqq d_N - 1]$$
$$= O(N^{-3/4}) , \qquad N \to \infty .$$

Define $a_N = N^{-3/4}(\log N)^{3/4}$, then using the condition that $F_1 \in \mathrm{Lip}\,(1)$ and Bernstein's inequality and (A1), for all $r = -d_N, \cdots, d_N$, we have

$$\mathrm{P}(|B_{1N}(\eta_{r,N})| > a_N)$$
$$\leqq \mathrm{P}(|\hat{S}(\eta_{r,N})| > a_N/6) + \mathrm{P}(|\hat{S}(\xi_p)| > a_N/6) + \mathrm{P}(|\hat{T}_1(\eta_{r,N}) - \hat{T}_1(\xi_p)| > a_N/3)$$
$$+ \mathrm{P}(|\hat{T}_2(\eta_{r,N}) - \hat{T}_2(\xi_p)| > a_N/3) = O(N^{-2}) , \qquad N \to \infty .$$

This implies that with probability 1,

$$\gamma_N \max[|B_{1N}(\eta_{r,N})|: -d_N \leqq r \leqq d_N] = O(N^{-3/4}(\log N)^{3/4}) , \qquad N \to \infty .$$

On the other hand,

$$\delta_N \max[|B_{2N}(\eta_{r,N})|: -d_N \leqq r \leqq d_N]$$
$$= \delta_N \max[|B'_{2N}(\eta_{r,N})|: -d_N \leqq r \leqq d_N] + O(N^{-1}) ,$$

where

$$B'_{2N}(t) = [L^*(t+\xi_p) - L^*(\xi_p)] - [L(t+\xi_p) - L(\xi_p)]$$

and $L^*(t)$ is defined in Lemma A1. Also, for a suitable choice of $c_1 > 0$ and for all $r = -d_N, \cdots, d_N$,

$$\mathrm{P}(|B'_{2N}(\eta_{r,N})| > c_1 N^{-3/4}(\log N)^{3/4}) = O(N^{-2}) , \qquad N \to \infty ,$$

by applying a method similar to that used by Geertsema ([4], Lemma 4.2). As a consequence, with probability 1,

$$\delta_N \max[|B_{2N}(\eta_{r,N})|: -d_N \leqq r \leqq d_N] = O(N^{-3/4}(\log N)^{3/4}) , \qquad N \to \infty .$$

This and the above results establish the proof of the lemma.

PROOF OF THEOREM 2.1. First, since both K and L are twice differentiable at ξ_p, thus with probability 1,

$$G_N(\hat{\xi}_{p_N}) - G_N(\xi_p) = [\gamma_N K'(\xi_p) + \delta_N L'(\xi_p)]$$
$$\cdot (\hat{\xi}_{p_N} - \xi_p) + O(N^{-1}(\log N)) , \qquad N \to \infty ,$$

by Lemma A1. Applying Lemma A2, we readily have, with probability 1,

$$\hat{G}(\hat{\xi}_{p_N})-\hat{G}(\hat{\xi}_p)=[\gamma_N K'(\xi_p)+\delta_N L'(\xi_p)] \cdot (\hat{\xi}_{p_N}-\xi_p)+O(N^{-3/4}(\log N)^{3/4}) , \qquad N\to\infty ,$$

and consequently,

$$\hat{\xi}_{p_N}-\xi_p=\frac{\hat{G}(\hat{\xi}_{p_N})-G(\xi_p)}{\gamma_N K'(\xi_p)+\delta_N L'(\xi_p)}+O(N^{-3/4}(\log N)^{3/4}) , \qquad N\to\infty .$$

This completes the proof.

PROOF OF THEOREM 2.2. Since Ψ is nondecreasing, therefore

$$P(\lambda_{\hat{G}}(t)<0)\leqq P(\hat{\theta}\leqq t)\leqq P(\lambda_{\hat{G}}(t)\leqq 0) .$$

Define $t_{z,N}=\theta+z\sigma_1 N^{-1/2}$ and let Φ denote the standard normal distribution, then it is enough to show that

$$\lim_{N\to\infty} P(\lambda_{\hat{G}}(t_{z,N})\leqq 0)=\Phi(z) , \qquad \text{for each} \quad z\in R .$$

Recall that $M=\alpha(n+m)(n+l)+\beta n(n+1)/2$ and write

$$P(\lambda_{\hat{G}}(t_{z,N})\leqq 0)=P\Big(M^{-1}\sigma_0^{-1}N^{1/2}\Big(\alpha\sum_{i=1}^{n+m}\sum_{j=1}^{n+l}[\Psi(Y_j-X_i-t_{z,N})-\lambda_K(t_{z,N})] +\beta\sum_{1\leqq i\leqq j\leqq n}\Big[\Psi\Big(\frac{W_i+W_j}{2}-t_{z,N}\Big)-\lambda_L(t_{z,N})\Big]\Big)\leqq z\Big)\to\Phi(z) ,$$

$$N\to\infty, \quad \text{for all} \quad z\in R .$$

Define

$$\hat{V}=\gamma_N(n+m)^{-1}\sum_{i=1}^{n+m}[\Psi_1(X_i+t_{z,N})-\lambda_K(t_{z,N})] +(n+l)^{-1}\sum_{j=1}^{n+l}[\Psi_2(Y_j-t_{z,N})-\lambda_K(t_{z,N})] -2\delta_N\sum_{i=1}^{n}[\Psi_3((W_i/2)-t_{z,N})-\lambda_L(t_{z,N})] .$$

Then it can be shown that

$$N^{1/2}\{[\lambda_{\hat{G}}(t_{z,N})-(\gamma_N\lambda_K(t_{z,N})+\delta_N\lambda_L(t_{z,N}))]-\hat{V}\}\xrightarrow{p}0 , \qquad N\to\infty ,$$

by utilizing the condition that Ψ is a bounded function and results similar to (A1) and the properties of the projection of a U-statistic. Finally applying the Lindeberg-Feller theorem for double arrays of random variables, the first assertion follows easily.

To establish a proof for the second assertion, we first note that

$$\text{(A6)} \qquad \lambda_{\hat{G}}(\hat{\theta})-\lambda_{\hat{G}}(\theta)=(\hat{\theta}-\theta)M^{-1}\Big[\alpha\sum_{i=1}^{n+m}\sum_{j=1}^{n+l}\frac{\partial\Psi(Y_j-X_i-t)}{\partial t}\Big|_{t=\bar{\theta}}$$

$$+\beta \sum_{1\leq i\leq j\leq n} \frac{\partial \Psi((W_i+W_j)/2-t)}{\partial t}\bigg|_{t=\tilde{\theta}}\bigg]$$

where $|\tilde{\theta}-\theta|\leq|\hat{\theta}-\theta|$. Hence we have

$$N^{1/2}(\hat{\theta}-\theta)$$

$$=\frac{-N^{1/2}\lambda_{\hat{G}}(\theta)}{M^{-1}\left[\alpha \sum_{i=1}^{n+m}\sum_{j=1}^{n+l} \frac{\partial \Psi(Y_j-X_i-t)}{\partial t}\Big|_{t=\tilde{\theta}}+\beta \sum_{1\leq i\leq j\leq n} \frac{\partial \Psi((W_i+W_j)/2-t)}{\partial t}\Big|_{t=\tilde{\theta}}\right]} .$$

From our previous argument, we have seen that

$$-N^{1/2}\lambda_{\hat{G}}(\theta)\xrightarrow{d} N(0, \sigma_0^2) .$$

Also, using the projection technique in connection with the property of $\hat{\theta}$,

$$M^{-1}\left[\alpha \sum_{i=1}^{n+m}\sum_{j=1}^{n+l} \frac{\partial \Psi(Y_j-X_i-t)}{\partial t}\Big|_{t=\tilde{\theta}}+\beta \sum_{1\leq i\leq j\leq n} \frac{\partial \Psi((W_i+W_j)/2-t)}{\partial t}\Big|_{t=\tilde{\theta}}\right]$$
$$\xrightarrow{p} \gamma \int_{-\infty}^{\infty} \frac{\partial \Psi(x-t)}{\partial t}\Big|_{t=\theta} dK(x)+\delta \int_{-\infty}^{\infty} \frac{\partial \Psi(x-t)}{\partial t}\Big|_{t=\theta} dL(x) .$$

This finishes the proof.

PROOF OF THEOREM 2.3. The proof is similar to that given in the proof of Theorem 2.1. The key steps of the proof are to show

(A7) $$\gamma_{\hat{G}}^{-1}(p_N)-\theta=O(N^{-1/2}(\log N)^{1/2}) , \qquad N\to\infty ,$$

with probability 1 and for constant $c>0$,

(A8) $$\sup_{|t|\leq cN^{-1/2}(\log N)^{1/2}} |(\gamma_{\hat{G}}(t+\theta)-\gamma_{\hat{G}}(\theta))-(\gamma_G(t+\theta)-\gamma_G(\theta))|$$
$$=O(\gamma_N-\gamma)+O(\delta_N-\delta)+O(N^{-3/4}(\log N)^{3/4}) , \qquad N\to\infty ,$$

with probability 1. Using (A7), (A8), conditions on $\gamma_G(d)$ and the continuity of Ψ, the assertion of the theorem follows.