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Simple compact topological lattices

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Introduction

The purpose of this paper is two-fold. First, we will construct a collection $\mathcal{S}$ of infinite, compact, zero dimensional topological lattices which are simple in the category $\mathcal{CTL}$ of compact topological lattices. Second, we will construct a collection T of infinite, compact, zero dimensional topological lattices which are simple in $\mathcal{L}$, the category of all lattices.

We are able to construct a $\mathcal{CTL}$-simple lattice which is modular. This is noteworthy because Numakura proved in [3] that a compact, distributive, zero dimensional topological lattice has enough $\mathcal{CTL}$-morphisms onto the two point discrete space $\mathbf{2}=\{0, 1\}$ to separate points, thereby providing a representation for such lattices. However, since our example has only the two trivial images in $\mathcal{CTL}$, we see that Numakura's result cannot be substantially improved.

Without the assumption of zero dimensionality, a similar result was obtained by J. D. Lawson in [4], where he gave an example of a distributive connected topological lattice which admits only trivial homomorphisms into the unit interval.

Preliminaries

If a lattice $(L, \wedge, \vee)$ is endowed with a Hausdorff topology such that the meet is a continuous function from $L\times L$ into L, then we say L is a *topological* $\wedge$*-semilattice*. If the join is also continuous, then L is a *topological lattice*. Moreover, if the topology on L is compact, then L is said to be a *compact topological* ($\wedge$-semi) *lattice*. If the topological ($\wedge$-semi) lattice L is such that every point of L has a neighborhood base system of open-closed sets, then L is a *zero dimensional topological* ($\wedge$-semi) *lattice*.

Let L and M be objects in $\mathcal{CTL}$; a $\mathcal{CTL}$-morphism $\varphi: L \to M$ is a continuous map which preserves arbitrary joins and arbitrary meets (cf., [2]).

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Note that for finite topologically descrete lattices, $\mathscr{CTL}$-simple and $\mathscr{L}$-simple are equivalent so we denote such lattices *simple.*[1]

We should first note that there are numerous examples of $\mathscr{L}$-simple infinite lattices in the literature. In particular, we have the following two examples:

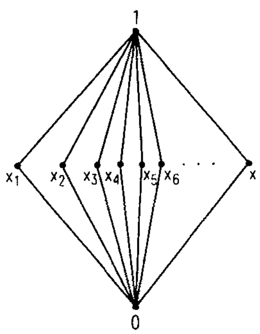


Fig. 1

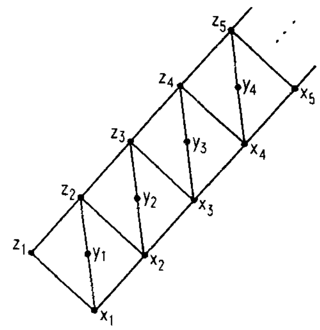


Fig. 2

However, it is not difficult to show that the lattice L_1 of Figure 1 cannot be made into a topological lattice, and although the lattice L_2 of Figure 2 is a topological lattice, it is not compact. Moreover, the obvious one point compactification of L_2 is of no help since what we end up with is not a topological lattice.

We begin by constructing a collection of $\mathscr{L}$-simple lattices. Let $\{(S_i, \vee_i, \wedge_i)$: $i = 1, 2, \ldots\}$ be a sequence of discrete finite simple lattices such that the cardinality of each S_i is greater than two. Let $\leq_i$ denote the partial ordering on S_i, and 1_i, 0_i denote the largest and smallest elements of S_i respectively. Let s_i be some fixed element of S_i-$\{0_i, 1_i\}$. Let ω be the infinite discrete lattice of all nonnegative integers with the natural order, which we denote by $\leq_0$. We now create a new lattice L. As a set, $L = \omega \cup \{S_i\colon i = 1, 2, \ldots\}$ with the following identifications: $0 = 0_1$, $1 = 0_2 = s_1$, $2 = 0_3 = s_2 = 1_1$, and in general, $n = 0_{n+1} = s_n = 1_{n-1}$ whenever $n \geq 3$. Therefore, every element of L may be considered as an element of at least one S_i.

The partial ordering on L will be the one obtained by "patching" together the partial orderings $\leq_i$ with $\leq_0$. That is, if $G_i = \{(x, y) \in S_i \times S_i \mid x \leq_i y\}$ and $G_0 = \{(x, y) \in \omega \times \omega \mid x \leq_0 y\}$ are the graphs of their respective partial orders then by the

[1] For further background or other concepts subsequently used, the reader may refer to S. Willard, *General Topology*, Addison-Wesley Publishing Co., Reading, Mass. (1970), and G. Birkoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, Providence, R.I. (1967).

identifications above, the graph of the partial ordering $\leq$ on L is the transitive closure of $(\bigcup_{i=1}^{\infty} G_i) \cup G_0$, which is by definition $\{(x, y) \in L \times L \mid$ there exists a collection

$$(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_{n+1}) \in \left(\bigcup_{i=1}^{\infty} G_i\right) \cup G_0,$$

n a positive integer, and $x = x_0$, $y = x_{n+1}\}$.

In order to clarify the above construction, we exhibit a specific example by letting every S_i be the five-point simple modular lattice M_5 with s_i being one of the three unrelated points. Thus L may be represented in the plane as follows:

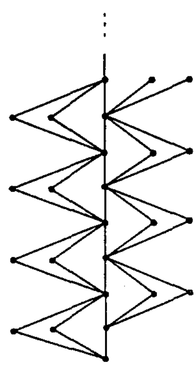


Fig. 3

Returning to the general case, note that when the partial order on L is restricted to ω it is exactly $\leq_0$, and if restricted to any S_i it is exactly $\leq_i$. Moreover, it is tedious but not difficult to show L is a well defined lattice.

To simplify what follows, for any lattice M we define $(\uparrow x) = \{y \in M \mid y \geq x\}$ and $[u, w] = \{y \in M \mid u \leq y \leq w\}$.

For our purposes, the following two lemmas contain the most relevant information about L.

(1) LEMMA. *For any $l \in L$ there exist only finitely many elements unrelated to l.*

Proof. For $l \in L$ let $U(l)$ be the set of elements of L unrelated to l. Suppose l can be considered as an element of S_i. Then for any $x \in (\uparrow 1_i)$, $x \geq l$. Thus $U(l) \subset L - (\uparrow 1_i)$. However, $\bigcup_{j=i+2}^{\infty} S_j \subset (\uparrow 1_i)$ which implies that $U(l) \subset \bigcup_{j=1}^{i+1} S_j$. But, $\bigcup_{j=1}^{i+1} S_j$ is finite. Hence so is $U(l)$.

(2) LEMMA. *L is $\mathscr{L}$-simple.*

Proof. Let φ be any lattice homomorphism of L other than the identity map. Then there exist $a, b \in L$ such that $a \neq b$, $\varphi(a) = \varphi(b)$. If a and b can be considered as elements of the same S_i then $\varphi(a) = \varphi(b)$ implies that $\varphi(S_i) = \varphi(a)$ since S_i is simple. But 0_{i+1} and s_{i+1} are identified with s_i and 1_i respectively. Therefore $\varphi(0_{i+1}) = \varphi(a) = \varphi(s_{i+1})$ implies that $\varphi(S_{i+1}) = \varphi(a)$ since S_{i+1} is also simple. Similarly, if $i \geq 2$ then s_{i-1} and 1_{i-1} are identified with 0_i and s_i respectively. Thus $\varphi(S_{i-1}) = \varphi(a)$ since S_{i-1} is simple. By the same reasoning, $\varphi(S_{i+1}) = \varphi(a)$ implies that $\varphi(S_{i+2}) = \varphi(a)$. Now continuing in this manner, we obtain $\varphi(S_i) = \varphi(a)$ for all $i = 1, 2, \ldots$. That is, $\varphi(L) = \varphi(a)$.

If a and b cannot be considered as elements of the same S_i, then suppose $a \in S_j$ and $b \in S_n$ such that $j < n$. Let k be the largest integer such that $a \in S_k$. Note that $k < n$ since otherwise $a, b \in S_n$. Also note that $a < 1_k$. Now if $b \in S_{k+1}$ (clearly $b \neq 1_k$ and $b \neq s_k$) then by definition of the order relation $a < 1_k < a \vee b$ if $a \nleq s_k$, and, since $a \notin S_{k+1}$ we have $a < s_k < a \vee b$ if $a \leq s_k$. But $\varphi(a) = \varphi(a \vee b)$ so again we identify two points from the same S_i – either a and s_k, or, a and 1_k. Therefore, by the first case above, $\varphi(L) = \varphi(a)$. If $b \notin S_{k+1}$ but can be considered as a member of S_n, $n > k+1$, then a similar reasoning applies since $a < 1_k = 0_{k+1} < b$. Thus any homomorphism besides the identity map identifies all elements of L.

To complete our construction, let $\hat{L}$ be the ideal completion of the lattice L obtained above. We now show that the set of all such lattices $\hat{L}$ is the collection $\mathscr{S}$ referred to in the introduction.

Let $\mathscr{E}$ be the embedding map from L into $\hat{L}$ such that $\mathscr{E}(l) = \{x \in L \mid x \leq l\}$. Now from (1), each collection of unrelated elements in L is finite. With this fact it is not difficult to show that $\hat{L} - \mathscr{E}(L)$ consists of merely one element; namely, the ideal L. This ideal, being the largest element of $\hat{L}$, shall be denoted by Π. Also note that any collection of unrelated elements in $\hat{L}$ is finite.

Now $\hat{L}$ is an algebraic lattice (cf., [1]). Thus, as Hofman, Mislove, and Stralka have shown in [2], $\hat{L}$ can be given a topology which makes it a compact zero dimensional topological $\bigcap$-semilattice. This topology, when applied to $\hat{L}$, is generated by declaring $(\uparrow \mathscr{E}(l))$ to be open and closed for every $l \in L$. But there are only finitely many elements $x \in \hat{L}$ such that $[\mathscr{E}(l), x] = \{\mathscr{E}(l), x\}$ for each $\mathscr{E}(l)$ in L, which implies that each $\mathscr{E}(l)$ is open and closed. Finally, note that Π is a limit point of $\hat{L} - \{\Pi\}$, with a neighborhood base of Π consisting of all sets of the form $(\uparrow \mathscr{E}(l))$, $l \in L$.

(3) THEOREM: *L is a $\mathscr{C}\mathscr{T}\mathscr{L}$-simple compact zero dimensional topological lattice.*

Proof. Since we have already noted that $\hat{L}$ is a compact zero dimensional topological $\bigcap$-semilattice, we need only show that $\hat{L}$ is a topological lattice and $\mathscr{CTL}$-simple.

To show that $\hat{L}$ is a topological lattice it suffices to show that the join is continuous with one coordinate constant; that is, if $\{z_i: i=1,2,\ldots\}$ is a sequence in $\hat{L}$ converging to $z\in\hat{L}$ then for any $x\in\hat{L}$ the sequence $\{z_i\vee x: i=1,2,\ldots\}$ converges to $z\vee x$. Moreover, since this condition is trivial for sequences that are eventually constant, we may assume $\{z_i: i=1,2,\ldots\}$ is eventually nonconstant. From above, every element of $\hat{L}-\{\Pi\}$ is open. Thus the only convergent sequences that are eventually nonconstant must converge to Π. Let $\{z_i: i=1,2,\ldots\}$ be such a sequence and $x\in\hat{L}$. Now $z_i\leq z_i\vee x$, so $\{z_i\vee x: i=1,2,\ldots\}$ converges to Π which is exactly what is desired. Thus L is a topological lattice.

To show that $\hat{L}$ is $\mathscr{CTL}$-simple note that $\mathscr{C}(L)=\hat{L}-\{\Pi\}$ is a sublattice of $\hat{L}$ and is also lattice-isomorphic to L. Therefore using (2), a $\mathscr{CTL}$-morphism φ on $\hat{L}$ that identifies any two distinct elements of $\mathscr{C}(L)$ must identify all of $\mathscr{C}(L)$. Moreover, with Π being a limit point of $\mathscr{C}(L)$, it follows from φ being continuous and $\varphi(\hat{L})$ being Hausdorff that $\varphi(\hat{L})$ is a single point. However, excluding the identity map, there seems to be another alternative to the morphisms described above; that is, a morphism ψ could identify Π with some element of $\mathscr{C}(L)$. But such a map would certainly identify at least two elements of $\mathscr{C}(L)$ since $\mathscr{C}(L)$ has no maximal elements. Thus $\psi(\hat{L})$ would also be a single point.

In particular, if we again consider L as in Figure 3 then it is quite easy to see that the ideal completion $\hat{L}$ of L is also modular. In fact, this lattice may also be constructed in the plane with the usual plane topology as follows:

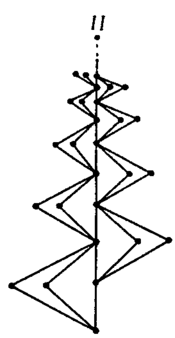


Fig. 4

Furthermore, note that the width of the lattice of Figure 4 is 4 and the breadth is 2.

Putting our construction in proper perspective, we remark that in order to obtain an infinite $\mathscr{C}\mathscr{T}\mathscr{L}$-simple compact topological lattice it is not enough to merely construct the lattice of ideals of any infinite $\mathscr{L}$-simple lattice. Such a lattice of ideals will be a compact topological $\bigcap$-semilattice by [2], but in general, the join will not be continuous.

For instance, consider $\hat{L}_2$, the ideal completion of L_2 in Figure 2. Besides the natural copy of L_2 embedded in $\hat{L}_2$, there are two other elements: the ideal consisting of L_2 (denoted Π), and the ideal consisting of only the x_i's (denoted I). If we endow $\hat{L}_2$ with the same sort of topology as we did $\hat{L}$ above (that is, declaring $(\uparrow\mathscr{E}(l))$ open and closed for all $l \in L_2$ where $\mathscr{E}: L_2 \to \hat{L}_2$ is the embedding map), then by definition of the topology, all finite intersections of the set $\{(\uparrow\mathscr{E}(x_i)): i = 1, 2, \ldots\} \cup \{[\hat{L}_2 - (\uparrow\mathscr{E}(y_i))]: j = 1, 2, \ldots\} \cup \{[\hat{L}_2 - (\uparrow\mathscr{E}(z_k))]: k = 1, 2, \ldots\}$ forms a basic neighborhood system for I. Using this neighborhood system, it is easy to show that the sequences $\{\mathscr{E}(x_i): i = 1, 2, \ldots\}$ and $\{\mathscr{E}(y_i): i = 1, 2, \ldots\}$ both converge to I. Thus, if the join were continuous the sequence $\{\mathscr{E}(x_i) \vee \mathscr{E}(y_i): i = 1, 2, \ldots\}$ would also converge to I. However this is not the case. The join of the two ideals $\mathscr{E}(x_i)$ and $\mathscr{E}(y_i)$ produces $\mathscr{E}(z_i)$ for all $i = 1, 2, \ldots$, and this sequence, $\{\mathscr{E}(z_i): i = 1, 2, \ldots\}$, converges to Π.

Finally, we note that none of the elements of $\mathscr{S}$ are $\mathscr{L}$-simple. When all lattice homomorphisms are considered, $\hat{L}$ has three distinct homomorphisms: the two obvious maps and another, η, such that $\eta(\hat{L}) = \{0, 1\}$ with $\eta(\hat{L} - \{\Pi\}) = 0$, $\eta(\Pi) = 1$.

However, using any $\hat{L} \in \mathscr{S}$ we can construct an infinite compact zero dimensional topological lattice that is $\mathscr{L}$-simple. For $\hat{L} \in \mathscr{S}$ we can represent $\hat{L}$ by the following diagram:

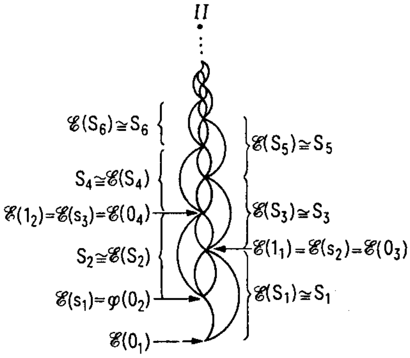
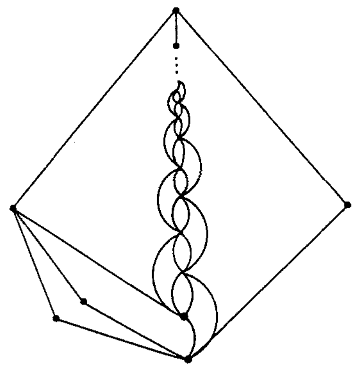


Fig. 5

With the diagram of Figure 5, it is easy to construct a collection T of $\mathscr{L}$-simple infinite compact zero dimensional topological lattices by attaching five other open and closed points to $\hat{L}$. An example of such a lattice is represented in Figure 6. (It is quite easy to show that this representation is what we claim.)



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