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AN ASYMPTOTIC-NUMERICAL ANALYSIS FOR THE LOWER BOUND DYNAMIC BUCKLING ESTIMATES*

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ABSTRACT: A finite element asymptotic analysis for determining the lower bound dynamic buckling estimates of imperfection-sensitive structures under step load of infinite duration is presented. The lower bound dynamic buckling loads and the corresponding displacements are sought in the form of asymptotic expansions based on the static stability criterion and they can be determined by solving numerically (FEM) several linear problems with a single nonsingular sub-stiffness matrix.

KEY WORDS: dynamic buckling, lower bound estimate, FEM implementation

1 INTRODUCTION

The nonlinear dynamic analysis[1] is generally used to obtain numerically the exact dynamic buckling load λ_d of an imperfection-sensitive (autonomous) structural system under step load of infinite duration. In view of numerical difficulties in solving the nonlinear initial-value problem associated with the dynamic buckling response of the system[2~6], the lower bound dynamic buckling estimate $\tilde{\lambda}_d$ based on the static stability criterion associated with the vanishing of the total potential energy on a certain equilibrium point of the unstable postbuckling path has been established [2~6]. As an attempt to determine the lower dynamic buckling loads and corresponding displacements based on the simple static stability criterion [2~6], a perturbation method [7] in forms of the sufficient notation of functional analysis, has also been presented and the effect of various possible sources of structural imperfections on the dynamic buckling loads of perfect structures has been considered. However, the previous paper [7] dealt only with a general formulation and simple examples.

The objective of this paper is to establish the finite element implementation of the previous paper[7]. We will first transfer the approach in Ref.[7] into the more versatile framework of the finite element method, and then propose a partitioning procedure so that the perturbation expansions can be carried out by solving numerically several linear problems with a single nonsingular sub-stiffness matrix. Finally, we take a column on a linear elastic foundation subjected to axial step load of infinite duration as an example to illustrate the present method.

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2 THEORY

The perturbation method [7] is summarized and slightly modified in this section. For details, readers are referred to Ref.[7]. Let the potential energy of an imperfect structure be given by $\Phi(\boldsymbol{u}, \lambda, \boldsymbol{w})$ where $\boldsymbol{u}$ denotes the additional displacement of the structure from the stress free reference configuration, the single scalar variable λ determines the magnitude of prescribed external loads on the structure, and $\boldsymbol{w}$ denotes the structural imperfection of the structure. Furthermore, let U and W denote, respectively, admissible function spaces of displacement and structural imperfection of the structure. We can introduce two inner products in the two spaces denoted by $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ for $\boldsymbol{u}_i \in U$ $(i = 1, 2)$ and $[\boldsymbol{w}_1, \boldsymbol{w}_2]$ for $\boldsymbol{w}_i \in W$ $(i = 1, 2)$. The potential energy Φ may be rewritten as $\Phi(\boldsymbol{u}, \lambda, \varepsilon\bar{\boldsymbol{w}})$ where ε is the imperfection amplitude and $\bar{\boldsymbol{w}}$ is normalized imperfection pattern, i.e., $[\bar{\boldsymbol{w}}, \bar{\boldsymbol{w}}] = 1$.

According to the static stability criterion[2~6], for each magnitude ε of a given normalized imperfection pattern $\bar{\boldsymbol{w}}$, the lower bound dynamic buckling load λ_d and the corresponding displacement can be found by solving the following nonlinear equations simultaneously (in unknowns $\boldsymbol{u}$ and λ):

$$\Phi_u(\boldsymbol{u}, \lambda, \varepsilon\bar{\boldsymbol{w}})\delta\boldsymbol{u} = 0 \tag{1}$$

for all admissible variations $\delta\boldsymbol{u}$ and

$$\Phi(\boldsymbol{u}, \lambda, \varepsilon\bar{\boldsymbol{w}}) = 0 \tag{2}$$

subject to the condition that the static equilibrium position determined by the solution of Eqs.(1) and (2) is unstable.

We assume that for the perfect structure there exists a trivial major solution $\boldsymbol{u} = \boldsymbol{u}_0(\lambda) = 0$ as the load increases from zero. Let $\lambda = \lambda_c$ be the buckling load for the perfect structure and it is assumed to be simple with corresponding buckling mode $\boldsymbol{u}_1$ normalized by $(\boldsymbol{u}_1, \boldsymbol{u}_1) = 1$. We assume also that [8,9]

$$a_{\xi\lambda} \equiv \Phi^c_{uu\lambda}\boldsymbol{u}_1{}^2 < 0 \qquad a_\varepsilon \equiv \Phi^c_{uw}\boldsymbol{u}_1\bar{\boldsymbol{w}} \neq 0 \tag{3}$$

where superscript "c" denote the corresponding derivatives of potential energy function Φ calculated at $(\boldsymbol{u}, \lambda, \boldsymbol{w}) = (0, \lambda_c, 0)$.

The solutions to the system consisting of Eqs.(1) and (2) are sought in the following forms

$$\left.\begin{aligned} \boldsymbol{u} &= \xi\boldsymbol{u}_1 + \xi^2\boldsymbol{u}_2 + \xi^3\boldsymbol{u}_3 + \cdots \\ \lambda &= \lambda_c + \lambda_1\xi + \lambda_2\xi^2 + \cdots \\ \varepsilon &= \varepsilon_2\xi^2 + \varepsilon_3\xi^3 + \cdots \end{aligned}\right\} \tag{4}$$

where $(\boldsymbol{u}_1, \boldsymbol{u}_i) = 0$ $(i = 2, 3, \cdots)$.

Substituting Eqs.(4) into Eqs.(1) and (2), expanding them, respectively, into series of ξ, letting the coefficients vanish separately, the resulting equations are grouped as follows

$$\left.\begin{aligned} &\Phi^c_{uu}\boldsymbol{u}_2 + \varepsilon_2\Phi^c_{uw}\bar{\boldsymbol{w}} + \lambda_1\Phi^c_{uu\lambda}\boldsymbol{u}_1 + \frac{1}{2}\Phi^c_{uuu}\boldsymbol{u}_1{}^2)\delta\boldsymbol{u} = 0 \\ &\varepsilon_2\Phi^c_{uw}\boldsymbol{u}_1\bar{\boldsymbol{w}} + \frac{1}{2}\lambda_1\Phi^c_{uu\lambda}\boldsymbol{u}_1{}^2 + \frac{1}{6}\Phi^c_{uuu}\boldsymbol{u}_1{}^3 = 0 \\ &(\boldsymbol{u}_2, \boldsymbol{u}_1) = 0 \end{aligned}\right\} \tag{5}$$

and

$$\left.\begin{aligned}
&[\Phi^c_{uu}\boldsymbol{u}_3+\varepsilon_3\Phi^c_{uw}\bar{\boldsymbol{w}}+\lambda_2\Phi^c_{uu\lambda}\boldsymbol{u}_1+\varepsilon_2(\lambda_1\Phi^c_{uw\lambda}\bar{\boldsymbol{w}}+\Phi^c_{uuw}\boldsymbol{u}_1\bar{\boldsymbol{w}})+\\
&\qquad\lambda_1\Phi^c_{uu\lambda}\boldsymbol{u}_2+\frac{1}{2}\lambda_1{}^2\Phi^c_{uu\lambda\lambda}\boldsymbol{u}_1+\Phi^c_{uuu}\boldsymbol{u}_1\boldsymbol{u}_2+\\
&\qquad\frac{1}{2}\lambda_1\Phi^c_{uuu\lambda}\boldsymbol{u}_1{}^2+\frac{1}{6}\Phi^c_{uuuu}\boldsymbol{u}_1{}^3]\delta\boldsymbol{u}=0\\
&\varepsilon_3\Phi^c_{uw}\boldsymbol{u}_1\bar{\boldsymbol{w}}+\frac{1}{2}\lambda_2\Phi^c_{uu\lambda}\boldsymbol{u}_1{}^2+\varepsilon_2(\Phi^c_{uw}\boldsymbol{u}_2\bar{\boldsymbol{w}}+\lambda_1\Phi^c_{uw\lambda}\boldsymbol{u}_1\bar{\boldsymbol{w}}+\frac{1}{2}\Phi^c_{uuw}\boldsymbol{u}_1{}^2\bar{\boldsymbol{w}})+\\
&\qquad\lambda_1\Phi^c_{uu\lambda}\boldsymbol{u}_1\boldsymbol{u}_2+\frac{1}{2}\Phi^c_{uu}\boldsymbol{u}_2{}^2+\frac{1}{4}\lambda_1{}^2\Phi^c_{uu\lambda\lambda}\boldsymbol{u}_1{}^2+\frac{1}{2}\Phi^c_{uuu}\boldsymbol{u}_1{}^2\boldsymbol{u}_2+\\
&\qquad\frac{1}{6}\lambda_1\Phi^c_{uuu\lambda}\boldsymbol{u}_1{}^3+\frac{1}{24}\Phi^c_{uuuu}\boldsymbol{u}_1{}^4=0\\
&(\boldsymbol{u}_3,\boldsymbol{u}_1)=0
\end{aligned}\right\}\tag{6}$$

and so on.

Finally, by the discussion of stability on the equilibrium position (4) in Ref.[7], the lower bound dynamic buckling loads are determined as

$$\tilde{\lambda}_d=\lambda_c+\lambda_1\xi+\lambda_2\xi^2+\cdots\tag{7}$$

where ξ satisfies $\lambda_1\xi<0$ if $\lambda_1\neq 0$ or $\lambda_2<0$ if $\lambda_1=0$.

3 FEM IMPLEMENTATION OF SOLVING THE PERTURBATION EQUATIONS

In this section, we give an algorithm for FEM implementation of solving the perturbation equations in Section 2. The principle of the algorithm is to respectively compute λ_1, ε_2, and $\boldsymbol{u}_2$, by Eqs.(5). Next calculate λ_2, ε_3 and $\boldsymbol{u}_3$, by Eqs.(6).

We refer now to a finite element modelling of the structure at hand. Let $\boldsymbol{u}$ be the vector of the nodal displacements which defines the displacement field $\boldsymbol{u}$ through a prefixed set of shape functions. Then, the tangent stiffness matrix $\boldsymbol{K}^c$ et al. at the bifurcation point can be defined, as follows

$$\left.\begin{aligned}
&\Phi^c_{uu}\boldsymbol{u}\delta\boldsymbol{u} &&\equiv \boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}^c\delta\boldsymbol{u}\\
&\Phi^c_{uw}\bar{\boldsymbol{w}}\delta\boldsymbol{u} &&\equiv \boldsymbol{b}^{c\mathrm{T}}_{\varepsilon}\delta\boldsymbol{u}\\
&\Phi^c_{uu\lambda}\boldsymbol{u}_1\delta\boldsymbol{u} &&\equiv \boldsymbol{b}^{c\mathrm{T}}_{\lambda}\delta\boldsymbol{u}\\
&\Phi^c_{uu\lambda}\boldsymbol{u}_1\boldsymbol{u}_{i+1} &&\equiv \boldsymbol{b}^{c\mathrm{T}}_{\lambda}\boldsymbol{u}_{i+1}\\
&(\boldsymbol{u},\boldsymbol{v}) &&\equiv \boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{v}
\end{aligned}\right\}\tag{8}$$

After the discretization, the linear problem and the orthogonality conditions in (5) and (6) lead to the mixed problems for $i=1,2$

$$\begin{bmatrix}\boldsymbol{K}^c & \boldsymbol{b}^c_{\varepsilon} & \boldsymbol{b}^c_{\lambda}\\ 0 & \boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{b}_{\varepsilon} & \frac{1}{2}\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{b}^c_{\lambda}\\ \boldsymbol{b}_1{}^{\mathrm{T}} & 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{u}_{i+1}\\ \varepsilon_{i+1}\\ \lambda_i\end{bmatrix}=\begin{bmatrix}\boldsymbol{e}_i\\ f_i\\ 0\end{bmatrix}\tag{9}$$

where $\boldsymbol{b}_1{}^{\mathrm{T}} = \boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{M}$.

It should be remarked that the coefficient matrix in system (9) is the same for $i = 1, 2, \cdots$ and their right-hand vectors depends only on the previously computed vectors $\boldsymbol{u}$; coefficients λ and ε. More specifically, although the stiffness matrix $\boldsymbol{K}^c$ at the bifurcation point is singular, the coefficient matrix in systems (9) is non-singular due to condition (3). Direct solution of system (9) with Gaussian elimination might require full pivoting strategy in order to avoid severe accumulation of roundoff errors. However, full pivoting destroys the sparse structure of the coefficient matrix, and such a procedure would be impractical.

In fact, the system above can be solved by a partition of the coefficient matrix. System (9) may be written as

$$\left.\begin{aligned} \boldsymbol{K}^c\boldsymbol{u}_{i+1} + \varepsilon_{i+1}\boldsymbol{b}_\varepsilon^c + \lambda_i\boldsymbol{b}_\lambda^c &= \boldsymbol{e}_i \\ \varepsilon_{i+1}\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{b}_\varepsilon^c + \frac{1}{2}\lambda_i\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{b}_\lambda^c &= f_i \\ \boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{u}_{i+1} &= 0 \end{aligned}\right\} \tag{10}$$

Firstly, by premultiplying Eq.(10.1) by the buckling mode $\boldsymbol{u}_1$, one obtains (note that $\boldsymbol{K}^c\boldsymbol{u}_1 = 0$)

$$\varepsilon_{i+1}\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{b}_\varepsilon^c + \lambda_i\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{b}_\lambda^c = \boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{e}_i \tag{11}$$

From Eqs.(10.2) and (11), we deduce that

$$\varepsilon_{i+1} = \frac{2f_i - \boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{e}_i}{\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{b}_\varepsilon^c} \qquad \lambda_i = \frac{2(\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{e}_i - f_i)}{\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{b}_\lambda^c} \tag{12}$$

Then Eq.(10.1) can be rewritten as

$$\mathrm{K}^c\boldsymbol{u}_{i+1} = -\varepsilon_{i+1}\boldsymbol{b}_\varepsilon^c - \lambda_i\boldsymbol{b}_\lambda^c + \boldsymbol{e}_i \equiv \boldsymbol{g}_i \tag{13}$$

The coefficient matrix $\boldsymbol{K}^c$ of Eq.(13) is of rank $(n-1)$, and the n equations are consistent, since $\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{g}_i = 0$. The solution to Eq.(13) can be written as

$$\boldsymbol{u}_{i+1} = \boldsymbol{h}_i + c_i\boldsymbol{u}_1 \tag{14}$$

The first vector, is a particular solution to Eq.(13), and the second is the buckling mode. In order to determine the coefficient c_i, one can use Eq.(10.3). The substitution of Eq.(14) in Eq.(10.3) gives

$$c_i = -\frac{\boldsymbol{b}_1{}^{\mathrm{T}}\boldsymbol{h}_i}{\boldsymbol{u}_1{}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{u}_1} \tag{15}$$

Therefore, the basic requirement for determining $\boldsymbol{u}_{i+1}$ is to determine a particular solution $\boldsymbol{h}_i$ to Eq.(13). The difficulty lies in the fact that the matrix $\boldsymbol{K}^c$ is of rank $(n-1)$ and can not be inverted. Let j be the biggest element in the buckling mode $\boldsymbol{u}_1$. Analysis shows that the sub-stiffness matrix $\bar{K}$ obtained by deleting the jth row and the jth column from $\boldsymbol{K}^c$ is non-singular at the bifurcation point. Furthermore, a special solution $\boldsymbol{h}_i$ to Eq.(13) with the jth component equal to 0 can be obtained by solving the following equation

$$\bar{\boldsymbol{K}}\bar{\boldsymbol{h}}_i = \bar{\boldsymbol{g}}_i \tag{16}$$

where $\bar{\boldsymbol{h}}_i = (h_{i1}, h_{i2}, \cdots, h_{i,j-1}, h_{i,j+1}, \cdots, h_{in})^{\mathrm{T}}$, and $\bar{\boldsymbol{g}}_i = (g_{i1}, g_{i2}, \cdots, g_{i,j-1}, g_{i,j+1}, \cdots, g_{in})^{\mathrm{T}}$ which is $(n-1)$-dimensional vector obtained by removing the jth component of g_i. Then the particular solution to the Eq.(13) is in the form $\boldsymbol{h}_i = (h_{i1}, h_{i2}, \cdots, h_{i,j-1}, 0, h_{i,j+1}, \cdots, h_{in})^{\mathrm{T}}$.

Note that $\bar{\boldsymbol{K}}$ is symmetric, bound, sparse, and positive matrix, therefore, one can solve Eq.(16) by Cholesky decomposition. Furthermore, all the perturbation expansions can be determined by solving numerically several linear problems with the same coefficient matrix.

A basic point concerning the implementation of the present formulation will be the computations of the buckling load λ_c, and the corresponding buckling mode $\boldsymbol{u}_1$ of the perfect structure. Their calculation leads to a linear eigenvalue problem [10]

$$\boldsymbol{K}(\lambda)\boldsymbol{u} = 0 \tag{17}$$

where $\boldsymbol{K}(\lambda)$ is the tangent stiffness matrix. For the determination of the buckling mode, we put the normalized condition

$$\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{u} = 1 \tag{18}$$

One can easily calculate the buckling load λ_c by means of inverse iterative method and then the corresponding buckling mode $\boldsymbol{u}_1$ which satisfies (18) can be easily obtained.

4 AN EXAMPLE

We consider the simply supported column on the linear-elastic foundation[7]. Because the imperfection patterns of the stiffness of bending and the stiffness of the foundation have no effect up to the order $O(\xi^3)$ on the lower bound dynamic buckling loads[7], we investigate only other two imperfections: the initial shape imperfection in form of the classical buckling mode and the load eccentricity e. The analytic dependence of the lower bound dynamic buckling load $\tilde{\lambda}_d$, the corresponding displacement $\boldsymbol{u}$ and the imperfection amplitude ε on ξ (the projection of $\boldsymbol{u}$ on the buckling mode $\boldsymbol{u}_1$) have been established by the writer in Ref.[7].

For finite element discretization of the problem, two-mode elements of equal length based on cubic Hermite interpolation of the displacement $\boldsymbol{u}$ are used. Integration of the element potential energy is done numerically, but with a sufficient number of Gaussian integration points to achieve exact integration. The computations for the buckling load $\bar{\lambda}_c$ and imperfection sensitivity-coefficients $\bar{\lambda}_2$ and $\bar{\varepsilon}_3$ of our finite element model were performed. We find that the convergence of these approximate results to the exact buckling load λ_c, the imperfection sensitivity-coefficients λ_2 and ε_3 is encouragingly good. The results for $n = 10$ and $n = 20$, respectively, are listed in Table 1. The buckling load $\bar{\lambda}_c$ is 2.000 013 4 and 2.000 000 8 for $n = 10$ and $n = 20$, respectively.

Table 1 Convergence of finite element solutions for the lower bound dynamic buckling estimates of the column on the elastic foundation

Combination of imperfections		Exact solutions		Finite element solutions			
				$n = 10$		$n = 20$	
a	b	λ_2	ε_3	$\bar{\lambda}_2$	$\bar{\varepsilon}_3$	$\bar{\lambda}_2$	$\bar{\varepsilon}_3$
1.0	0.0	−0.238 732 40	0.039 788 74	−0.238 726 00	0.039 787 39	−0.238 732 00	0.039 788 65
−0.37	0.93	−0.238 732 40	−0.014 743 46	−0.238 726 00	−0.014 742 80	−0.23873210	−0.014 743 43
0.0	1.0	−0.238 732 40	−0.015 873 41	−0.238 726 00	−0.015 872 67	−0.238 732 10	−0.015 873 37

5 CONCLUSIONS

We have proposed a finite element asymptotic analysis for determining the lower dynamic buckling loads and corresponding displacements of imperfection-sensitive structures under step load of infinite duration. The various possible sources of structural imperfections have been simultaneously considered and treated. The method can give higher-order approximation to the lower bound dynamic buckling loads and the corresponding displacements without solving the system of differential equation of motion numerically. The FEM implementation of the method is characterized by a much lower computational cost, since the computations of the perturbation expansions can be carried by solving numerically several linear problems with a single (positive-definite and symmetric) sub-stiffness matrix. The numerical example illustrate the characteristics and effectiveness of the method.

REFERENCES

1 Budiansky B, Roth RS. Axisymmetric dynamic buckling of clamped shallow spherical shells. Collected Papers on Instability of Shell Structures, NASA TN D-1510, 1962
2 Kounadis AN. Nonlinear dynamic buckling of discrete dissipative or nondissipative systems under step loading. *AIAA J*, 1991, 29(2): 280~289
3 Kounadis AN, Mallis J, Raftoyiannis J. Dynamic buckling estimates for discrete systems under step loading. *ZAMM*, 1991, 71(10): 391~402
4 Kounadis AN. Nonlinear dynamic buckling and stability of autonomous structural systems. *Int J Mechanical Sciences*, 1993, 35(8): 643~656
5 Gantes C, Kounadis AN. Energy-based dynamic buckling estimates for autonomous dissipative systems. *AIAA J*, 1995, 33(7): 1342~1349
6 Simitses GJ. Dynamic Stability of Suddenly Loaded Structures. Berlin: Springer-Verlag, 1990
7 Wu B. A method for determining the lower bound dynamic buckling loads of imperfection-sensitive structures. *ZAMM*, 1997, 77: in press
8 Budiansky B. Theory of buckling and post-buckling behaviour of elastic structures. *Advances in Applied Mechanics*, 1974, 14: 1~65
9 Ikeda K, Murota K. Critical initial imperfection of structures. *Int J Solids and Structures*, 1990, 26(8): 865~886
10 Riks E, Brogan FA, Rankin CC. Numerical aspects of shell stability analysis. in: Krätzig WB, Oñate E eds. Computational Mechanics of Nonlinear Response of Shells. Berlin: Springer-Verlag, 1990. 125~151