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PLANE SURFACE SUDDENLY SET IN MOTION IN A VISCOELASTIC FLUID WITH FRACTIONAL MAXWELL MODEL*

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ABSTRACT: The fractional calculus approach in the constitutive relationship model of viscoelastic fluid is introduced. The flow near a wall suddenly set in motion is studied for a non-Newtonian viscoelastic fluid with the fractional Maxwell model. Exact solutions of velocity and stress are obtained by using the discrete inverse Laplace transform of the sequential fractional derivatives. It is found that the effect of the fractional orders in the constitutive relationship on the flow field is significant. The results show that for small times there are appreciable viscoelastic effects on the shear stress at the plate, for large times the viscoelastic effects become weak.

KEY WORDS: viscoelastic fluid, fractional calculus, Stokes problem, fractional Maxwell model

1 INTRODUCTION

There are very few cases in which the exact analytic solutions of Navier-Stokes equations can be obtained. These are even rare if the constitutive equations for the viscoelastic fluid are considered. Although there are many models used to describe viscoelastic behavior of the fluid, the fluid of differential type have received special attention[1~6]. Recently the fractional calculus has achieved a great success in the description of complex dynamics. In particular it has proved to be a valuable tool to handle viscoelastic behavior. The starting point of the fractional derivative model of viscoelastic fluid is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so called Riemann-Liouville fractional calculus operators. This generalization allows one to define precisely non-integer order integrals or derivatives. Li Jian, Song Daoyun and Jian Tiqian used the fractional model to analyze the characteristics of sesbania gum and xanthan gum in their experiment and obtained satisfactory results[7~9]. Fractional derivatives have been found to be quite flexible in describing viscoelastic behavior[10~13] .

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This paper will study the flow near a wall suddenly set in motion for a non-Newtonian viscoelastic fluid with the fractional Maxwell model. Firstly, the fractional calculus approach in the constitutive relationship model of viscoelastic fluid is introduced. The fractional Maxwell model of viscoelastic fluid is developed. Then, by using the discrete inverse Laplace transform of the sequential fractional derivatives to the governing equations, we obtain the exact solutions of the velocity and stress. It is found that the effect of the fractional orders in the constitutive relationship on the flow field is significant and for small times there are appreciable viscoelastic effects on the shear stress at the plate, for large times the viscoelastic effects become weak.

2 THE FRACTIONAL MAXWELL MODEL AND BASIC EQUATIONS

First, we begin by recalling the definition of a fractional integral of order $-p$ of a function $f(t)$. This is given by the Riemann-Liouville integral operator[12]

$$\mathrm{D}_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau)\mathrm{d}\tau \qquad p>0 \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function. Next, the fractional derivative of order p is defined as

$$\mathrm{D}_t^{p} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\mathrm{D}_t^{p-n} f(t)\right) \qquad 0<n-p<1 \tag{2}$$

Now, following Palade et al.[12], we shall use the shorthand notation

$$\mathrm{D}_t^{q} f(t) = \frac{\mathrm{d}^q f}{\mathrm{d}t^q} \tag{3}$$

for the fractional integral or differential operation of order q on any function $f(t)$. Using the notation, we observe from [12] that the composition rule for integration and differentiation obeys the simple form

$$\frac{\mathrm{d}^p}{\mathrm{d}t^p}\frac{\mathrm{d}^q}{\mathrm{d}t^q} = \frac{\mathrm{d}^{p+q}}{\mathrm{d}t^{p+q}} \tag{4}$$

for all numbers p and q, whether they are positive or negative.

The fluid considered in this paper is a viscoelastic fluid with the fractional derivative Maxwell model[14] and its constitutive equation is given by

$$\sigma + \lambda^\alpha \frac{\mathrm{d}^\alpha \sigma}{\mathrm{d}t^\alpha} = G\lambda^\beta \frac{\mathrm{d}^\beta \varepsilon}{\mathrm{d}t^\beta} \tag{5}$$

where σ is the shear stress, ε is the shear strain, $\lambda = \mu/G$ is a relaxation time, where G is a shear modulus, μ is viscosity. Also, α and β are fractional calculus parameters such that $0 \le \alpha \le \beta \le 1$. For $\alpha > \beta$ the relaxation function is increasing, which is in general not reasonable[14], and one has to require that $\alpha \le \beta$. When $\alpha = \beta = 1$, it may be simplified as the ordinary Maxwell model; when $\alpha = 0$, $\beta = 1$, it may be simplified as the classical Newtonian fluid.

We shall interpret the derivative on the right hand side of (5) as a fractional integral of order $\beta - 1$ of the ordinary derivative of ε, i.e.

$$\frac{\mathrm{d}^\beta \varepsilon}{\mathrm{d}t^\beta} = \frac{\mathrm{d}^{\beta-1}}{\mathrm{d}t^{\beta-1}}\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right) = \frac{\mathrm{d}^{\beta-1}\dot{\varepsilon}}{\mathrm{d}t^{\beta-1}} \tag{6}$$

where $\dot{\varepsilon}$ is the shear rate. Thus, we can write (5) as follows

$$\sigma + \lambda^{\alpha}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} = G\lambda^{\beta}\frac{\mathrm{d}^{\beta-1}\dot{\varepsilon}}{\mathrm{d}t^{\beta-1}} \tag{7}$$

We are interested in the flow of a viscoelastic fluid modeled by (7) over an infinite flat plate, which is driven by the impulsive motion of the flat plate with a constant velocity U. We select the direction of motion of the wall as the x axis and the direction perpendicular to the wall as the y axis and assume that the side effects of the wall are neglected, namely, the wall is infinitely long. After that, we seek a solution for the velocity field of the form

$$\bar{\boldsymbol{v}} = u(y,t)\boldsymbol{i} \tag{8}$$

where u is the velocity in the x coordinate direction and $\boldsymbol{i}$ is its unit vector. The fluid is set into motion through the action of the stress at the plate. For this problem the constitutive relationship becomes

$$\sigma_{yx} + \lambda^{\alpha}\frac{\partial^{\alpha}\sigma_{yx}}{\partial t^{\alpha}} = G\lambda^{\beta}\frac{\partial^{\beta-1}}{\partial t^{\beta-1}}\left(\frac{\partial u}{\partial y}\right) \tag{9}$$

The momentum equation is

$$\rho\frac{\partial u}{\partial t} = \frac{\partial \sigma_{yx}}{\partial y} \tag{10}$$

where ρ is the density of the fluid. From (9) and (10), we can obtain the basic equation

$$\rho\frac{\partial u}{\partial t} + \rho\lambda^{\alpha}\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}} = G\lambda^{\beta}\frac{\partial^{\beta-1}}{\partial t^{\beta-1}}\left(\frac{\partial^2 u}{\partial y^2}\right) \tag{11}$$

For this problem the initial and boundary conditions are

$$u(y,0) = 0 \qquad \text{for} \quad y > 0 \tag{12}$$

$$u(0,t) = U \qquad \text{for} \quad t > 0 \tag{13}$$

$$u \to 0 \qquad \text{for} \quad y \to \infty \tag{14}$$

3 SOLUTION OF BASIC EQUATIONS

3.1 The Velocity Field

Let us introduce dimensionless variables: $u^* = u/U$, $y^* = (yU\rho)/\mu$, $t^* = (tU^2\rho)/\mu$, $\eta = (U^2\rho)/G$, in which U and $\mu/(U^2\rho)$ denote characteristic velocity and time, respectively, η is a dimensionless parameter. Using the first mean value theorem of the integral, it can easily be proved that the operator D_t^{α} has the fractional time dimensions $\left[\mu/(U^2\rho)\right]^{-\alpha}$. Thus, the dimensionless equation and its boundary and initial conditions are obtained as follows (for simplicity, the dimensionless mark "*" will be omitted hereinafter)

$$\frac{\partial u}{\partial t} + \eta^{\alpha}\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}} = \eta^{\beta-1}\frac{\partial^{\beta-1}}{\partial t^{\beta-1}}\left(\frac{\partial^2 u}{\partial y^2}\right) \tag{15}$$

$$u(y,0) = 0 \qquad \text{for} \quad y > 0 \tag{16}$$

$$u(0,t) = 1 \qquad \text{for} \quad t > 0 \tag{17}$$

$$u \to 0 \qquad \text{for} \quad y \to \infty \tag{18}$$

Let us suppose that $\bar{u}(y,s) = \mathrm{L}\{u(y,t),s\} = \int_0^\infty \mathrm{e}^{-st}u(y,t)\mathrm{d}t$ is the image function of $u(y,t)$, where s is a transform parameter. The type of the initial condition (16) suggests that the fractional derivative in Eq.(15) must be interpreted as a properly chosen sequential fractional derivative[15]. Using the Laplace transform properties for sequential fractional derivatives we can obtain

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}y^2} - \frac{1+\eta^\alpha s^{\alpha+1}}{\eta^{\beta-1}s^{\beta-1}}\bar{u} = 0 \tag{19}$$

$$\bar{u}(0,s) = 1/s \tag{20}$$

$$\bar{u} \to 0 \qquad \text{for} \quad y \to \infty \tag{21}$$

Solving the above equations yields

$$\bar{u}(y,s) = \frac{1}{s}\exp\left[-sy\left(\frac{1+\eta^\alpha s^\alpha}{\eta^{\beta-1}s^\beta}\right)^{1/2}\right] \tag{22}$$

In order to avoid the burdensome calculations of residues and contour integrals, we will apply the discrete inverse Laplace transform method to obtain the velocity distribution. Firstly, we can rewrite (22) in a series form by using Taylor theorem.

$$\bar{u}(y,s) = \frac{1}{s} + \sum_{n=1}^{\infty}\frac{(-y)^n\eta^{\frac{n(1+\alpha-\beta)}{2}}}{n!}\sum_{m=0}^{\infty}\frac{\Gamma(n/2+1)\,\eta^{-\alpha m}}{m!\,\Gamma(n/2-m+1)}s^{n-\frac{\beta-\alpha}{2}n-m\alpha-1} \tag{23}$$

Here, we have used the well-known property of the Gamma function

$$\frac{1}{\Gamma(-k)} = 0 \qquad k = 0,\ 1,\ 2,\ 3,\ \cdots \tag{24}$$

Applying the inverse Laplace transform to (23), we obtain

$$u(y,t) = 1 + \sum_{n=1}^{\infty}\frac{(-y)^n\eta^{\frac{n(1+\alpha-\beta)}{2}}t^{\frac{n(\beta-\alpha)}{2}-n}}{n!}\cdot \mathrm{H}^{1,1}_{1,3}\left[-\eta^{-\alpha}t^\alpha\left|\begin{matrix}\left(-\frac{n}{2},0\right)\\(0,1),\left(-\frac{n}{2},-1\right),\left(n-\frac{n(\beta-\alpha)}{2},\alpha\right)\end{matrix}\right.\right] \tag{25}$$

in which $\mathrm{H}^{m,n}_{p,q}(z)$ denotes H function[11]. To obtain (25), we used the property of H function

$$\sum_{n=0}^{\infty}\frac{(-z)^n\prod_{j=1}^{p}\Gamma(a_j+A_jn)}{n!\prod_{j=1}^{q}\Gamma(b_j+B_jn)} = \mathrm{H}^{1,p}_{p,q+1}\left[z\left|\begin{matrix}(1-a_1,A_1),\cdots(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\cdots(1-b_q,B_q)\end{matrix}\right.\right] \tag{26}$$

3.2 The Stress Field

Since the fluid is set into motion through the action of the stress at the plate, the calculation of the stress field is needed. From (9) the dimensionless stress can be represented by

$$F+\eta^{\alpha}\frac{\partial^{\alpha}F}{\partial t^{\alpha}}=\eta^{\beta-1}\frac{\partial^{\beta-1}}{\partial t^{\beta-1}}\left(\frac{\partial u}{\partial y}\right) \tag{27}$$

where $F(y,t)=(\sigma_{yx})\,/\,(\rho U^2)$. The Laplace transform of (27) is

$$\bar{F}(y,s)+\eta^{\alpha}s^{\alpha}\bar{F}(y,s)=\eta^{\beta-1}s^{\beta-1}\frac{\mathrm{d}\bar{u}(y,s)}{\mathrm{d}y} \tag{28}$$

Substituting (22) into (28), we have

$$\bar{F}(y,s)=-\frac{1}{s}\left(\frac{\eta^{\beta-1}s^{\beta}}{1+\eta^{\alpha}s^{\alpha}}\right)^{1/2}\exp\left[-sy\left(\frac{1+\eta^{\alpha}s^{\alpha}}{\eta^{\beta-1}s^{\beta}}\right)^{1/2}\right] \tag{29}$$

Applying the discrete inverse Laplace transform method again, we obtain

$$F(y,t)=-\sum_{k=0}^{\infty}\frac{(-y)^k\eta^{\frac{(k-1)(1-\beta)}{2}}t^{\frac{(k-1)(\beta-\alpha)}{2}}}{k!}\cdot$$

$$\mathrm{H}_{1,3}^{1,1}\left[-\eta^{-\alpha}t^{\alpha}\left|\begin{matrix}\left(\frac{1}{2}-\frac{k}{2},0\right)\\(0,1),\left(\frac{1}{2}-\frac{k}{2},-1\right),\left(k-\frac{(k-1)(\beta-\alpha)}{2},\alpha\right)\end{matrix}\right.\right] \tag{30}$$

In order to obtain the value of the shear stress at the plate, taking $y=0$ in (29), we obtain the following expression

$$\bar{F}_p(s)=-\frac{1}{s}\left(\frac{\eta^{\beta-1}s^{\beta}}{1+\eta^{\alpha}s^{\alpha}}\right)^{1/2} \tag{31}$$

Similarly, we obtain the formula to calculate the shear stress at the plate.

$$F_p(t)=-\frac{1}{\sqrt{\pi}}\eta^{\frac{\beta-\alpha-1}{2}}t^{\frac{\alpha-\beta}{2}}\cdot\mathrm{H}_{1,2}^{1,1}\left[\eta^{-\alpha}t^{\alpha}\left|\begin{matrix}\left(\frac{1}{2},1\right)\\(0,1),\left(\frac{\beta-\alpha}{2},\alpha\right)\end{matrix}\right.\right] \tag{32}$$

4 DISCUSSION AND RESULTS

4.1 The Relationship between Stress Field and Velocity Field

Substituting (22) into (29), we obtain

$$\bar{F}(y,s)=-\left(\frac{\eta^{\beta-1}s^{\beta}}{1+\eta^{\alpha}s^{\alpha}}\right)^{1/2}\bar{u}(y,s) \tag{33}$$

In the same way, we can easily obtain the Laplace inversion of the first term of the right hand side of Eq.(33)

$$g(t)=\mathrm{L}^{-1}\left\{\left(\frac{\eta^{\beta-1}s^{\beta}}{1+\eta^{\varepsilon}s^{\alpha}}\right)^{1/2},t\right\}=\frac{1}{\sqrt{\pi}}\eta^{\frac{\beta-\alpha-1}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(1/2+n)\left(-\eta^{-\alpha}\right)^n t^{n\alpha-\frac{\beta-\alpha}{2}-1}}{n!\,\Gamma\left[n\alpha-(\beta-\alpha)/2\right]} \tag{34}$$

Using the convolution theorem of the Laplace transform to (33), we have

$$F(y,t) = -\frac{1}{\sqrt{\pi}}\eta^{\frac{\beta-\alpha-1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(1/2+n)(-\eta^{-\alpha})^n}{n!\,\Gamma[n\alpha-(\beta-\alpha)/2]} \int_0^t (t-\tau)^{n\alpha-\frac{\beta-\alpha}{2}-1} u(y,\tau)\mathrm{d}\tau \tag{35}$$

Using the definition of the fractional calculus, (35) can be rewritten as

$$F(y,t) = -\frac{1}{\sqrt{\pi}}\eta^{\frac{\beta-\alpha-1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(1/2+n)(-\eta^{-\alpha})^n}{n!} \cdot \mathrm{D}_t^{-n\alpha+\frac{\beta-\alpha}{2}} u(y,t) \tag{36}$$

The physical meaning of (35) and (36) is that the stress at a given point at any time in the viscoelastic fluid depends on the time history of the velocity profile at that point, and this time history can be depicted by the fractional calculus.

4.2 The Case $\alpha = 0,\ \beta = 1$

In the case of $\alpha = 0$, $\beta = 1$, Eq.(15) may be simplified as

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial y^2} \tag{37}$$

It is obvious that the fluid may be simplified as a viscous Newtonian fluid.

Substituting $\alpha = 0$, $\beta = 1$ into (25), we have

$$u(y,t) = \sum_{n=0}^{\infty} \frac{\left(-\sqrt{2}yt^{-1/2}\right)^n}{n!\,\Gamma(1-n/2)} = W_{-1/2,1}\left(-\sqrt{2}yt^{-1/2}\right) = \mathrm{erfc}\left(-\frac{y}{\sqrt{2t}}\right) \tag{38}$$

in which $W_{\rho,\mu}(z) = \sum\limits_{n=0}^{\infty} \dfrac{z^n}{n!\,\Gamma(\rho n+\mu)}$ $z \in C$, is Wright function[16]. This is the classical Rayleigh's similarity solution of (37). It is obvious that the result of this paper includes the classical Newtonian fluid as a special case.

Similarly, substituting $\alpha = 0$, $\beta = 1$ into (36), we have

$$F(y,t) = -\frac{1}{\sqrt{2}}\mathrm{D}_t^{1/2} u(y,t) \tag{39}$$

Here we used a useful particular value of Gamma function: $\Gamma(1/2) = \sqrt{\pi}$ and the equivalent formula: $1/\sqrt{2} = 1 + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n\Gamma(1/2+n)}{\Gamma(1/2)n!}$. The formula (39) just is the result of Bagley and Torvik[17], which is obtained for Newtonian fluid. Therefore the result of this paper is a generalization of the results given by Bagley and Torvik[17].

4.3 Computing Results

The dimensionless velocity can be evaluated by (25). Figure 1 and Fig.2 are the velocity distributions for several selected parameters α and β, respectively. The curves in Fig.1 and Fig.2 are plotted at the fixed time $t = 4$ and $\eta = 10$. The effect of the fractional orders α and β in the constitutive relationship on the flow field is significant. The greater the α, the more slowly the velocity changes near the plate. The effect of β on the velocity is opposite to that of α. But it seems that in a critical point their effect on velocity will change direction.

Figure 3 is the velocity distribution for several selected parameters η at the fixed time. It represents the relationship between parameter η and velocity.

The shear stress at the plate can be calculated by (32). Figure 4 shows the variation of the shear stress at the plate for various values of time. For small times, there are appreciable viscoelastic effects. For large times the viscoelastic effects become weak. It is clearly seen from Fig.4 that for the values of $(tU^2\rho)/\mu \geq 6$ the fluid behaves as if it was a Newtonian fluid.

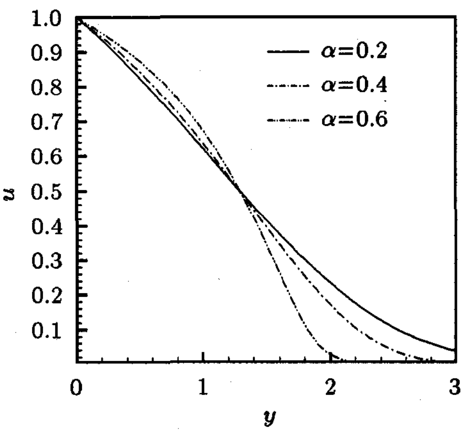

Fig.1 Velocity distribution for three selected parameters α. $t = 4$, $\beta = 0.8$, $\eta = 10$

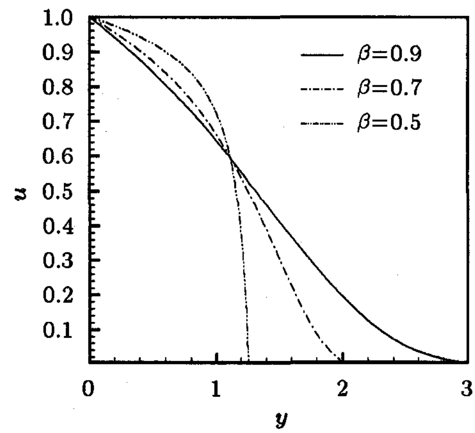

Fig.2 Velocity distribution for three selected parameters β. $t = 4$, $\alpha = 0.5$, $\eta = 10$

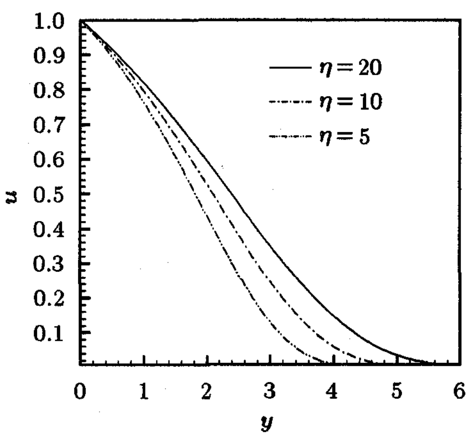

Fig.3 Velocity distribution for three selected parameters η. $t = 4$, $\alpha = 0.4$, $\beta = 0.8$

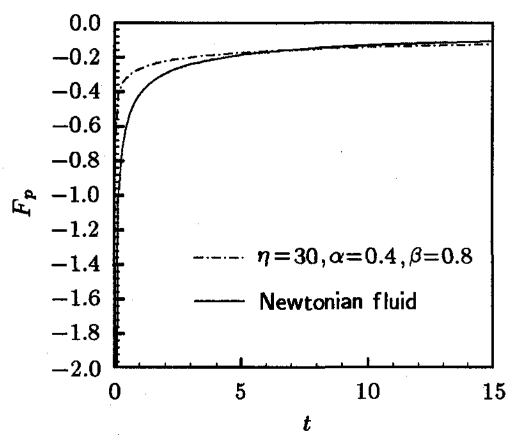

Fig.4 The variation of the shear stress at the plate with respect to time

5 CONCLUSIONS

We have presented here results for the flow field of a non-Newtonian viscoelastic fluid with fractional Maxwell model near a wall suddenly set in motion. Exact solutions of velocity and stress are obtained by using the discrete inverse Laplace transform of the sequential fractional derivatives. The results show that for large times the viscoelastic effects become weak.

The fractional calculus approach in the constitutive relationship model of viscoelastic fluid is introduced. The effect of the fractional orders in constitutive relationship on the flow field is significant. The fractional constitutive relationship model is more useful than the classical Maxwell model for describing the properties of viscoelastic fluid.

The model and the analytical method employed in this paper may be useful in theoretical and engineering analyses.

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