Ukrainian Mathematical Journal, Vol. 50, No. 4, 1998

# STOCHASTIC INTEGRATION AND ONE CLASS OF GAUSSIAN RANDOM PROCESSES

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UDC 519.21

We consider one class of Gaussian random processes that are not semimartingales but their increments can play the role of a random measure. For an extended stochastic integral with respect to the processes considered, we obtain the Itô formula.

## 1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\{w(t);\ t \in [0;\,1]\}$ be a Wiener process on $[0;\,1]$, and let $\mathcal{F} = \sigma(w)$. The main object of investigation is the class $\Gamma$ of Gaussian random processes on the segment $[0;\,1]$ with the following properties:

(i) every element $\gamma \in \Gamma$ is jointly Gaussian with $w$;

(ii) for every $\gamma \in \Gamma$, there exists $c > 0$ such that

$$\forall\, n \geq 1 \quad \forall\, a_1, \ldots, a_n \in \mathbb{R}, \qquad 0 = t_0 < t_1 < \ldots < t_n = 1:$$

$$M\left(\sum_{k=0}^{n-1} a_k(\gamma(t_{k+1}) - \gamma(t_k))\right)^2 \leq c \sum_{k=0}^{n-1} a_k^2 (t_{k+1} - t_k);$$

(iii) for every $\gamma \in \Gamma$ and arbitrary $t \in [\,0;\,1]$,

$$M\gamma(t) = 0, \qquad \gamma(0) = 0.$$

In what follows, the elements of $\Gamma$ are called integrators. This name is explained by the fact that condition (ii) enables one to integrate functions from $\Gamma$ over random processes from $L_2([0;\,1])$. Indeed, according to the Kolmogorov criterion, condition (ii) implies that every element of $\Gamma$ has a continuous modification with probability one. Therefore, the sum

$$\sum_{k=0}^{n-1} a_k(\gamma(t_{k+1}) - \gamma(t_k))$$

can be regarded as the integral $\int_0^1 f\, d\gamma$ of the step function

$$f = \sum_{k=0}^{n-1} a_k \chi_{[t_k;\, t_{k+1})}$$

Institute of Mathematics, Ukrainian Academy of Sciences, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 50, No. 4, pp. 485–495, April, 1998. Original article submitted December 13, 1996.

 0041–5995/98/5004–0550 $20.00 

over the process $\gamma$, and the validity of the inequality enables us to conclude that the mapping

$$f \mapsto \int_0^1 f\,d\gamma \in L_2(\Omega, \mathcal{F}, P)$$

can be extended to a continuous linear operator on the entire space $L_2([0; 1])$. The result of this extension is called an integral over the process $\gamma$. The class $\Gamma$ contains certain martingales and processes with smooth trajectories. At the same time, it also contains some other objects.

***Example 1.*** Let $L \subset L_2([0; 1])$. Denote

$$\mathcal{F}_L = \sigma\left(\int_0^1 \varphi\,dw,\ \varphi \in L\right).$$

Then the process

$$\gamma(t) = M(w(t)/\mathcal{F}_L), \qquad t \in [0; 1],$$

is an integrator. Indeed, it suffices to verify the validity of condition (ii). According to the Jensen inequality, we have

$$M\left(\sum_{k=0}^{n-1} a_k(\gamma(t_{k+1}) - \gamma(t_k))\right)^2 = M\left(\sum_{k=0}^{n-1} a_k M(w(t_{k+1}) - w(t_k)/\mathcal{F}_L)\right)^2$$

$$= M\left(M\left(\sum_{k=0}^{n-1} a_k(w(t_{k+1}) - w(t_k))/\mathcal{F}_L\right)\right)^2$$

$$\le MM\left(\left(\sum_{k=0}^{n-1} a_k(w(t_{k+1}) - w(t_k))\right)^2 /\mathcal{F}_L\right) = \sum_{k=0}^{n-1} a_k^2(t_{k+1} - t_k).$$

In Sec. 2, we consider the internal structure of integrators and, in particular, the presence or absence of a quadratic variation. In Sec. 3, we construct an extended stochastic integral with the use of an integrator and give the Itô formula for this integral.

## 2. Construction of Integrators and Quadratic Variation in the Mean

Here and below, we use the following notation:

$$\forall \varphi \in L_2([0; 1]) \qquad (\varphi; \xi) := \int_0^1 \varphi\,dw,$$

$(\cdot;\cdot)$ and $\|\cdot\|$ are the standard scalar product and norm in $L_2([0; 1])$, and $\xi$ can be regarded as a generalized Gaussian random element in $L_2([0; 1])$ with mean value zero and identity correlation operator [1]. An arbitrary process $\gamma \in \Gamma$ is jointly Gaussian with $\xi$ and, hence, can be represented as follows:

$$\gamma(t) = (g(t); \xi), \qquad t \in [0; 1]. \tag{1}$$

In this case, condition (ii) means that there exists a linear continuous operator $A \in L(L_2([0; 1]))$ such that

$$\forall\, t \in [0; 1] \quad g(t) = A(\chi_{[0;t]}). \tag{2}$$

For instance, in Example 1, the operator $A$ is an orthogonal projector onto a subspace generated by $L$. In turn, it is easy to verify that, by virtue of equalities (1) and (2), any operator $A \in L(L_2([0; 1]))$ can be associated with a random process from $\Gamma$. In particular, taking isometry as $A$, we obtain a new Wiener process as $\gamma$.

Since the processes from $\Gamma$ admit the integration of functions from $L_2([0; 1])$ over these processes, it is natural to expect that these processes possess properties similar to the properties of semimartingales or processes with smooth trajectories. Both semimartingales and processes with smooth trajectories have a quadratic variation. Let $\gamma \in \Gamma$. For the partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of the segment $[0; 1]$, we form the sum

$$S = \sum_{k=0}^{n-1} (\gamma(t_{k+1}) - \gamma(t_k))^2.$$

The corresponding mathematical expectation has the form

$$V = MS = \sum_{k=0}^{n-1} \| g(t_{k+1}) - g(t_k) \|^2.$$

It turns out that, even among projectors, one can find those for which expressions $V$ constructed by a sequence of partitions whose diameter converges to zero have no limit for the corresponding functions $g$.

***Example 2.*** To construct an integrator that does not have a quadratic variation in the mean, we specify an orthonormal basis $\{e_k; k \geq 1\}$ in a subspace the projector onto which would be associated with the required process. For every $n \geq 0$, we denote by $L_n$ the subspace of $L_2([0; 1])$ generated by the family of indicators $\left\{\chi_{[i/2^n; i+1/2^n]}, i = 0, \ldots, 2^n - 1\right\}$. We set $e_1 \equiv 1$ and define vectors $e_2, \ldots, e_{2^{10}+1}$ as follows:

$$e_k = 2^5 \left(\chi_{[(k-2)/2^{10}; (k-2)/2^{10}+1/2^{11})} - \chi_{[(k-2)/2^{10}+1/2^{11}; (k-1)/2^{10})}\right).$$

Analogously, we construct vectors $e_{2^{10}+2}, \ldots, e_{2^{100}+2^{10}_1}$ by partition of the segment $[0; 1]$ into $2^{100}$ equal parts. Continuing this procedure, we obtain an orthonormal sequence of functions possessing the following properties:

$$\forall\, n \geq 0 \quad \forall\, i = 2 + 2^{10} + \ldots + 2^{10^n}, \ldots, 1 + 2^{10} + \ldots + 2^{10^{n+1}}: \; e_i \in L_{10^{n+1}+1}, \qquad e_i \perp L_{10^{n+1}}.$$

Let $A$ be the projector onto the subspace generated by the sequence $\{e_k; k \geq 1\}$ and let $\gamma$ be the integrator corresponding to $A$. Then, for every $n \geq 1$, the expression for $V_n$ constructed for the partition $0 < 1/2^n < \ldots < (2^n - 1)/2^n < 1$ has the form

$$V_n = \sum_{i=0}^{2^n-1} \sum_{k=1}^{\infty} \left( \int_{i/2^n}^{(i+1)/2^n} e_k(s)\, ds \right)^2.$$

Therefore,

$$V_{10} = \sum_{i=0}^{2^{10}-1} \sum_{k=1}^{\infty} \left( \int_{i/2^{10}}^{(i+1)/2^{10}} ds \right)^2 = \frac{1}{2^{10}},$$

$$V_{11} = \sum_{i=0}^{2^{11}-1} \left[ \left( \int_{i/2^{11}}^{(i+1)/2^{11}} ds \right)^2 + 2^{10} \left( \int_{i/2^{11}}^{(i+1)/2^{11}} ds \right)^2 \right] = \frac{1}{2^{11}} + \frac{1}{2},$$

$$V_{100} = \sum_{i=0}^{2^{100}} \left[ \left( \int_{i/2^{100}}^{(i+1)/2^{100}} ds \right)^2 + 2^{10} \left( \int_{i/2^{100}}^{(i+1)/2^{100}} ds \right)^2 \right] = \frac{1}{2^{100}} + \frac{1}{2^{90}},$$

etc. Thus,

$$V_{10^j} \to 0, \quad j \to \infty, \qquad V_{10^j+1} \to \frac{1}{2}, \; j \to \infty.$$

Hence, the sequence $\{ V_n;\, n \geq 1 \}$ does not have a limit and, consequently, the integrator $\gamma$ does not have a quadratic variation in the mean.

Let us consider sufficient conditions for the integrator $\gamma$, in terms of the operator $A$, to have a quadratic variation in the mean.

**Lemma 1.** *If $A$ is a Hilbert–Schmidt operator, then $\gamma$ has a quadratic variation in the mean equal to zero.*

***Proof.*** The operator $A^*A$ is nuclear. Consider the corresponding Riesz representation:

$$A^*A = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k,$$

where $\{ e_k;\, k \geq 1 \}$ is the orthonormal sequence,

$$\lambda_k \geq 0, \quad k \geq 1, \qquad \sum_{k=1}^{\infty} \lambda_k < +\infty.$$

For an arbitrary partition $0 = t_0 < t_1 < \ldots < t_n = 1$, the corresponding expression for $V$ has the form

$$V = M \sum_{i=0}^{n-1} (\gamma(t_{i+1}) - \gamma(t_i))^2 = \sum_{i=0}^{n-1} \left( \sum_{k=1}^{\infty} \lambda_k \left( \int_{t_i}^{t_{i+1}} e_k(s)\,ds \right)^2 \right) = \sum_{k=1}^{\infty} \lambda_k \left( \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} e_k(s)\,ds \right)^2 \right)$$

$$\leq \sum_{k=1}^{\infty} \lambda_k \left( \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} e_k^2(s)\,ds\,(t_{i+1} - t_i) \right) \leq \max_{i=0,\ldots,n-1} (t_{i+1} - t_i) \sum_{k=1}^{\infty} \lambda_k.$$

Therefore, $V$ converges to zero as the diameter of the partition tends to zero. The lemma is proved.

***Remark.*** The statement of Lemma 1 becomes clear if we note that, in this case, the process $\gamma$ has the form

$$\gamma(t) = \int_0^t (A^*\xi)(s)ds, \quad t \in [0; 1],$$

where $A^*\xi$ is an ordinary random element in $L_2([0; 1])$. Thus, in the case under consideration, the quadratic variation of $\gamma$ is equal to zero.

**Lemma 2.** *Let the restriction of the operator $A$ onto the space $C([0; 1])$ be a continuous linear operator that maps $C([0; 1])$ into itself. Then the corresponding process $\gamma$ has a quadratic variation in the mean.*

***Proof.*** It follows from the conditions of the lemma that the operator $A$ can be associated with the function $\mu\colon [0; 1] \times \mathcal{B}([0; 1]) \to \mathbb{R}$ ($\mathcal{B}([0; 1])$) that is the $\sigma$-algebra of Borel subsets of the segment $[0; 1]$ such that

(i) for any $t \in [0; 1]$, $\mu(t, \cdot)$ is a finite charge on $\mathcal{B}([0; 1])$;

(ii) for any $\Delta \in \mathcal{B}([0; 1])$, $\mu(\cdot, \Delta)$ is a Borel function on $[0; 1]$;

(iii) for every $f \in C([0; 1])$ and $t \in [0; 1]$,

$$(Af)(t) = \int_0^1 f(s)\mu(t, ds).$$

Since the operator $A$ is continuous on $L_2([0; 1])$, it follows from Lebesgue's dominated convergence theorem that

$$\forall\, t \in [0; 1] \quad \left(A\chi_{[0; t]}\right)(\tau) = \mu(\tau, [0; t]) \quad (\mathrm{mod}\, \lambda),$$

where $\lambda$ is the Lebesgue measure. Therefore, for any partition $0 = t_0 < t_1 < \ldots < t_n = 1$, the corresponding expression for $V$ has the form

$$V = \sum_{i=0}^{n-1} \left\| A\chi_{[t_i; t_{i+1}]} \right\|^2 = \sum_{0}^{n-1} \int_0^1 \mu(\tau, [t_i; t_{i+1}])^2 d\tau = \int_0^1 \sum_{i=0}^{n-1} \mu(\tau, [t_i; t_{i+1}])^2 d\tau.$$

Since, by condition, there exists a number $c$ such that

$$\forall\, t \in [0; 1] \quad |\mu|(\tau, [0; 1]) \le c,$$

where $|\mu|$ is the variation of the charge $\mu$, we conclude that $V$ has the following limit as the diameter of partition tends to zero:

$$\int_0^1 \sum_{t_{\tau k}} \mu\left(\tau, \{t_{\tau k}\}\right)^2 d\tau.$$

Here, for every $\tau$, $\{ t_{\tau k};\, k \ge 1 \}$ is the set of all atoms of the charge $\mu(\tau, \cdot)$ (this set is always at most countable). The lemma is proved.

Note that the conditions of Lemma 2 are satisfied by integrators of various types.

***Example 3.*** The integrator

$$\gamma(t) = \int_0^t w(s)\,ds, \qquad t \in [\,0;\,1\,],$$

is associated with the operator $A$ acting according to the rule

$$\forall\, \varphi \in L_2([0;1]) \qquad (A\varphi)(t) = \int_0^1 \varphi(s)\,ds.$$

Therefore, $\gamma$ satisfies the conditions of Lemma 2, and the corresponding charges are the following:

$$\mu(\tau, \Delta) = \lambda([\tau;\,1\,] \cap \Delta), \qquad \tau \in [\,0;\,1\,], \qquad \Delta \in \mathcal{B}([0;1]).$$

Thus, as expected, the quadratic variation in the mean of the process $\gamma$ is equal to zero. Similarly, one can verify that the Wiener process is associated with the collection of charges

$$\mu(\tau, \cdot) = \delta_\tau, \qquad \tau \in [\,0;\,1\,],$$

and the martingale $w(t) - 2tw(1)$, $t \in [\,0;\,1\,]$ (with filtration different from the Wiener filtration) is associated with the collection of charges

$$\mu(\tau, \cdot) = \delta_\tau - 2\tau\lambda, \qquad \tau \in [\,0;\,1\,].$$

## 3. Extended Stochastic Integral and the Itô Formula for Integrators

Recall the necessary information on an extended stochastic integral over a Wiener process. It is now more convenient to reason in terms of a generalized random element $\xi$, replacing integrals of nonrandom functions over $w$ by a scalar product with $\xi$. An arbitrary random element $x$ in $L_2([0;1])$ that has a finite second moment of the norm can be uniquely represented as the sum [1]

$$x = \sum_{k=0}^{\infty} T_k(\xi, \ldots, \xi). \tag{*}$$

Here, $\{T_k(\xi, \ldots, \xi), k \geq 0\}$ are multiple Wiener integrals with symmetric kernels taking values in $L_2([0;1])$ over the corresponding powers of $[0;1]$, and $T_k$, $k \geq 1$, are interpreted as $k$-linear symmetric Hilbert–Schmidt forms on $L_2([0;1])$ with values in $L_2([0;1])$. For every $k \geq 0$, on $L_2([0;1])$ we consider the $(k+1)$-linear Hilbert–Schmidt form

$$S_{k+1}(\varphi_1, \ldots, \varphi_{k+1}) = (\varphi_1; T_k(\varphi_2, \ldots, \varphi_{k+1})),$$

$$\varphi_1, \ldots, \varphi_{k+1} \in L_2([0;1]).$$

Let $\wedge S_{k+1}$ be the symmetrization of $S_{k+1}$ over all arguments.

***Definition 1*** [1]. *An extended stochastic integral of* $x$ *is defined as the sum of the series*

$$\sum_{k=0}^{\infty} \wedge S_{k+1}(\xi, \ldots, \xi)$$

*provided that this series converges in mean square.*

Denote

$$\int_0^1 x(s)\,dw(s) = (x; \xi).$$

In the case where $x$ is a random process consistent with a flow of $\sigma$-algebras generated by a Wiener process, the extended stochastic integral of $x$ coincides with the Itô integral.

In addition to an extended stochastic integral, we need the notion of stochastic derivative, which can also be defined with the use of expansion (*). For a random variable $\alpha$ with finite second moment, we consider its representation in the form of a sum of multiple Wiener integrals:

$$\alpha = \sum_{k=0}^{\infty} R_k(\xi, \ldots, \xi).$$

***Definition 2*** [1]. *A random element* $\zeta$ *in* $L_2([0;1])$ *with finite second moment is called a stochastic derivative of the random variable* $\alpha$ *if, for any* $\varphi \in L_2([0;1])$, *the series*

$$\sum_{k=0}^{\infty} k R_k(\varphi, \xi, \ldots, \xi)$$

*converges in mean square and*

$$(\zeta; \varphi) = \sum_{k=0}^{\infty} k R_k(\varphi, \xi, \ldots, \xi) \quad (\operatorname{mod} P).$$

Denote $\zeta = D\alpha$.

The stochastic derivative of a random element in a Hilbert space and higher derivatives are defined in a similar way. We denote by $W^k(H)$ (if $H = \mathbb{R}$, we simply write $W^k$) the collection of all random elements in a Hilbert space $H$ that have the stochastic derivative of order $k$. At present, there are many works where the properties of extended stochastic integrals and stochastic derivatives and methods for their determination in Gaussian and more general cases are studied (see [2–5] and the bibliography therein). Here, we present several relations necessary for what follows (see, e.g., [1]).

If $x \in W^1(L_2([0;1]))$, then the extended stochastic integral of $x$ is defined and

$$M(x;\xi) = 0, \qquad M(x;\xi)^2 = M\|x\|^2 + M\operatorname{tr}(Dx)^2. \tag{3}$$

In (3), we is taken into account that the stochastic derivative of a random element $x \in W^1(L_2([0;1]))$ is a random Hilbert–Schmidt operator. If a random variable $\alpha \in W^1$ and a random element $x$ in $L_2([0;1])$ are such that $x$

and $\alpha x$ belong to the domain of definition of the extended stochastic integral and $\alpha(x;\xi)$ has a finite second moment, then

$$(\alpha x;\xi) = \alpha(x;\xi) - (x; D\alpha). \tag{4}$$

Formulas (3) and (4) yield the following property of extended stochastic integrals:

**Proposition 1.** *Let* $x \in W^1(L_2([0;1]))$ *and let* $K$ *be a Hilbert–Schmidt operator in* $L_2([0;1])$. *Then* $Kx \in W^1(L_2([0;1]))$ *and*

$$(Kx;\xi) = (x; K^*\xi) - \operatorname{tr} K^* Dx. \tag{5}$$

The validity of (5) easily follows from relation (4) for a finite-dimensional operator $K$, and the possibility of passing to the limit is guaranteed by (3).

In a similar way, we can obtain the following statement about the approximation of an extended stochastic integral, which is true not only in the Gaussian case [5]:

**Proposition 2.** *Let* $\{K_n; n \ge 1\}$ *be a sequence of Hilbert–Schmidt operators in* $L_2([0;1])$ *with kernels* $\{h_n; n \ge 1\}$ *strongly convergent to the identity operator. Then for arbitrary* $x \in W^1(L_2([0;1]))$, *the following equality holds:*

$$\int_0^1 x\,dw = \lim_{n\to\infty}\left(\int_0^1 x(s)\int_0^1 h_n(s,\tau)\,dw(\tau)\,ds - \int_0^1\int_0^1 (Dx(s))(\tau)h_n(s,\tau)\,d\tau\,ds\right),$$

*where the convergence is understood in the mean-square sense.*

Now let $\gamma \in \Gamma$ and let $A$ be the corresponding continuous linear operator.

***Definition 3.*** *A random element* $x$ *in* $L_2([0;1])$ *with finite second moment belongs to the domain of definition of the extended stochastic integral over* $\gamma$ *if* $Ax$ *belongs to the domain of definition of the extended stochastic integral over* $w$ *and*

$$\int_0^1 x\,d\gamma := (Ax;\xi).$$

Let us consider several examples.

***Example 4.*** Let

$$\gamma(t) = w(1)\cdot t, \qquad t \in [0;1].$$

In this case, the corresponding operator $A$ acts as follows:

$$\forall\, \varphi \in L_2([0;1])$$

$$(A\varphi)(t) = \int_0^1 \varphi(s)\,ds, \qquad t \in [0;1].$$

Therefore, for a random element $x$ in $L_2([0;1])$ that has a finite second moment, the stochastic integral over $\gamma$ is defined if the random process identically equal to

$$\int_0^1 x(s)\,ds$$

belongs to the domain of definition of the extended stochastic integral over $w$. In this case,

$$\int_0^1 x\,d\gamma = \int_0^1\left(\int_0^1 x(s)\,ds\right)dw.$$

In particular, if $x \in W^1(L_2([0;1]))$, then

$$\int_0^1 x\,d\gamma = \int_0^1 x(s)\,ds\cdot w(1) - \iint_0^1 Dx(s,\tau)\,ds\,d\tau.$$

Therefore, in the case where $\gamma$ has smooth trajectories, the stochastic integral over $\gamma$ does not coincide with the Stieltjes integral.

The next example shows that integrators and stochastic integrals over them appear in the course of the solution of problems of filtration for ordinary stochastic differential Itô equations.

***Example 5.*** Let $L \subset L_2([0;1])$ and let $\mathcal{F}_L$ be constructed in the same way as in Example 1. A random element $x$ in $L_2([0;1])$ that belongs to the domain of definition of the extended stochastic integral over $w$ satisfies the equality

$$M\left(\int_0^1 x(s)\,dw(s)/\mathcal{F}_L\right) = \int_0^1 M(x(s)/\mathcal{F}_L)\,d\gamma(s),$$

where, on the right-hand side, we have the stochastic integral of the process $\{M(x(s)/\mathcal{F}_L), s\in[0;1]\}$ over the integrator $\{\gamma(s) = M(w(s)/\mathcal{F}_L),\ s\in[0;1]\}$. One can verify this relation by using expansion (1) and the fact that multilinear forms of $\xi$ (i.e., multiple Wiener integrators) satisfy the equality

$$M(S_k(\xi,\ldots,\xi)/\mathcal{F}_L) = S_k(P_L\xi, P_L\xi,\ldots,P_L\xi),$$

where $P_L$ is the orthogonal projector in $L_2([0;1])$ onto the linear subspace generated by $L$.

We now write the Itô formula for processes from $\Gamma$.

**Theorem.** *Let $\gamma \in \Gamma$ and let an operator $A$ satisfy one of the following conditions:*

*(i) $A$ is a Hilbert–Schmidt operator,*

*(ii)* $\exists c>0\colon \forall t\in[0;1]\ \ \forall\varphi\in L_2([0;1])\cap C([0;t])$:

$$AA^*(\varphi) \in C([0;t]),$$

$$\max_{[0;t]} |AA^*(\varphi)| \le c \max_{[0;t]} |\varphi|.$$

*Then for any twice differentiable function* $F\colon [0;1] \times \mathbb{R} \to \mathbb{R}$ *with bounded derivatives, the equalities*

$$F(t,\gamma(t)) = F(0,0) + \int_0^t F_1'(s,\gamma(s))ds + \int_0^t F_2'(s,\gamma(s))d\gamma(s) + \frac{1}{2}\operatorname{tr} A^*\Psi_T A, \quad s \in [0;1], \tag{6}$$

*and*

$$F(t,\gamma(t)) = F(0,0) + \int_0^t F_1'(s,\gamma(s))ds + \int_0^t F_2'(s,\gamma(s))d\gamma(s) + \frac{1}{2}\int_0^t AA^*(F_{22}''(s\vee\cdot,\gamma(s\vee\cdot)))(s)\,ds \tag{6a}$$

*are true in cases (i) or (ii), respectively. Here,* $\Psi_t$ *is the integral operator in* $L_2([0;1])$ *with the kernel*

$$\chi_{[0;t]}(s)\chi_{[0;t]}(r)\cdot F_{22}''(s\vee r,\gamma(s\vee r)), \qquad s,r \in [0;1].$$

***Proof.*** For a nonnegative even function $h \in C^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} h(x)dx = 1,$$

we set $h_n(s,\tau) = nh(n(r-s))$, $\tau, s \in [0;1]$, $n \ge 1$. Then the integral operators $\{K_n;\, n \ge 1\}$ corresponding to the kernels $\{h_n;\, n \ge 1\}$ in $L_2([0;1])$ satisfy the conditions of Proposition 2. Moreover, for every $n \ge 1$, $K_n$ is a continuous linear operator in $C([0;1])$. For $n \ge 1$, we consider the new random process

$$\gamma_n(t) = \int_0^t \int_0^1 h_n(s,\tau)d\gamma(\tau)ds, \qquad t \in [0;1].$$

Since

$$\gamma_n(t) = \int_0^t n\int_0^1 h(n(\tau-s))d\gamma(\tau)ds = \int_0^t n\left(h(n(1-s))\gamma(1) - n\int_0^1 h'(n(\tau-s))\gamma(\tau)d\tau\right)ds$$

$$= \gamma(1)\,n\int_0^t h(n(1-s))ds - n^2\int_0^1\left(\int_0^t h'(n(\tau-s))ds\right)\gamma(\tau)d\tau$$

$$= \gamma(1)\,n\int_0^t h(n(1-s))ds + n\int_0^1 h(n(\tau-t))\gamma(\tau)d\tau - n\int_0^1 h(n\tau)\gamma(\tau)d\tau \qquad (\operatorname{mod} P),$$

$\gamma_n$ converges uniformly on $[0;1]$ to $\gamma$ as $n \to \infty$ with probability one. Therefore, for any $t \in [0;1]$,

$$F(t, \gamma_n(t)) = \lim_{n\to\infty} F(t, \gamma_n(t)) \qquad (\mathrm{mod}\, P).$$

According to the Newton–Leibniz formula, we have

$$F(t, \gamma_n(t)) = F(0,0) + \int_0^t F_1'(s, \gamma_n(s))ds + \int_0^t F_2'(s, \gamma_n(s))\int_0^1 h_n(s,\tau)d\gamma(\tau)ds.$$

We transform the second term as follows:

$$\int_0^t F_2'(s, \gamma_n(s))\int_0^1 h_n(s,\tau)d\gamma(\tau)ds$$

$$= \int_0^t F_2'(s, \gamma_n(s))\int_0^1 h_n(s,\tau)d\gamma(\tau)ds - \int_0^t\int_0^1 DF_2'(s,\gamma_n(s))(\tau)A^*(h_n(s,\cdot))(\tau)d\tau ds$$

$$+ \int_0^t\int_0^1 DF_2'(s,\gamma_n(s))(\tau)A^*(h_n(s,\cdot))(\tau)d\tau ds$$

$$= \int_0^1 K_n^*\big(F_2'(\cdot,\gamma_n(\cdot))\chi_{[0;t]}\big)(\tau)d\gamma(\tau) + \int_0^t\int_0^1 DF_2'(s,\gamma_n(s))(\tau)A^*(h_n(s,\cdot))(\tau)d\tau ds.$$

Under the assumptions made for the functions $h$ and $F$, the random elements $K_n^*(F_2'(\cdot,\gamma_n(\cdot))\chi)[0;t])$, $n \ge 1$, are stochastically differentiable; as $n\to\infty$, they converge in mean square, together with their stochastic derivatives, to the random element $F_2'(\cdot,\gamma_n(\cdot))\chi_{[0;t]}$. Therefore,

$$\lim_{n\to\infty}\int_0^1 K_n^*\big(F_2'(\cdot,\gamma_n(\cdot))\chi_{[0;t]}(\cdot)\big)(\tau)d\gamma(\tau) = \int_0^1 F_2'(s,\gamma(s))\chi_{[0;t]}(s)d\gamma(s) = \int_0^t F_2'(s,\gamma(s))d\gamma(s),$$

where the convergence is understood in the mean-square sense. We now consider

$$\int_0^t\int_0^1 DF_2'(s,\gamma_n(s))(\tau)A^*(h_n(s,\cdot))(\tau)d\tau ds$$

$$= \int_0^t F_{22}''(s,\gamma_n(s))\int_0^s\left(\int_0^1 A^*(h_n(r,\cdot))(\tau)A^*(h_n(s,\cdot))(\tau)d\tau\right)dr ds$$

$$= \frac{1}{2}\iint_0^t F_{22}''(s\vee r,\gamma_n(s\vee r))\left(\int_0^1 A^*(h_n(r,\cdot))(\tau)A^*(h_n(s,\cdot))(\tau)d\tau\right)ds dr.$$

In this case, the statement of the theorem follows from Lebesgue's dominated convergence theorem in case (ii), and from the properties of nuclear operators in case (i) (see [6]). The theorem is proved.

***Example 6.*** Let $\gamma = w$ be a Wiener process. In this case, $A$ is the identity operator. Condition (ii) is satisfied, and relation (6a) turns into the standard Itô formula for Wiener processes.

***Remark.*** Note that if condition (ii) is satisfied, then the process $\gamma$ has differentiable trajectories with probability one. However, there is a term containing the second derivative of $F$ because the integral over $\gamma$ is an extended stochastic integral.

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