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Abstract commutative ideal theory without chain condition

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A multiplicative lattice is a complete lattice L on which there has been defined a commutative, associative multiplication which distributes over arbitrary joins (i.e., $A(\bigvee_\alpha B_\alpha) = \bigvee_\alpha AB_\alpha$) and has greatest element I as a multiplicative identity. In [16], M. Ward and R. P. Dilworth extended the Noether decomposition theory to suitably defined multiplicative lattices; however, further development was not possible because of the lack of a proper abstraction of principal ideals. In [7], Dilworth defined such a principal element and extended the Krull Intersection Theorem and Principal Ideal Theorem to what he called a Noether lattice. A Noether lattice is an abstraction of the lattice of ideals of a Noetherian commutative ring. For the development of the theory of Noether lattices, the reader is referred to [1], [5]–[7], and [10]. In this paper we introduce the r-lattice as an abstraction of the lattice of ideals of a commutative ring. An r-lattice is a special type of compactly generated multiplicative lattice. It is part of the folklore of the subject that compactly generated lattices can replace lattices with ACC for many aspects of the subject including the theory of localization. We make this statement precise.

In the first section we study principal elements in some detail. We show, for example, under quite mild hypothesis that a weak meet principal element is principal. r-lattices are introduced in the second section. Some basic results for r-lattices are established and a theory of localization is developed. In the third section, distributive r-lattices are studied. It is shown that any quasi-local distributive r-lattice is the lattice of ideals of a very special type of commutative monoid with zero. This result is used to show that any distributive r-lattice may be embedded in the lattice of ideals of a commutative ring. It is also shown that a distributive r-lattice domain is representable as the lattice of ideals of a commutative ring if and only if it satisfies the weak union condition. In the fourth section, r-lattices in which the principal elements are products of prime elements are studied.

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1. Principal elements

Let L be a multiplicative lattice and for A, $B \in L$ let $(A:B) = \bigvee \{X \in L \mid XB \leq A\}$. Following Dilworth [7] we define $M \in L$ to be meet (join) principal if $A \wedge MB = M((A:M) \wedge B)$ $(A \vee (B:M) = (AM \vee B:M))$. M is weak meet (join) principal if $A \wedge M = M(A:M)$ $(A \vee (0:M) = (MA:M))$ for all $A \in L$. Finally, M is said to be (weak) principal if M is both (weak) meet and (weak) join principal.

For L a multiplicative lattice and $A \in L$, $L/A = \{B \in L \mid B \geq A\}$ is a multiplicative lattice with multiplication $C \circ D = CD \vee A$. The following results about principal elements are well-known and easy to establish (see [1], [5]–[7], and [10]).

PROPOSITION 1.1. *Let L be a multiplicative lattice and let $M \in L$. Then*
(1) *M meet (join) principal implies M is weak meet (join) principal,*
(2) *M is weak meet principal if and only if $A \leq M$ implies $A = MC$ for some $C \in L$,*
(3) *M is weak join principal if and only if $AM \leq BM$ implies $A \leq B \vee (0:M)$,*
(4) *M is join principal if and only if $M \vee A$ is weak join principal in L/A for all $A \in L$,*
(5) *if L is modular, then M (weak) meet principal implies $M \vee A$ is (weak) meet principal in L/A,*
(6) *if L is modular, then M is principal if and only if it is weak principal.*

A multiplicative lattice will be called quasi-local if it has a unique maximal element ($\neq I$). The first theorem shows that under quite general circumstances a weak meet principal element is join irreducible.

THEOREM 1.2. *Let (L, M) be a quasi-local multiplicative lattice. Assume every element of L is a join of weak join principal elements. Then every weak meet principal element is join irreducible. In particular, if L is generated by (weak) principal elements, then the following are equivalent:*
(1) *K is (weak) principal*
(2) *K is weak meet principal, and*
(3) *K is join irreducible.*

Proof. Let $K \in L$ be weak meet principal. We may assume $K \neq 0$. Let $K = \bigvee A_\alpha$. Then $A_\alpha \leq K$ and K weak meet principal gives $A_\alpha = K(A_\alpha : K)$, so $K = \bigvee A_\alpha = \bigvee (K(A_\alpha : K)) = K(\bigvee (A_\alpha : K))$. If $\bigvee (A_\alpha : K) = I$, then since L is quasi-local some $(A_{\alpha_0} : K) = I$ and so $K = A_{\alpha_0}$. Hence we may assume $\bigvee (A_\alpha : K) \leq M$ so $K = MK$. Let $0 \neq A \leq K$ be weak join principal. Then $A = K(A:K) =$

$(MK)(A:K)=M(K(A:K))=MA$. But A is weak join principal, so $I \leq M \vee (0:A)$, and hence $A=0$ since L is quasi-local. Thus K is join irreducible. The second statement of the theorem is now immediate.

Recall that an element A of a multiplicative lattice is compact if $A \leq \bigvee B_\alpha$ implies $A \leq B_{\alpha_1} \vee \cdots \vee B_{\alpha_n}$ for some subset $\{\alpha_1, \ldots, \alpha_n\}$. The following result [9] will prove useful.

THEOREM 1.3. *Let L be a multiplicative lattice in which I is compact. Suppose $A \in L$ is weak principal. Then $A = \bigvee A_\alpha$ implies $A = A_{\alpha_1} \vee \cdots \vee A_{\alpha_n}$ for some finite subset $\{\alpha_1, \ldots, \alpha_n\}$.*

Proof. Now A weak meet principal implies $A_\alpha = A(A_\alpha : A)$, hence $A = \bigvee A_\alpha = \bigvee (A(A_\alpha : A)) = A(\bigvee (A_\alpha : A))$. But A is also weak join principal so $I = \bigvee (A_\alpha : A) \vee (0:A)$. I is compact so $I = (A_{\alpha_1} : A) \vee \cdots \vee (A_{\alpha_n} : A) \vee (0:A)$ for some finite subset $\{\alpha_1, \ldots, \alpha_n\}$. Hence $A = A(A_{\alpha_1} : A) \vee \cdots \vee A(A_{\alpha_n} : A) \vee A(0:A) = A_{\alpha_1} \vee \cdots \vee A_{\alpha_n}$.

The next theorem is an abstract version of Nakayama's Lemma.

THEOREM 1.4. *Let (L, M) be a quasi-local multiplicative lattice and suppose B is a finite join of join principal elements. Then for C and $D \neq I$, $B \leq C \vee DB$ implies $B \leq C$. In particular, $MB = B$ implies $B = 0$.*

Proof. By passing to L/C it suffices to prove $MB = B$ implies $B = 0$. Suppose $B \neq 0$ and let $B = A_1 \vee \cdots \vee A_n$ be a minimal representation of B as a finite join of join principal elements of L. Now $B = MB$ gives $A_1 \vee \cdots \vee A_n = MA_1 \vee A_2 \vee \cdots \vee A_n$. Let $E = A_2 \vee \cdots \vee A_n$, then in L/E $A_1 \vee E = (M \vee E) \circ (A_1 \vee E)$. But since $A_1 \vee E$ is join principal in L/E, we get $A_1 \vee E \leq E$. Hence $A_1 \leq E = A_2 \vee \cdots \vee A_n$, so $B = A_2 \vee \cdots \vee A_n$. This contradiction shows that $B = 0$.

For results on join principal elements in Noether lattices, see [4], [11], and [13]. In the general case we have

THEOREM 1.5. *Let (L, M) be a quasi-local multiplicative lattice and suppose $J \in L$ is join principal and is a finite join of principal elements each of which has zero annihilator. Then J is principal. In particular, if J is join principal and is finite join of principal elements, then $J \vee P$ is principal in L/P for every prime P of L.*

Proof. Let $J = A_1 \vee \cdots \vee A_n$ be a minimal representation of J as a finite join of principal elements with $0 = (0:A_1) = \cdots = (0:A_n)$. Assume $n > 1$ and put $C = A_2 \vee \cdots \vee A_n$. Now $(0:A) = 0$, so

$$\begin{aligned} A &= (A^2:A) = (AC \vee A_1^2:A) = C \vee (A_1^2:A) = C \vee (A_1^2:A_1 \vee C) \\ &= C \vee ((A_1^2:A_1) \wedge (A_1^2:C)) = C \vee (A_1 \wedge (A_1^2:C)) = C \vee A_1((A_1^2:C):A_1). \end{aligned}$$

If $((A_1^2:C):A_1) \neq I$, then by Nakayama's Lemma we get $A = C$, a contradiction and hence A is principal. We may suppose $((A_1^2:C):A_1) = I$ so $A_1 \leq (A_1^2:C)$ and hence $A_1C \leq A_1^2$. Since A_1 is weak join principal and $(0:A_1) = 0$ we get $C \leq A_1$. This contradiction proves that A is principal.

2. r-lattices

A multiplicative lattice L is called an r-lattice if it is modular, principally generated, compactly generated (every element is a join of compact elements) and has I compact. Thus an r-lattice is a generalization of a Noether lattice, in fact, a Noether lattice is just an r-lattice satisfying ACC. For any commutative ring R, $L(R)$, the lattice of ideals of R is an r-lattice. More generally, if R is a graded ring over a torsionless grading monoid, then $L_G(R)$, the lattice of graded ideals of R, is an r-lattice. In section three, we shall see that $L(S)$, the lattice of ideals of an r-semigroup is also an r-lattice.

A key result is that a principal element of an r-lattice is compact.

THEOREM 2.1. *Let L be an r-lattice and let S be a set of principal elements which generates L under joins. Then $A \in L$ is compact if and only if it is a finite join of elements of S.*

Proof. If A is compact, then clearly A is a finite join of elements of S. Conversely, let B be principal and let $B = \bigvee B_\alpha$ where B_α is compact. Then by Theorem 1.3, $B = B_{\alpha_1} \vee \cdots \vee B_{\alpha_n}$. Since a finite join of compact elements is compact, we have shown that any principal element is compact. Thus any finite join of elements of S is compact.

Thus in an r-lattice, the product of two compact elements is compact. A non-empty subset S of compact elements is called sub-multiplicatively closed if for any A, $B \in S$, there exists an element C in S with $C \leq AB$. If actually $AB \in S$, we simply say that S is multiplicatively closed. The following theorem allows us to construct prime elements.

THEOREM 2.2. *Let L be an r-lattice and let S be a sub-multiplicatively closed subset of L. Suppose $A \in L$ and $T \not\leq A$ for every $T \in S$. Then there exists an element*

$B \geq A$ maximal with respect to the property that $T \not\leq B$ for every $T \in S$. Further, any such B is prime.

Proof. By Zorn's Lemma, such a B exists. Suppose B is not prime, then there exist, $C, D \in L$ with $CD \leq B$ but $C \not\leq B$ and $D \not\leq B$. Then $C \vee B > B$ and $D \vee B > B$ so there exist $T_1, T_2 \in S$ with $T_1 \leq C \vee B$ and $T_2 \leq D \vee B$. Since S is sub-multiplicatively closed, there exists $U \in S$ with $U \leq T_1T_2$. Hence $U \leq T_1T_2 \leq (C \vee B)(D \vee B) \leq CD \vee B \leq B$, a contradiction.

Let L be an r-lattice and let $A \in L$. A is called a zero-divisor if $(0:A) \neq 0$. We define $\sqrt{A} = \bigvee \{X \in L \text{ principal} \mid X^n \leq A \text{ for some integer } n\}$. The next two theorems use Theorem 2.2.

THEOREM 2.3. *Suppose L is an r-lattice, then any minimal prime P of L consists of zero divisors, that is, any compact element contained in P is a zero divisor.*

Proof. Let S denote the set of elements AB where $A \not\leq P$ is compact and B is a principal non-zero divisor of L. Then S is multiplicatively closed and $0 \notin S$. By Theorem 2.2, 0 can be enlarged to a prime element Q with $C \not\leq Q$ for every $C \in S$. Now $Q \leq P$ so $Q = P$ by minimality and clearly every compact element contained in $Q = P$ is a zero divisor.

THEOREM 2.4. *Suppose L is an r-lattice and $A \in L$. Then $\sqrt{A} = \bigwedge \{P \in L \mid P \geq A \text{ is a prime minimal over } A\}$.*

Proof. Since any prime containing A can be shrunk to a prime minimal over A (Zorn's Lemma), it suffices to show $\sqrt{A} = \bigwedge \{P \in L \mid P \geq A \text{ is prime}\}$. Of course, $\sqrt{A} \leq \bigwedge \{P \in L \mid P \geq A \text{ is prime.}\}$. Suppose X is principal with $X \not\leq \sqrt{A}$, then $X^n \not\leq A$ for every integer $n > 0$. Let S be the multiplicatively closed set $\{X^n\}_{n=1}^{\infty}$. By Theorem 2.2, there exists a prime element P_0 with $P_0 \geq A$ but $X^n \not\leq P_0$ for every $n > 0$. Thus $\sqrt{A} = \bigwedge \{P \in L \mid P \geq A \text{ is prime}\}$.

The next theorem is a generalization of Cohen's theorem which states that a commutative ring is Noetherian if every prime ideal is finitely generated.

THEOREM 2.5. *An r-lattice L satisfies ACC if and only if every prime element is compact.*

Proof. Suppose every prime element of L is compact. Let $S = \{B \in L \mid B \text{ is not compact}\}$. Assume $B \neq \varnothing$. By Zorn's Lemma, S contains a maximal element P. If P is prime, the it is compact. Suppose not, then there exist principal elements

$A, B \in L$ with $AB \leq P$ but $A \not\leq P$ and $B \not\leq P$. Now $P \vee A > P$ so $P \vee A$ is compact. Put $J = (P:A)$, then $J \geq B$ and $J \geq P$ so $J > P$ is compact. Now A is principal, so $P \wedge A = A(P:A) = AJ$ and hence is compact. Since L is modular $P/P \wedge A \cong P \vee A/A$. Therefore $P = T \vee (P \wedge A)$ where T is compact. Thus P is compact.

An important property of r-lattices is that they can be localized at sub-multiplicatively closed sets. Previously localizations for Noether lattices had been defined using primary decomposition [7].

Suppose L is an r-lattice and S is a sub-multiplicatively closed set of L. We define $A \leq B\ (S)$ for $A, B \in L$ if for every principal element $X \leq A$, there exists $T \in S$ such that $TX \leq B$ and $A = B(S)$ if $A \leq B(S)$ and $B \leq A(S)$. Using the fact that S is sub-multiplicatively closed (and hence consists of compact elements) it is easily seen that $=(S)$ is an equivalence relation. For $A \in L$, define $A_S = \{B \in L \mid B = A(S)\}$ and let L_S be the set of equivalence classes of elements of L. L_S is a partially ordered set with the partial order $A_S \leq B_S$ if and only if $A \leq B(S)$.

The following proposition, whose proof will be omitted, shows that L_S is again an r-lattice.

PROPOSITION 2.6. *Let L be an r-lattice and S a sub-multiplicatively closed set. Then*

(1) *for any set $\{A_\alpha\} \subseteq L$, $(\bigvee A_\alpha)_S = \bigvee A_{\alpha S}$,*

(2) *$(A_1 \wedge \cdots \wedge A_n)_S = A_{1S} \wedge \cdots \wedge A_{nS}$ for any finite subset of L,*

(3) *the product $A_S B_S = (AB)_S$ makes L_S a multiplicative lattice,*

(4) *for $A, C \in L$ with C compact $(A:C)_S = (A_S : C_S)$,*

(5) *M principal in L implies M_S is principal in L_S,*

(6) *A compact in L implies A_S is compact in L_S.*

Consider the map $\theta: L \to L_S$ defined by $\theta(A) = A_S$. Then θ is a multiplicative lattice homomorphism which further satisfies $\theta(\bigvee A_\alpha) = \bigvee \theta(A_\alpha)$. For $B \in L_S$, we define $\theta^{-1}(B) = \bigvee \{X \in L \mid \theta(X) \leq B\}$. As for rings, we have the following result, the proof of which is rather similar to the ring case.

THEOREM 2.7. *Let L be an r-lattice and S a sub-multiplicatively closed set. Then*

(1) *for $B \in L_S$, $(\theta^{-1}(B))_S = B$,*

(2) *there is a one-to-one correspondence between the primes of L_S and those primes P of L for which $T \not\leq P$ for every $T \in S$.*

Let L be an r-lattice and P a prime element of L. Let S be the multiplicatively closed set consisting of all principal elements $X \not\leq P$. Then L_S has P_S as its unique maximal element and the prime elements of L_S are in one-to-one correspondence with the prime elements of L contained in P. For such localizations we use the notation L_P. The next theorem will be very useful.

THEOREM 2.8. *Let L be an r-lattice and let $J, K \in L$. Then $J = K$ if and only if $J_M = K_M$ for every maximal element M of L.*

Proof. If $J = K$, then $J_M = K_M$ for every maximal element M. Conversely, suppose $J_M = K_M$ for every maximal element of L. Suppose $J \not\leq K$, so there exists a principal element $E \leq J$, but with $E \not\leq K$. Now $E_M \leq J_M = K_M$ so $(K:E)_M = (K_M:E_M) = I_M$ for every maximal element M since E is compact. Thus $(K:E)$ is contained in no proper maximal element, hence $(K:E) = I$. Hence $E \leq K$, a contradiction. Thus $J \leq K$.

We are now in a position to characterize principal elements in r-lattices.

THEOREM 2.9. *In a quasi-local r-lattice L, for $J \in L$,* the following are equivalent

(1) *J is principal,*

(2) *J is weak meet principal,*

(3) *J is join irreducible.*

In an r-lattice L, an element J is principal if and only if J is compact and J_M is principal in L_M for every maximal element M of L.

Proof. The first statement follows from Theorem 1.2. By Theorem 2.1 any principal element is compact. Since J is compact, $(A:J)_M = (A_M:J_M)$ for any maximal element M and any A in L. Thus by Theorem 2.8, it follows that J satisfies the meet and join principal laws in L if and only if J_M does in L_M for each maximal element M of L.

In a commutative ring, an ideal which is weak meet principal is called a multiplication ideal. Since localization preserves multiplication ideals, it follows from Theorem 2.9, that a multiplication ideal is locally principal. The second statement of Theorem 2.9 for rings is found in [14].

Now that we have discussed r-lattice localization let us see what has been done in the case of commutative rings. Suppose R is a commutative ring and S a multiplicatively closed set of R. Then $\bar{S} = \{(a) \mid a \in S\}$ is a multiplicatively closed set in $L(R)$. It is easily seen that the map $J_S \to J_{\bar{S}}$ is a multiplicative lattice isomorphism from $L(R_S)$ to $L(R)_{\bar{S}}$.

3. Distributive r-lattices

In this section we study distributive r-lattices. Distributive multiplicative lattices have been studied in [3], [5], [6], [12], [15], and [17]. By a semigroup we mean a commutative semigroup with 0 and 1 written multiplicatively. A non-empty subset J of a semigroup S is called an ideal if $JS = J$. The set theoretic union of any set of ideals is an ideal, as is the set theoretic intersection. Thus $L(S)$, the lattice of ideals of S, is a completely distributive lattice. $L(S)$ is a multiplicative lattice under the usual product of ideals. The units of S form an abelian group and the non-units form the unique maximal ideal of S, hence $L(S)$ is quasi-local.

Let S be a semigroup. Any ideal (a) is meet principal, but such a principal ideal need not be even weak join principal. Consider the semigroup $S = \{0, 1, x, y\}$ with multiplication $x = x^2$, $y = y^2$, and $xy = 0$. Then (x) and (y) are meet principal, but not weak join principal. Note that (x, y) is both meet and join principal, but is not join irreducible.

A principal ideal (x) of S is (weak) join principal if and only if $(x)(b) = (x)(c) \neq 0$ implies $(b) = (c)$. Thus $L(S)$ is principally generated if and only if for every $x \in S$, $xb = xc \neq 0$ implies $b = \lambda c$ for some unit $\lambda \in S$. A principally generated semigroup will be called an r-semigroup. It is clear that if S is an r-semigroup, then $L(S)$ is an r-lattice. The proof of the following lemma may be found in [3].

LEMMA 3.1. *Let L be a multiplicative lattice. Then L is isomorphic to the lattice of ideals of a semigroup if and only if*

(A) *L is distributive,*

(B) *L is quasi-local, and*

(C) *there exists a set $\bar{S}$ of weak meet principal elements of L which generates L under joins, is closed under products, and whose elements are join irreducible and compact.*

THEOREM 3.2. *Let L be a quasi-local distributive r-lattice. Then L is isomorphic to the lattice of ideals of an r-semigroup.*

Proof. The weak meet principal elements of a quasi-local r-lattice are principal, join irreducible and compact. The result now follows from Lemma 3.1.

In [3], we showed that any distributive local Noether lattice could be embedded in the lattice of ideals of a Noetherian ring. We extend this result to r-lattices.

THEOREM 3.3. *A distributive r-lattice is embeddable in the lattice of ideals of a commutative ring.*

Proof. Let L be a distributive r-lattice. The map $L \to \Pi L_M$ given by $A \to (A_M)$ where M ranges over all maximal elements of L is an embedding. Thus it suffices to embed L_M in the lattice of ideals of a commutative ring. Hence we may assume that L is a quasi-local distributive r-lattice. Thus $L \cong L(S)$ where S is an r-semigroup. Let k be a fixed field and let $k[X, S]$ be the semigroup ring over S with coefficients in k. We denote the generator in $k[X, S]$ determined by $a \in S$ by X_a. Let R be the ring $k[X, S]/(X_0)$. We claim that $L \cong L(S)$ can be embedded in $L(R)$. Let J be an ideal in S. We send J to $(X_j)_{j \in J} + (X_0)$, an ideal in R. This map is well-defined, order preserving, injective, preserves arbitrary joins, products, and meets, sends (0) to (X_0) and S to R, and sends principal elements to principal ideals.

In [12], it is shown that a distributive Noether lattice is representable as the lattice of ideals of a ring if and only if it satisfies the weak union condition. (Recall that a multiplicative lattice L satisfies the weak union condition if given A, B, $C \in L$ with $A \not\leq B$ and $A \not\leq C$, there exists a principal element $E \leq A$ with $E \not\leq B$ and $E \not\leq C$.) We extend this result, by a different method, to an arbitrary distributive r-lattice domain.

THEOREM 3.4. *Let L be an r-lattice domain, then the following are equivalent:*

(1) *L is distributive and satisfies the weak union condition,*
(2) *for every maximal element M, L_M is totally ordered,*
(3) $(A:B) \vee (B:A) = I$ *for A, B compact,*
(4) $(A \vee B):C = (A:C) \vee (B:C)$ *for C compact,*
(5) $C:(A \wedge B) = (C:A) \vee (C:B)$ *for A, B compact,*
(6) $A(B \wedge C) = AB \wedge AC$ *for all A, B, C,*
(7) $(A \vee B)(A \wedge B) = AB$ *for all A, B,*
(8) *every compact element of L is principal,*
(9) *L is representable as the lattice of ideals of a Prüfer domain.*

Proof. (1)$\Rightarrow$(2). For M a maximal element, L_M is a quasilocal distributive r-lattice domain satisfying the weak union condition. Thus L_M is isomorphic to the lattice of ideals of an r-semigroup (Theorem 3.2). However, it is easily seen that a semigroup satisfying the weak union condition must be totally ordered. The implications (2)$\Rightarrow$(3), (3)$\Rightarrow$(2), (2)$\Rightarrow$(4), and (2)$\Rightarrow$(6) follow by localization, since these properties hold if and only if they hold locally for each L_M, M a maximal element. (4)$\Rightarrow$(3): For A and B compact, $A \vee B$ is compact and hence $I = (A \vee B : A \vee B) = (A : A \vee B) \vee (B : A \vee B) = (A:B) \vee (B:A)$. (5)$\Rightarrow$(3): For A and B compact, $I = (A \wedge B : A \wedge B) = (A \wedge B : A) \vee (A \wedge B : B) = (B:A) \vee (A:B)$. (6)$\Rightarrow$(7): $AB \leq (A \vee B)(A \wedge B) = ((A \vee B)A) \vee ((A \vee B)B) \leq AB$. (7)$\Rightarrow$(8): It suffices to show that the join of two principal elements is principal. Let A and B be principal, then $(A \vee B)(A \wedge B) = AB$ and AB being the product of two principal elements is principal. It is easily verified that in a domain, any factor of a principal element is still principal. Thus $A \vee B$ is principal. (Note that in this case we have also shown that the intersection of two compact elements is compact.) (8) $\Rightarrow$ (9): Let S be the set of non-zero compact (in this case principal) elements of L. Then $(S, \geq)$ is a partially ordered cancellation monoid under multiplication; moreover, $\geq$ (the order $\leq$ reversed) is actually a lattice order. Let $(S^*, \geq)$ be the group of quotients of S with $\geq$ the partial order induced by $\geq$. Then $(S^*, \geq)$ is lattice ordered. By the Krull–Kaplansky–Jaffard–Ohm theorem [8], S^* is the group of divisibility of a Bézout domain R. It now follows as in [2], that $L \cong L(R)$. (9) $\Rightarrow$ (1): The lattice of ideals of a Prüfer domain is distributive and satisfies the weak union condition. (9) $\Rightarrow$ (5): It is well-known that a Prüfer domain satisfies 5).

The extension of Theorem 3.4 to non-domains evidently requires a better understanding of arithmetical rings. There are many other conditions equivalent to representability of a distributive r-lattice domain involving conditions satisfied by a Prüfer domain, see [8]. We remark that Theorem 3.4 may be used to prove the representability theorem for distributive Noether lattices.

4. π-lattices

A ring in which every principal ideal is a product of prime ideals is called a π-ring. It is well-known that a π-ring is a finite direct product of π-domains and special principal ideal rings [8]. A multiplicative lattice L will be called a π-lattice if there exists a set S of elements of L (not necessarily principal) which generate L under joins such that every element of S is a finite product of prime elements. For example, the usual case would be L the lattice of ideals of a ring R and S the set of principal ideals of R. We begin with an elementary, but useful, lemma.

LEMMA 4.1. *Every minimal prime of a π-lattice is weak meet principal.*

Proof. First note that since 0 is a product of primes, minimal primes do exist. Let P be a minimal prime and let $A \leq P$ where $A \in S$. Since A is a product of primes and P is minimal, $A = PB$ for some $B \in L$. Since S generates L under joins, the result follows.

A multiplicative lattice L is said to satisfy the union condition on primes if for any set $P_1, \ldots, P_n$ of primes in L and any $A \in L$ with $A \nleq P_1, \ldots, P_n$, there exists a principal element $E \leq A$ with $E \nleq P_1, \ldots, P_n$.

THEOREM 4.2. *Let (L, M) be a quasi-local modular principally generated π-lattice. Then either L is a domain or L has only finitely many prime elements.*

Proof. By Theorem 1.2, every principal element is a product of primes. If $\dim L = 0$, then every principal element is a power of M. Thus L is a special principal element lattice. Thus we may suppose $\dim L > 0$. Since 0 is a product of primes, L has a finite number of minimal primes $P_1, \ldots, P_n$. By Lemma 4.1, the P_i's are weak meet principal and hence principal by Theorem 1.2. Now $\dim L > 0$ implies $M \not\leq P_1, \ldots, P_n$.

First suppose that there exists a principal element $E \leq M$ with $E \not\leq P_1, \ldots, E \not\leq P_n$. (This is indeed the case when L satisfies the union condition on primes.) Let $A = \sqrt{0} = P_1 \wedge \cdots \wedge P_n$, then L/A is a principally generated modular quasi-local π-lattice and $E \vee A$ is a principal element in L/A which is not a zero divisor. Let $E = Q_1 \cdots Q_t$ be a representation of E as a product of primes in L. Then in $\bar{L} = L/A$, $\bar{E} = E \vee A = \bar{Q}_1 \cdots \bar{Q}_t$. Since $\bar{E}$ is a non-zero divisor (i.e., $(\bar{O}:\bar{E}) = \bar{0}$), each $\bar{Q}_i$ is a factor of a principal element and hence is principal. Say $\bar{Q}_1 \geq \bar{P}_1$, then $\bar{P}_1 = \bar{Q}_1\bar{P}_1$. But $\bar{P}_1$ is principal and hence $\bar{I} = \bar{Q}_1 \vee (\bar{0}:\bar{P}_1)$. Therefore $P_1 = A$, so L has a unique minimal prime P. Now L/P is a principally generated quasi-local π-domain, so L/P contains a non-zero principal prime $Q \vee P$ where $Q \in L$ is principal. In L we have $Q = Q_1 \cdots Q_t$ a product of primes. In L/P, $Q \vee P = \bar{Q} = \bar{Q}_1 \cdots \bar{Q}_t$, so $t = 1$ and hence Q is prime. Now Q is a principal prime and $Q > P$ implies $P = PQ$, so $P = 0$ since P is principal. Thus L is a domain.

Thus we may assume that every principal element of L contained in M is contained in one of the minimal principal primes $P_1, \ldots, P_n$. It easily follows that every principal element is a product of the primes $P_1, \ldots, P_n$. Thus every prime element in L is a finite join of the primes $P_1, \ldots, P_n$ and hence L has only finitely many primes.

COROLLARY 4.3. *A quasi-local principally generated modular π-lattice satisfying the union condition on primes is either a domain or a special principal element lattice.*

COROLLARY 4.4. *A quasi-local r-lattice L which is a π-lattice is either a domain or a Noether lattice with a finite number of prime elements.*

Proof. Suppose L is not a domain. By the proof of Theorem 4.2 it follows that every prime in L is a finite join of principal elements. By Theorem 2.5 L is a Noether lattice.

We next globalize Corollary 4.3.

THEOREM 4.5. *Let L be an r-lattice satisfying the union condition on primes. Then L is a π-lattice if and only if it is a finite direct product of π-domains and special principal element lattices.*

Proof. Now $0 = P_1^{e_1} \cdots P_n^{e_n}$ where the P_i are the distinct minimal primes of L. Since any localization of a π-lattice is a π-lattice, by Corollary 4.3 we get that the P_i's are comaximal and hence the $P_i^{e_i}$'s are comaximal. Thus $L \cong L/P_1^{e_1} \times \cdots \times L/P_n^{e_n}$. Each $L/P_i^{e_i}$ is an π-r-lattice satisfying the union condition on primes and has a unique minimal prime. If $\dim L/P_i^{e_i} = 0$, then $L/P_i^{e_i}$ is quasi-local and hence a special principal element lattice. If $\dim L/P_i^{e_i} > 0$, then $P_i/P_i^{e_i}$ is locally 0 and hence 0 by Theorem 2.8. In this case $L/P_i^{e_i}$ is a π-domain. The converse is clear.

By a UFD lattice we mean a principally generated multiplicative lattice domain in which every principal element is a product of principal primes. If R is a commutative domain, then $L(R)$ can be a UFD without R being a UFD. For example, for any Dedekind domain R, $L(R)$ is a UFD lattice. In fact, $L(R)$ is a UFD if and only if R is a π-domain. First we need the following

THEOREM 4.6. *Let L be an r-lattice domain. Then L is a UFD if and only if every non-zero prime of L contains a non-zero principal prime.*

Proof. If L is a UFD lattice, then every non-zero prime element contains a non-zero principal prime. For the converse, let $S = \{0 \neq J \in L \mid J \text{ is a product of principal primes}\}$. Now $S \neq \varnothing$ is a multiplicatively closed set and $0 \notin S$. Hence 0 can be enlarged to an element K maximal with respect to exclusion of S and K is prime (Theorem 2.2). By hypothesis K contains a principal prime; thus we must have $K = 0$. Therefore every non-zero principal element contains a non-zero principal element which is a product of principal primes. Let $J \in L$ be any non-zero principal element, then $J \geq Q_1 \cdots Q_t$ where $Q_1, \ldots, Q_t$ are principal primes. Now since J is principal, $JA = Q_1 \cdots Q_t$ for some $A \in L$. Thus it suffices to show that S is "saturated." Suppose AB is a product of principal primes, $AB = Q_1 \cdots Q_n$, we show that A is a product of principal primes. The case $n = 1$ is clear. Now $AB = Q_1 \cdots Q_n \leq Q_1$, so say $B \leq Q_1$. Then $B = CQ_1$ for some $C \in L$ since Q_1 is principal. Hence $Q_1 \cdots Q_n = AB = ACQ_1$, so $AC = Q_2 \cdots Q_n$. By induction A is a product of principal primes.

COROLLARY 4.7. *Let L be an r-lattice. Then L is a UFD if and only if it is a π-domain.*

Proof. Clearly any UFD is a π-lattice. Conversely, suppose L is a π-domain. Then L is generated under joins by a set S of elements (not necessarily principal) each of which is a product of primes. Let P be a non-zero prime of L. Then L_P is a π-lattice domain in which every principal element is a product of principal primes (by Theorem 2.9, the set $S_P = \{A_P \mid A \in S\}$ necessarily contains all principal elements of L_P). Thus there is a non-zero prime $Q \leq P$ such that Q_P is principal in L_P. Thus Q_P has rank 1 in L_P and hence rank $Q = 1$ in L. A slight modification of Lemma 4.1 shows that Q is weak meet principal in L. Since L is a domain, Q is actually principal.

We remark that Corollary 4.3 is not true without the union condition on primes. Let (R, π) be a DVR and let T be the graded ring $R[X]/(\pi X)$. Then $L_G(T)$ is a quasi-local π-lattice with dimension one which is not a domain. More generally one can show that a quasi-local distributive π-r-lattice is either the lattice of ideals of a free semigroup or of the form RL_K/A where $(X_1 \cdots X_K)^n \leq A \leq X_1, \ldots, X_K$ and $X_1, \ldots, X_K$ is the minimal basis for RL_K. (RL_K is the distributive regular local Noether lattice of dimension K.)

It is classical that a domain is a Dedekind domain if every ideal is a product of primes. A remarkable result is that if every ideal generated by two elements is a product of primes, then the ring is a ZPI ring [8]. Actually, more is true, if $R = R_0 \oplus R_1 \oplus \cdots$ is a graded ring in which every homogeneous ideal generated by two homogeneous elements is a product of homogeneous primes, then R is a ZPI ring. We generalize this result to r-lattices.

LEMMA 4.8. *Let (L, M) be a principally generated modular quasi-local multiplicative lattice. Suppose that every join of two principal elements of L is a product of primes, then every element of L is a power of M and M is principal.*

Proof. We may suppose $M \neq 0$. We first note that L contains a non-zero principal prime P. If $P = M$ we are done, so assume $P < M$. Then there exists a principal element Y with $Y \not\leq P$. By hypothesis, $P \vee Y = P_1^{n_1} \cdots P_t^{n_t}$ and $P \vee Y^2 = Q_1^{m_1} \cdots Q_s^{m_s}$ where $P_1, \ldots, P_t, Q_1, \ldots, Q_s \in L$ are primes. Passing to L/P, we get $P \vee Y = \bar{Y} = \bar{P}_1^{n_1} \cdots \bar{P}_t^{n_t}$ and $P \vee Y^2 = \bar{Y}^2 = \bar{Q}_1^{m_1} \cdots \bar{Q}_s^{m_s}$. Now L/P is a domain and $\bar{Y}$ is principal, so $s = 2t$ and after rearranging $\bar{P}_1 = \bar{Q}_1, \ldots, \bar{P}_t = \bar{Q}_t$. Then $P \vee Y^2 = P_1^{2n_1} \cdots P_t^{2n_t} = (P_1^{n_1} \cdots P_t^{n_t})^2 = (P \vee Y)^2$ and hence $P \leq P^2 \vee Y$. Now in L/P^2, $P = P \vee P^2 \leq Y \vee P^2$ and $Y \vee P^2$ is principal so $P = P \vee P^2 = (C \vee P^2) \circ (Y \vee P^2) = CY \vee P^2$ for some $C \in L$. $P \geq CY$ implies $C \leq P$ since P is prime and $Y \not\leq P$. Hence $P = PY \vee P^2 = P(Y \vee P)$ so $I = Y \vee P \vee (0:P)$ since P is principal. $P \neq 0$ implies $I = Y \vee P \leq M$, a contradiction. Therefore $M = P$ is principal. It follows that every element of L is a power of M.

THEOREM 4.9. *Let L be an r-lattice and let S be a set of elements (not necessarily principal) which generates L under joins. Assume that every join of two elements of S is a product of primes. Then L is isomorphic to the lattice of ideals of a ZPI ring.*

Proof. Let $0=P_1^{e_1}\cdots P_n^{e_n}$ where $P_1,\ldots,P_n$ are the distinct minimal prime elements of L. By Lemma 4.8, each maximal element contains a unique minimal prime, so the $P_1,\ldots,P_n$ are comaximal and hence so are $P_1^{e_1},\ldots,P_n^{e_n}$. Thus $L\cong L/P_1^{e_1}\times\cdots\times L/P_n^{e_n}$. If $\dim L/P_i^{e_i}=0$, then $L/P_i^{e_i}$ is quasi-local with maximal element $P_i/P_i^{e_i}$. Let (R,π) be a DVR, then $L/P_i^{e_i}\cong L(R/\pi^{e_i}R)$. If $\dim L/P_i^{e_i}>0$, then $P_i/P_i^{e_i}$ is locally 0 and hence 0, so $L/P_i^{e_i}$ is a domain. Since $L/P_i^{e_i}$ is a domain it is a UFD and since $\dim. L/P_i^{e_i}=1$, every prime of $L/P_i^{e_i}$ is principal so by Theorem 2.5, $L/P_i^{e_i}$ is a Noether lattice. By Theorem 2.9, every element of $L/P_i^{e_i}$ is principal. It follows that $L/P_i^{e_i}$ is isomorphic to the lattice of ideals of PID (Theorem 3.4 or [12]). If $L/P_i^{e_i}\cong L(R_i)$, then $L\cong L/P_1^{e_1}\times\cdots\times L/P_n^{e_n}\cong L(R_1)\times\cdots\times L(R_n)\cong L(R_1\times\cdots\times R_n)$. Since $R_1\times\cdots\times R_n$ is a product of PID's and special principal ideal rings, it is a ZPI ring.

Theorem 4.9 yields what might be called the ultimate characterization of ZPI rings.

COROLLARY 4.10. *Let R be a commutative ring and S a set of ideals of R (not necessarily principal) which generates R under sums. Assume that every sum of two ideals of S is a product of prime ideals. Then R is a finite direct product of Dedekind domains and special principal ideal rings.*

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