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Mailbox

Every distributive groupoid is trimedial

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Distributive groupoids have been considered by various authors. In a paper of Ježek and Kepka [1] the problem arose whether any distributive groupoid is trimedial. Here a groupoid is called distributive if it satisfies the following identities

$$(xy)z=(xz)(yz) \tag{1}$$

and

$$x(yz)=(xy)(xz). \tag{2}$$

It is called trimedial if it satisfies the identity

$$(xy)(zx)=(xz)(yx). \tag{3}$$

Obviously any commutative groupoid is trimedial. Partial answers to the problem are given in [1] and [2], where it is shown that any distributive cancellation groupoid is trimedial, and there are more results in recent work of Kepka and Némec, for example that every distributive division groupoid is trimedial. Here the affirmative answer to the problem is given without additional assumptions for the groupoid.

I would like to thank the referee for helpful comments.

LEMMA. *A distributive groupoid satisfies the following identities:*

$$(xy)(xx)=(xx)(yx) \tag{4}$$

$$(xy)x=x(yx) \tag{5}$$

$$(xy)(zx)=((xy)(zx))((xz)(yx)). \tag{6}$$

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Proof.

$$
\begin{aligned}
(4)\colon (xy)(xx) &= ((xy)x)((xy)x) && \text{by (2)}\\
&= ((xx)(yx))((xx)(yx)) && \text{by (1)}\\
&= (xx)((yx)(yx)) && \text{by (2)}\\
&= (xx)(y(xx)) && \text{by (2)}\\
&= ((xx)y)((xx)(xx)) && \text{by (2)}\\
&= ((xx)y)((xx)x) && \text{by (1)}\\
&= (xx)(yx) && \text{by (2)}
\end{aligned}
$$

$$
\begin{aligned}
(5)\colon\quad (xy)x &= (xx)(yx) && \text{by (1)}\\
&= (xy)(xx) && \text{by (4)}\\
&= x(yx) && \text{by (2)}
\end{aligned}
$$

$$
\begin{aligned}
(6)\colon (xy)(zx) &= ((xy)z)((xy)x) && \text{by (2)}\\
&= [((xy)z)(xy)][((xy)z)x] && \text{by (2)}\\
&= [(xy)(z(xy))][((xy)x)(zx)] && \text{by (5), (1)}\\
&= [(xy)((zx)(zy))][((xy)(zx))(x(zx))] && \text{by (2), (1)}\\
&= ((xy)(zx))[((xy)(zy))(x(zx))] && \text{by (2)}\\
&= ((xy)(zx))[((xz)y)((xz)x)] && \text{by (1), (5)}\\
&= ((xy)(zx))((xz)(yx)) && \text{by (2)}
\end{aligned}
$$

THEOREM. *Any distributive groupoid is trimedial.*

Proof. For a groupoid $(G, \cdot)$ let $(G, *)$ be the dual of it, which is defined by $x * y = y \cdot x$. Let $(G, \cdot)$ be a distributive groupoid, then so is $(G, *)$, therefore $(G, \cdot)$ and $(G, *)$ fulfill equation (6) of the lemma. This yields

$$
\begin{aligned}
(xy)(zx) &= ((xy)(zx))((xz)(yx))\\
&= ((x * y) * (z * x)) * ((x * z) * (y * x))\\
&= (x * y) * (z * x)\\
&= (xz)(yx).
\end{aligned}
$$

REFERENCES

[1] J. JÉZEK and T. KEPKA, *Semigroup representations of commutative idempotent abelian groupoids.* Comm. Math. Univ. Carolinae *16* (1975), 487–500.

[2] J. SOUBLIN, *Etude algébraique de la notion de moyenne.* J. Math. Pures et Appl. *50* (1971), 53–264.

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