Amalgamation properties and interpolation theorems for equational theories

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We classify a family of 216 interpolation principles for formulas in equational theories and show that those in a natural subfamily which are not theorems are equivalents of four standard diagrammatic properties, including the Amalgamation Property.

0. Introduction

Separation and interpolation principles have been studied by logicians for several years now, and so to place this work in a general context, let us make the following very abstract definitions, later bringing them down to earth.

DEFINITION 0.1. Let $\Gamma_1, \Gamma_2, \Gamma$ be subsets of a boolean algebra B. Then the $(\Gamma_1, \Gamma_2, \Gamma)$-*Separation Principle*, written $\mathrm{Sep}(\Gamma_1, \Gamma_2, \Gamma)$ for short, is the following: whenever $a \in \Gamma_1$, $b \in \Gamma_2$ and $a \wedge b = 0$, there is $c \in \Gamma$ with $a \leqslant c$ and $c \wedge b = 0$. The $(\Gamma_1, \Gamma_2, \Gamma)$-*Interpolation Principle*, written $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ for short, is the following: whenever $a \in \Gamma_1$, $b \in \Gamma_2$ and $a \leqslant b$ there is $c \in \Gamma$ such that $a \leqslant c \leqslant b$.

Clearly $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ is just $\mathrm{Sep}(\Gamma_1, -\Gamma_2, \Gamma)$ where $-\Gamma_2 = \{-b : b \in \Gamma_2\}$, so that the two notions are coextensive. Which one to use is a matter of taste, convenience and history.

In practice B is often the Lindenbaum algebra of formulas reduced modulo some theory: then we replace each equivalence class $[\varphi]$ by its representative formula φ, the ordering $[\varphi] \leqslant [\psi]$ of the algebra becomes a valid implication $\varphi \to \psi$, and O denotes falsity. We also abuse notation mildly by using Γ_1, etc., for classes of *formulas*.

Several standard examples of separation principles occur in Recursion Theory, Descriptive Set Theory and Model Theory: however, they are usually *symmetric* in the sense that $\Gamma_1 = \Gamma_2$, and *strict* in the sense that $\Gamma_1 \cap -\Gamma_2 = \Gamma$. In contrast, we shall often want to dispense with these assumptions, and this is why we phrase the definition with *three* parameters as opposed to the usual *one*.

The best known interpolation theorem in Model Theory is probably Craig's Theorem: this can be summarised as $\mathrm{Int}(E^2, U^2, F)$ where F is the set of all first order sentences in a language L and E^2 [respectively U^2] is the set of existential [universal] *second-order* sentences over L. We shall, however, be interested only in first order formulas, of low quantifier complexity, and so we begin by looking at an interpolation theorem for such formulas, Herbrand's Theorem.

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1. Herbrand's theorem

Throughout this paper L will denote a fixed first order language (with certain relation, function and constant symbols). We shall be concerned with various syntactical classes of L-formulas, which we shall use so often that we shall give them special names.

Let O be the class of open (i.e.quantifier-free) formulas of L, and E [respectively U] be the class of existential [universal] formulas of L, i.e. those whose prenex normal form is $\exists\bar{x}\varphi$ [$\forall\bar{x}\varphi$] with φ open. Let O^+, E^+, U^+ denote the corresponding classes of positive formulas (i.e. their quantifier-free parts are positive open formulas).

There are obvious inclusions between these classes which we can summarise in the following diagram.

$$\begin{tikzcd}
E & & U \\
& O \arrow[ul] \arrow[ur] & \\
E^+ \arrow[uu] & & U^+ \arrow[uu] \\
& O^+ \arrow[ul] \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

We shall be concerned with classifying the various interpolation principles relating these classes, relative to a fixed L-theory T. Unless otherwise specified, we shall consider only those T whose axioms are universally quantified atomic formulas. We shall call such a theory T *equational* – this is the natural extension to languages with relation symbols of the usual notion. The class of models of T is closed under products, substructures, and homomorphic images, and admits the construction of free models and coproducts (= free products). We shall need these facts later.

Our notation is mostly standard or taken from [1], except that we shall use $S\vdash$ to mean S is inconsistent (you can regard the blank as being the void disjunction, which is falsity). We shall sometimes use the compactness theorem, often in the strong form which states:

if $S\vdash \bigvee_{i\in I}\theta_i$ then there is a finite subset S_0 of S and a finite subset J of I such that $S_0\vdash \bigvee_{i\in J}\theta_i$.

To begin with let us look at Herbrand's Theorem. In our notation this states $\mathrm{Int}(U, E, O)$, i.e. $\mathrm{Sep}(U, U, O)$. Written out more fully it says:

whenever $\theta(\bar{x})\in U$ and $\varphi(\bar{x})\in E$ are such that $T\vdash\theta(\bar{x})\to\varphi(\bar{x})$ there is $\psi(\bar{x})\in O$ such that $T\vdash\theta(\bar{x})\to\psi(\bar{x})$ and $T\vdash\psi(\bar{x})\to\varphi(\bar{x})$.

As usual the notation $\varphi(\bar{x})$ means that the list $\bar{x}$ includes the free variables of φ. Note that we allow free variables to occur in interpolation principles, but require the free variables of the interpolant to be restricted in the obvious way, as above. (Basically, one can regard each list $\bar{x}$ as being added as new constants to the language L and the interpolation as taking place for *sentences* in this enlarged language.) In fact we have to require that $\bar{x}$ is a nonvoid list of variables, unless L has at least one constant symbol – this occurs because we do not admit the empty set as a model.

Since we wish to refine the method later we shall give a proof of Herbrand's Theorem.

LEMMA 1.1 (Herbrand). $\mathrm{Int}(U, E, O)$, *i.e.* $\mathrm{Sep}(U, U, O)$.

Proof. Let $T \vdash \theta(\bar{x}) \to \varphi(\bar{x})$ with $\theta \in U$, $\varphi \in E$. Write $\varphi(\bar{x}) = \exists \bar{y} \alpha(\bar{x}, \bar{y})$ with α open. If $A \models T$ and $A \models \theta(\bar{a})$ then $A \models \alpha(\bar{a}, \bar{b})$ for some $\bar{b}$ in the substructure of A generated by $\bar{a}$. Hence $T + \theta(\bar{x}) \vdash \bigvee_{\bar{\tau}(\bar{x})} \alpha(\bar{x}, \bar{\tau}(\bar{x}))$ where $\bar{\tau}$ ranges over all lists of terms, of length that of $\bar{y}$. The result follows by compactness. Note that the interpolant $\psi(\bar{x})$ is a finite disjunct of certain *instances* $\alpha(\bar{x}, \bar{\tau}(\bar{x}))$ of $\exists \bar{y} \alpha(\bar{x}, \bar{y})$.

It is usually the case that if $\mathrm{Sep}(\Gamma_1, \Gamma_1, \Gamma)$ holds then $\mathrm{Sep}(-\Gamma_1, -\Gamma_1, \Gamma)$ will not hold. Thus we might expect $\mathrm{Sep}(E, E, O)$ not to hold, and it is in fact easy to construct T and L for which it fails. However, it is natural then to ask for what T it can hold, and we shall answer this in the next section.

But to conclude this section we shall prove two asymmetric interpolation theorems closely related to Herbrand's Theorem.

LEMMA 1.2. (1) $\mathrm{Int}(U, E^+, O^+)$;

(2) $\mathrm{Int}(U^+, E, O^+)$.

Proof. (1) By the construction of the interpolant ψ in the proof of Lemma 1.1, it is clearly positive if α is positive.

(2) Similarly ψ is negative if α is. Thus we obtain $\mathrm{Int}(U, E^-, O^-)$ in an obvious notation, and the result follows by manipulating negations.

2. Amalgamation properties

Three diagrammatic properties of the category of models of a theory have been intensively studied in recent years. They are the Amalgamation Property (AP), Congruence Extension Property (CEP), and Injections Transferable (IT). Precise definitions can be found in Taylor [9].

Recently these properties have been given syntactic characterisations for general theories T – see [2] and [3]. The characterisations, though fairly simple and reminiscent of separation principles, are not really appealing. We shall show that for equational (or in fact universal) T they reduce to interpolation principles.

THEOREM 2.1. *Let T be a universal L-theory. Then*

(1) *T has* AP *iff* $\mathrm{Sep}(E, E, O)$, *i.e.* $\mathrm{Int}(E, U, O)$, *holds for T;*

(2) *T has* CEP *iff* $\mathrm{Sep}(E^+, O, O^+)$, *i.e.* $\mathrm{Int}(E^+, O, O^+)$, *holds for T;*

(3) *T has* IT *iff* $\mathrm{Sep}(E^+, E, O^+)$ *i.e.* $\mathrm{Int}(E^+, U, O^+)$, *holds for T.*

Proof. The general idea is (a) to write down in a sensible way what it means for T to have the diagrammatic property, using Robinson's method of diagrams, and apply repeatedly the compactness theorem to obtain *some* syntactic characterisation, and then (b) knock this into shape using the fact that T is universal. We shall give the details for IT.

The procedure (a) was carried out in [2] and gave the following characterisation (for completeness we shall give a simpler, alternative proof in §5).

T has IT iff

(*) for all $\alpha_1(\bar{x}) \in E^+$, $\alpha_2(\bar{x}) \in E$ with $T+\alpha_1+\alpha_2 \vdash$, there are $\beta_1(\bar{x}) \in E$ and $\beta_2(\bar{x}) \in E^+$ with $T \vdash \beta_1 \vee \beta_2$ and $T+\beta_i+\alpha_i \vdash$, $i=1, 2$.

But since T is universal, the method of Herbrand's Theorem applied to $T \vdash \beta_1(\bar{x}) \vee \beta_2(\bar{x})$ yields $\gamma_1(\bar{x}) \in O$ and $\gamma_2(\bar{x}) \in O^+$ such that $\vdash \gamma_i \to \beta_i$ $(i=1, 2)$ and yet $T \vdash \gamma_1 \vee \gamma_2$. Thus $T+\neg\gamma_2 \vdash \gamma_1$ and so $T+\neg\gamma_2 \vdash \beta_1$, while $T+\gamma_2 \vdash \beta_2$. Hence $T+\gamma_2+\alpha_2 \vdash$, and $T+\neg\gamma_2+\alpha_1 \vdash$, i.e. $T+\alpha_1 \vdash \gamma_2$. Thus (*) implies

(**) for all $\alpha_1(\bar{x}) \in E^+$ and $\alpha_2(\bar{x}) \in E$ with $T+\alpha_1+\alpha_2 \vdash$ there is $\gamma_2(\bar{x}) \in O^+$ with $T+\alpha_1 \vdash \gamma_2$ and $T+\alpha_2+\gamma_2 \vdash$.

It is easy to see that all steps are reversible, so that (*) is equivalent to (**). But (**) is what we want.

Similarly we can prove (1) and (2) using the characterisations of AP and CEP given in [2].

Note that we cannot simplify our characterisation of AP for general T. For since $\mathrm{Sep}(E, E, O)$ depends only on the universal consequences of T, if it held for the theory T of real closed fields, it would hold for the theory $T \cap U$ of formally real fields, and so $T \cap U$ would have AP, which is false – yet T has AP (being model-complete).

It may seem surprising that the Amalgamation Property should have a characterisation as a simple interpolation principle 'dual' to Herbrand's Theorem. However, there are at least two sources of motivation for this result.

1. The known relation between interpolation theorems and AP for cylindric algebras – see for example Pigozzi [7]. However, this relates to Craig's Theorem, and so is useful mainly for the *idea* of a connection.

2. More crucially, the 'equational interpolation property' discussed by Jónsson, which we shall discuss now.

DEFINITION 2.2. T is said to have the *equational interpolation property* (EIP) if whenever $T \vdash \alpha(\bar{x}, \bar{y}) \to \beta(\bar{y}, \bar{z})$ there is $\gamma(\bar{y})$ such that $T \vdash \alpha(\bar{x}, \bar{y}) \to \gamma(\bar{y})$ and $T \vdash \gamma(\bar{y}) \to \beta(\bar{y}, \bar{z})$, where α, β, γ are conjunctions of atomic formulas.

In [5] Jónsson showed that if T has AP then T has EIP. In fact he deduced EIP from a weaker version of AP. We shall show that the weaker version is in fact *equivalent* to EIP, but before proving that, we shall discuss the weaker version in the next lemma (as the proof is routine, we shall omit it). For background on free algebras and coproducts, see Grätzer [4].

LEMMA 2.3. *The following are equivalent:*

(1) *given any diagram* $B \leftarrow A \rightarrow A * \langle X \rangle$ *of embeddings in* T *with* $\langle X \rangle$ *the free* T*-algebra on the set* X *and* $A \rightarrow A * \langle X \rangle$ *the canonical embedding into the coproduct* $A * \langle X \rangle$, *there are embeddings* $B \rightarrow C$ and $A * \langle X \rangle \rightarrow C$ such that

$$A \rightarrow B \rightarrow C = A \rightarrow A * \langle X \rangle \rightarrow C;$$

(2) *whenever* $A \rightarrow B$ *is an embedding in* T *and* X *is a set, then the natural morphism* $A * \langle X \rangle \rightarrow B * \langle X \rangle$ *is an embedding;*

(3) *if* $A \leqslant B$ *in* T *and* X *is a set, then* $A(X)$, *the subalgebra of* $B * \langle X \rangle$ *generated by* A *and* X, *is equal to* $A * \langle X \rangle$.

We call any one of these conditions the *flat amalgamation property* (FAP) for T.

LEMMA 2.4. *If* T *has* EIP *then* T *has* FAP.

Proof. Let $A \leqslant B \leqslant B * \langle X \rangle$. We must show that $A(X) = A * \langle X \rangle$, which means that for all $\bar{a} \in A$, $\bar{u} \in X$, and atomic $\beta(\bar{x}, \bar{u})$, if $A(X) \models \beta(\bar{a}, \bar{u})$ then $T + \Delta(A) \vdash \beta(\bar{a}, \bar{u})$ (by the definition of coproduct and free algebra). Let $A(X) \models \beta(\bar{a}, \bar{u})$. Since $A(X) \leqslant B * \langle X \rangle$, $B * \langle X \rangle \models \beta(\bar{a}, \bar{u})$, and so $T + \Delta(B) \vdash \beta(\bar{a}, \bar{u})$: hence there is $\alpha(\bar{x}, \bar{y})$ and $\bar{b} \in B$ such that $B \models \alpha(\bar{a}, \bar{b})$ and $T + \alpha(\bar{a}, \bar{b}) \vdash \beta(\bar{a}, \bar{u})$.

By EIP there is γ such that $T + \alpha(\bar{a}, \bar{b}) \vdash \gamma(\bar{a})$ and $T + \gamma(\bar{a}) \vdash \beta(\bar{a}, \bar{u})$. It follows easily that $A \models \gamma(\bar{a})$ and so $T + \Delta(A) \vdash \beta(\bar{a}, \bar{u})$.

The converse of this lemma, which is Jónsson's result, can be proved in a very similar way. Thus we know that EIP is equivalent to FAP. What makes this of interest to us is that an easy manipulation of disjunctions using propositional calculus and the fact that T is closed under product establishes that EIP is equivalent to the property $\mathrm{Int}(E^+, U^+, O^+)$. Hence we can add a fourth clause to Theorem 2.1.

THEOREM 2.5. *Let* T *be an equational theory. Then* T *has* FAP *iff* $\mathrm{Int}(E^+, U^+, O^+)$ *holds for* T.

3. Twelve strict interpolation principles

We have found four natural semantic properties of an equational theory T which are equivalent to interpolation principles for T of simple form. We now want to show that no other (different) semantic property has such a characterisation. To do this we *classify* all such simple interpolation principles.

To do this we need more insight into their structure. Let us again consider the diagram of classes of formulas. This is a partially ordered set under $\subseteq$. Now let us

$$\begin{tikzcd} E & & U \\ E^+ \arrow[u] & O \arrow[ul] \arrow[ur] & U^+ \arrow[u] \\ & O^+ \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

define $\Gamma_1 \cap \Gamma_2$ to be the set of formulas logically equivalent to a formula of Γ_1 and also one of Γ_2. We can now show that the diagram is a semilattice under $\cap$. To do this it suffices to prove that

$$E \cap U = O, \qquad U \cap E^+ = O^+ \quad \text{and} \quad U^+ \cap E = O^+.$$

Now if $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ is a theorem then $\Gamma_1 \cap \Gamma_2 \subseteq \Gamma$ (just interpolate for $\alpha \vdash \alpha$ when $\alpha \in \Gamma_1 \cap \Gamma_2$). Thus these three intersections follow from the forms of Herbrand's Theorem established in §1.

We can now begin the classification of interpolation principles, restricting attention, not surprisingly, to principles $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma_3)$ where $\Gamma_i \in \{E, E^+, U, U^+, O, O^+\}$, $i = 1, 2, 3$. We call such a principle *normal* if $\Gamma_1 \cap \Gamma_2 \subseteq \Gamma_3$, i.e. $\Gamma_1 \cap \Gamma_2$ (computed by the rules above) lies at or below Γ_3. Thus $\mathrm{Int}(E, U, O)$ is normal, as $E \cap U = O \subseteq O$, while $\mathrm{Int}(E, O, U^+)$ is not, as $E \cap O = O \nsubseteq U^+$. In fact all the interpolation principles considered so far have been normal. This is one reason for first considering this class. The other is that the normal principles are the natural ones – any non-normal interpolation principle leads to an identification of vertices in our semilattice, as we shall show in the next section.

Among the normal interpolation principles we distinguish the subclass of strict ones, where we call $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ *strict* if $\Gamma_1 \cap \Gamma_2 = \Gamma$, and *substrict* if $\Gamma_1 \cap \Gamma_2 \neq \Gamma$ (note that by normality always $\Gamma_1 \cap \Gamma_2 \subseteq \Gamma$). We shall be concerned in this section just with the strict interpolation principles. Given that Γ_1 and Γ_2 are arbitrary there are clearly 36 such: however if $\Gamma_1 \subseteq \Gamma_2$ or $\Gamma_2 \subseteq \Gamma_1$ then $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ is trivially true, and we shall omit these ones from the following table.

The 12 non-trivial strict interpolation principles

	E	E^+	U	U^+	O
E			EUO	EU^+O^+	
E^+			E^+UO^+	$E^+U^+O^+$	E^+OO^+
U	UEO	UE^+O^+			
U^+	U^+EO^+	$U^+E^+O^+$			U^+OO^+
O		OE^+O^+		OU^+O^+	

Note that from this point on we may often write $\Gamma_1\Gamma_2\Gamma$ for $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$, by analogy with notation for forms of the syllogism. We can immediately identify UEO, UE^+O^+ and U^+EO^+ as true (see §1) and clearly $U^+E^+O^+$, OE^+O^+ and U^+OO^+ are corollaries of these, hence also true. Among the six cases left, we have

$$\begin{aligned} EUO, &\quad \text{which is AP;} \\ E^+UO^+, &\quad \text{which is IT;} \\ E^+U^+O^+, &\quad \text{which is FAP;} \end{aligned}$$

and

$$E^+OO^+, \quad \text{which is CEP.}$$

This leaves two mysteries, EU^+O^+ and OU^+O^+. To solve these, we require a peculiar interpolation theorem.

LEMMA 3.1. *Let T be any equational theory. Then* $\mathrm{Int}(O, U^+, O^+)$ *holds for T.*

Proof. It will help to use the convenient (and almost self-explanatory) notation of Keisler [6]. Suppose that $T\vdash \phi(\bar{x}) \to \psi(\bar{x})$ where $\phi \in O$, $\psi \in U^+$. Clearly $\phi \in \vee \wedge (L)$ (where L is the language of T). However, we can in fact take $\phi \in \wedge (L)$, as the general case follows by interpolating for each disjunct. We can further decompose ϕ as $\phi = \phi^+ \wedge \neg \phi^-$ where $\phi^+ \in \wedge (L)^+$ and $\phi^- \in \vee (L)^+$. Thus

$$T + \phi^+(\bar{x}) \vdash \phi^-(\bar{x}) \vee \psi(\bar{x}).$$

Note that $T + \phi^+(\bar{x})$ is closed under product, and that $\phi^-(\bar{x})$ and $\psi(\bar{x})$ are positive. Thus, by a well known argument, either (a) $T + \phi^+(\bar{x}) \vdash \phi^-(\bar{x})$ or (b) $T + \phi^+(\bar{x}) \vdash \psi(\bar{x})$ (just argue by contradiction and take products). If (a) holds then $T + \phi(\bar{x}) \vdash$; thus, as usual in these Lyndon-type situations, the falsity symbol f is the interpolant, and since f is just the void disjunction, we regard $f \in O^+$. Otherwise, (b) holds and the interpolant is $\phi^+ \in O^+$.

A simple general argument shows that if $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ holds for T then so does $\mathrm{Int}(\exists\Gamma_1, \Gamma_2, \exists\Gamma)$: thus Lemma 3.1 implies that $\mathrm{Int}(E, U^+, E^+)$ holds. Now we can show that $\mathrm{Int}(E, U^+, O^+)$ is equivalent to $\mathrm{Int}(E^+, U^+, O^+)$. For, arguing schematically, if $E \vdash U^+$ then $E \vdash E^+ \vdash U^+$ and so $E \vdash E^+ \vdash O^+ \vdash U^+$: thus $E \vdash O^+ \vdash U^+$. Hence the mysteries are solved: OU^+O^+ is true and EU^+O^+ is another form for FAP.

This completes the classification of the strict interpolation principles.

4. The remaining interpolation principles

Recall that an interpolation principle $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ is called substrict if $\Gamma_1 \cap \Gamma_2 \subseteq \Gamma$ but $\Gamma_1 \cap \Gamma_2 \neq \Gamma$. In this case $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ is implied by $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma_1 \cap \Gamma_2)$, which is strict, and so we can classify substrict interpolation principles by the strict ones they are weakenings of. Discounting weakenings to trivialities (i.e. where some $\Gamma_i \subseteq \Gamma$) and weakenings of theorems, we arrive at the following table.

Substrict interpolation principles					
(IT)	E^+UO^+	implies	E^+UO	and	E^+UU^+
(FAP)	EU^+O^+	implies	EU^+O	and	EU^+E^+
(FAP)	$E^+U^+O^+$	implies	E^+U^+O		
(CEP)	E^+OO^+	implies	E^+OU^+		

Since OU^+O^+ is true, EU^+O is equivalent to EU^+O^+ (FAP), and E^+U^+O is equivalent to $E^+U^+O^+$ (again FAP). Since U^+OO^+ is true, E^+OU^+ is equivalent to E^+OO^+ (CEP). Also EU^+E^+ is true (by Lemma 3.1 and the subsequent remark).

It is an easy general result that $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ implies $\mathrm{Int}(\Gamma_1, \forall\Gamma_2, \forall\Gamma)$, and so $\mathrm{Int}(E^+, O, O^+)$ implies $\mathrm{Int}(E^+, U, U^+)$. Conversely, E^+UU^+ implies E^+OU^+, which is CEP (by the preceding paragraph). Thus E^+UU^+ is equivalent to CEP.

However, to characterise the one remaining principle, E^+UO, we have to go back to the beginning again.

THEOREM 4.1. *$\mathrm{Int}(E^+, U, O)$ holds for T iff T has* AP.

Proof. Since T is closed under products, T has AP precisely if

(*) whenever $B \leftarrow A \rightarrow C$ is a pair of embeddings in T there is an embedding $B \rightarrow D$ and a *morphism* $C \rightarrow D$ with $A \rightarrow B \rightarrow D = A \rightarrow C \rightarrow D$.

(We can amalgamate $B \leftarrow A \rightarrow C$ by taking the product of the maps given by (*) first for B and C and second for C and B.) By inserting a few $+$ superscripts in any proof of the characterisation of AP, such as in [2], one arrives at:

T has (*) iff for all $\theta_1(\bar{x}) \in E^+$ and $\theta_2(\bar{x}) \in E$ such that $T + \theta_1 + \theta_2 \vdash$ there is $\beta_1(\bar{x})$, $\beta_2(\bar{x}) \in E$ with $T \vdash \beta_1 \vee \beta_2$ and $T + \beta_i + \theta_i \vdash$, $i = 1, 2$.

This yields the result by the method of Theorem 2.1.

To sum up, each substrict interpolation principle which is not a theorem is equivalent to one of the four semantic properties.

To conclude, we have to classify the non-normal interpolation principles. As we said before, these lead to identifications of vertices in the following sense: we say that $\Gamma \subseteq \Gamma'$ *in T* if for each $\phi(\bar{x}) \in \Gamma$ there is $\psi(\bar{x}) \in \Gamma'$ such that $T \vdash \phi \leftrightarrow \psi$; and naturally we define $\Gamma = \Gamma'$ *in T* to be $\Gamma \subseteq \Gamma'$ and $\Gamma' \subseteq \Gamma$ in T.

It is then routine to show that if Γ and Γ' are vertices such that $\Gamma \nsubseteq \Gamma'$ (i.e. Γ is not below Γ' in the diagram) then $\Gamma \subseteq \Gamma'$ in T is equivalent to one of the following identifications:

the four simple ones $O = O^+$, $E^+ = O^+$, $E = O$, $U^+ = O^+$, and the compound one $E = O = O^+$ (where all vertices are identified).

Three of the simple ones correspond to natural semantic properties. In fact, given any universal theory T, it can be easily shown, using standard diagram and compactness arguments as in [1] or [8] and then some interpolation theorems, that

$O=O^+$	in T	iff every model of T is ultrasimple,
$E^+=O^+$	in T	iff every model of T is algebraically closed,
$E=O$	in T	iff every model of T is existentially closed, i.e. T is model-complete.

The fourth simple one, $U^+=O^+$, corresponds to a similar but less natural property. For definitions of the above concepts and some results relating them see [1].

It is clear that $O=O^+$ cannot hold for any nontrivial equational theory (as no nontrivial product is ultrasimple). However, $E^+=O^+$ holds for several equational theories, for example the theory of boolean algebras. The status of $E=O$ and $U^+=O^+$ for equational theories T is less clear: for example if $E=O$ in T then T is model-complete, has no nontrivial finite models, and no 1-element submodels of any (other) models.

Nevertheless, we can still classify the non-normal interpolation principles. The vast majority of them succumb to the following trivial lemma.

LEMMA 4.2. *Let T be any theory.*

(1) *Suppose that* $\Gamma_1\cap\Gamma_2\supseteq\Gamma$. *Then* $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ *holds for T iff* $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma_1\cap\Gamma_2)$ *holds for T (this is strict) and* $\Gamma_1\cap\Gamma_2\subseteq\Gamma$ *in T (an identification).*

(2) *Suppose that* $\Gamma_1\subseteq\Gamma_2$ *or* $\Gamma_2\subseteq\Gamma_1$. *Then* $\mathrm{Int}(\Gamma_1, \Gamma_2, \Gamma)$ *holds for T iff* $\Gamma_1\cap\Gamma_2\subseteq\Gamma$ *in T (an identification).*

In fact only four non-normal interpolation principles fail to succumb to this lemma: EUE^+, EUU^+, UEE^+ and UEU^+, and they can all be shown to be equivalent to $O=O^+$. Hence every non-normal interpolation principle is equivalent to an identification, possibly together with a strict interpolation principle, and so they are all classified.

The natural next step is to look at classes of formulas other than those we have considered. It does not seem very fruitful to study a finer classification of the kind of formulas we have dealt with – in fact several of the semantic properties have characterisations in terms of more delicate interpolation principles – but rather one should look at classes of larger quantifier complexity. It would be interesting to discover where interpolation principles corresponding to 'non-algebraic' semantic properties lie.

5. The characterisation of injections transferable

For completeness we give a brief proof of the characterisation of IT (which is simpler than that in [2]). The basic tool is the Compactness Theorem, used repeatedly in the manner of Lemma 1.1 of [1].

We say that $\phi(\bar{a})$ is *satisfiable over* A if $T+\Delta(A)+\phi(\bar{a})$ is consistent, and *persistently satisfiable over* A if it is satisfiable over B for each $B \geqslant A$ in T. It is easy to prove that T has IT iff for each $\phi(\bar{x})\in E$, $A\models T$ and $\bar{a}\in A$, if $\phi(\bar{a})$ is satisfiable over A then $\phi(\bar{a})$ is persistently satisfiable over A.

Now $\phi(\bar{a})$ is satisfiable over B iff not $B\models \bigvee_{\theta\in E^+\cap S(\phi)} \theta(\bar{a})$, where

$$S(\phi)=\{\theta(\bar{x}): T+\theta+\phi\vdash\}, \quad \text{i.e.} \quad B\models \bigwedge_{\theta\in E^+\cap S(\phi)} \neg\theta(\bar{a}).$$

Thus $\phi(\bar{a})$ is persistently satisfiable over A iff

$$T+\Delta_0(A)\vdash \bigwedge_{\theta\in E^+\cap S(\phi)} \neg\theta(\bar{a}),$$

i.e. for all $\theta(\bar{x})\in E^+$ with $T+\theta+\phi\vdash$,

$$T+\Delta_0(A)\vdash\neg\theta(\bar{a}), \quad \text{i.e.} \quad A\models \bigvee_{\beta\in E\cap S(\theta)} \beta(\bar{a}).$$

Hence T has IT iff for all $\phi(\bar{x})\in E$ and $\theta(\bar{x})\in E^+$ with $T+\theta+\phi\vdash$,
(*) given $A\models T$ and $\bar{a}\in A$,
if not

$$A\models \bigvee_{\alpha\in E^+\cap S(\phi)} \alpha(\bar{a}) \quad \text{then} \quad A\models \bigvee_{\beta\in E\cap S(\theta)} \beta(\bar{a}),$$

i.e.

$$A\models \bigvee_{\alpha\in E^+\cap S(\phi)} \alpha(\bar{a}) \vee \bigvee_{\beta\in E\cap S(\theta)} \beta(\bar{a}).$$

Thus (*) is equivalent to

(**) $\quad T\vdash \bigvee_{\alpha\in E^+\cap S(\phi)} \alpha(\bar{x}) \vee \bigvee_{\beta\in E\cap S(\theta)} \beta(\bar{x}),$

which by compactness is equivalent to

(***) there is $\alpha(\bar{x})\in E^+$ with $T+\alpha+\phi\vdash$ and $\beta(\bar{x})\in E$ with $T+\beta+\theta\vdash$

such that $T\vdash\alpha(\bar{x})\vee\beta(\bar{x})$.

This completes the proof of the characterisation of IT given in [2] and used in Theorem 2.1 of this paper.

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